Section 12.3 Adding / Subtracting Rational Expressions.

Ex $\frac{3}{11} + \frac{2}{11} = \frac{5}{11}$

denominators match?
Then keep that denominator
and add / subtract numerators.

In general $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

Ex $\frac{5}{12} + \frac{2}{21}$

Goal: make the denominators match...

denominators: 12 21

factor them: $12 = 2 \cdot 6 = 2 \cdot 2 \cdot 3$ $\quad 21 = 3 \cdot 7$

make the "least common denominator" LCD...

accounted for

$2 \cdot 2 \cdot 3 \cdot 7 = 84$

get our denoms to become 84

$= \frac{5}{12} \cdot \frac{7}{7} + \frac{2}{21} \cdot \frac{4}{4}$

$= \frac{35}{84} + \frac{8}{84}$

$= \frac{43}{84}$

look for simplification... none here in this example

To add/subtract rational expressions...

1) Identify denominators and factor them
2) Use this to write down the LCD.
3) Use that to "build up" the two expressions
4) Add/Subtract & look for simplification.

Ex $\frac{5}{x+8} - \frac{x}{x+8} = \frac{5-x}{x+8}$

denoms match...

$= \frac{-x+5}{x+8}$

$= \frac{-(x-5)}{x+8}$

Ex $\frac{1}{2x-1} - \frac{2x}{2x-1} = \frac{1-2x}{2x-1}$

$= \frac{-2x+1}{2x-1}$

$= \frac{-\cancel{(2x-1)} \cdot 1}{\cancel{(2x-1)} \cdot 1}$

$= \frac{-1}{1}$

$= -1, \quad x \neq \frac{1}{2}$

$\frac{2x-1}{2x-1} = 1$

only when

$2x-1 \neq 0$

$2x \neq 1$

$x \neq 1/2$

Ex $\quad \dfrac{3}{x-4} + \dfrac{2}{x+3}$

denoms: $\quad x-4 \qquad x+3$

factored: $\quad \underbrace{(x-4)} \qquad (x+3)$

LCD: $\quad (x-4)(x+3)$

$$= \frac{3}{(x-4)} \cdot \frac{(x+3)}{(x+3)} + \frac{2}{(x+3)} \frac{(x-4)}{(x-4)}$$

Now denom's match

$$= \frac{3x+9}{(x-4)(x+3)} + \frac{2x-8}{(x+3)(x-4)}$$

$\leftarrow$ don't multiply out denominators.

$$= \frac{5x+1}{(x-4)(x+3)}$$

No cancellation here.

$$\frac{3}{x-4} + \frac{2}{x+3} \overset{?}{=} \frac{5x+1}{(x-4)(x+3)}$$

pick some number... $x=5$

$$\frac{3}{5-4} + \frac{2}{5+3} \qquad\qquad \frac{5(5)+1}{(5-4)(5+3)}$$

$$\frac{3}{1} + \frac{2}{8} \qquad\qquad \frac{25+1}{(1)\cdot(8)}$$

$$3 + \frac{1}{4} \qquad\qquad \frac{26}{8}$$

$$3.25 \qquad\qquad \frac{13}{4}$$

$$\checkmark \qquad\qquad 3.25$$

Ex $\dfrac{x}{x-6} - \dfrac{10x-24}{x^2-6x}$

denom's ~~factors~~: $x-6$ $\quad x^2-6x$

factored: $(x-6)$ $\quad x\underbrace{(x-6)}_{\text{accounted for}}$

LCD: $(x-6)(x)$

$$= \frac{x}{(x-6)} \cdot \frac{(x)}{(x)} - \frac{10x-24}{x(x-6)}$$

Now denom's match

$$= \frac{x^2}{x(x-6)} - \frac{10x-24}{x(x-6)}$$

!!!

$$= \frac{x^2-10x+24}{x(x-6)}$$

Might still be cancellation... Factor numerator.

$$= \frac{\cancel{(x-6)}(x-4)}{x\cancel{(x-6)}}$$

$$= \frac{x-4}{x}, \quad x \neq 6$$

Ex $\dfrac{-1}{x-1} - 2$

$$= \frac{-1}{(x-1)} - \frac{2}{1}$$

$$= \frac{-1}{(x-1)} - \frac{2}{1} \cdot \frac{(x-1)}{(x-1)}$$

$$= \frac{-1}{(x-1)} - \frac{2x-2}{(x-1)}$$

$-1-(-2)$

$-1+2$

$$= \frac{-2x+1}{x-1} = \frac{-(2x-1)}{(x-1)}$$

Recall $\quad$ average speed $= \dfrac{\text{distance}}{\text{time}}$

$$s = \frac{d}{t}$$

$$t \cdot s = d$$

$$t = \frac{d}{s}$$

Ex $\quad$ We hiked a 6-mile trail. On the way out, average speed was x mph.
On the way back, we walked 1 mph faster, on avergge.

Let $T(x)$ be the total time it took us to get there and back. $\quad T:$ speed on the way there in mph $\longrightarrow$ total time in hr.

Write a formula for T.

$d = 6$, $s = x$, $t = \dfrac{6}{x}$

$d = 6$, $s = x+1$, $t = \dfrac{6}{x+1}$

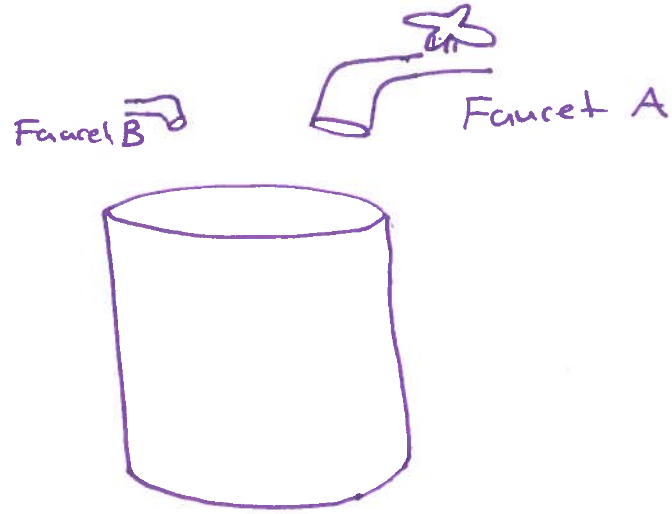
$$T(x) = \frac{6}{x} + \frac{6}{x+1}$$

$$= \frac{6}{x} \cdot \frac{(x+1)}{(x+1)} + \frac{6}{(x+1)} \cdot \frac{(x)}{(x)}$$

$$= \frac{6x+6}{x(x+1)} + \frac{6x}{x(x+1)} = \frac{12x+6}{x(x+1)} = \frac{6(2x+1)}{x(x+1)}$$

Ex

Water tank can hold 100 gal.



Suppose faucet A ~~to~~ take x minutes to fill the tank.

And suppose faucet B (working alone) takes 15 minutes longer than faucet A.

gallons/min

What will be the filling rate if both faucets run at the same time?

Faucet A's rate

$$\frac{100\text{ gal}}{x\text{ min}} = \frac{100}{x}\left(\frac{\text{gal}}{\text{min}}\right)$$

Facet B's rate

$$\frac{100}{x+15}\left(\frac{\text{gal}}{\text{min}}\right)$$

Working together: $\dfrac{100}{x} + \dfrac{100}{x+15}$

$$= \frac{100}{x}\cdot\frac{(x+15)}{(x+15)} + \frac{100}{(x+15)}\cdot\frac{(x)}{(x)}$$

$$= \frac{100x + 1500}{x(x+15)} + \frac{100x}{x(x+15)}$$

$$= \frac{200x + 1500}{x(x+15)} = \frac{100(2x+15)}{x(x+15)}$$

Adding and Subtracting Rational Expressions

1. Add or subtract the fractions/rational expressions.

a) $\frac{2}{3}+\frac{1}{4}$

denoms : 3, 4
factored : 3, 2·2
LCM: 3·2·2 = 12

$= \frac{2}{3}\cdot\frac{4}{4} + \frac{1}{4}\cdot\frac{3}{3}$

$= \frac{8}{12} + \frac{3}{12}$

$= \frac{11}{12}$

b) $\frac{3}{12}-\frac{5}{16}$

denoms: 12, 16
factored: 2·2·3, 2·2·2·2
LCM: 2·2·2·2·3 = 48

$= \frac{3}{12}\cdot\frac{4}{4} - \frac{5}{16}\cdot\frac{3}{3}$

$= \frac{12}{48} - \frac{15}{48}$

$= \frac{-3}{48}$

$= \frac{-1}{16}$

c) $\frac{8}{b^3}-\frac{5}{b^3}$

$= \frac{3}{b^3}$

d) $\frac{2z}{4-z}-\frac{3z-4}{4-z}$

$= \frac{-z+4}{4-z}$

$= \frac{-z+4}{-z+4}$

$= 1, \; z \neq 4$

e) $\frac{4}{4-n}+\frac{3}{2-n}$

denoms; 4-n, 2-n
factored: (4-n), (2-n)
LCM: (4-n)(2-n)

$= \frac{4}{(4-n)}\cdot\frac{(2-n)}{(2-n)} + \frac{3}{(2-n)}\cdot\frac{(4-n)}{(4-n)}$

$= \frac{8-4n}{(4-n)(2-n)} + \frac{12-3n}{(2-n)(4-n)}$

$= \frac{20-7n}{(4-n)(2-n)}$

f) $\frac{x}{x+4}+\frac{x+1}{x}$

denoms: x+4, x
factored : (x+4), (x)
LCM: (x+4)(x)

$= \frac{x}{(x+4)}\cdot\frac{(x)}{(x)} + \frac{(x+1)}{x}\cdot\frac{(x+4)}{(x+4)}$

$= \frac{x^2}{(x+4)(x)} + \frac{x^2+5x+4}{(x)(x+4)}$

$= \frac{2x^2+5x+4}{x(x+4)}$

g) $\dfrac{3n}{(4n-3)^2} - \dfrac{1}{4n-3}$

denoms: $(4n-3)^2$, $4n-3$
factored: $(4n-3)^2$, $(4n-3)$
LCM: $(4n-3)^2$

$$= \frac{3n}{(4n-3)^2} - \frac{1}{(4n-3)} \cdot \frac{(4n-3)}{(4n-3)}$$

$$= \frac{3n}{(4n-3)^2} - \frac{4n-3}{(4n-3)^2}$$

$$= \frac{-n+3}{(4n-3)^2}$$

h) $\dfrac{2x}{x-5} + \dfrac{2x-1}{3x^2-16x+5}$

denoms: $x-5$, $3x^2-16x+5$
factored: $(x-5)$, $(x-5)(3x-1$
LCM: $(x-5)(3x-1)$

$$= \frac{2x}{(x-5)} \cdot \frac{(3x-1)}{(3x-1)} + \frac{(2x-1)}{(x-5)(3x-1)}$$

$$= \frac{6x^2-2x}{(x-5)(3x-1)} + \frac{2x-1}{(x-5)(3x-1)}$$

$$= \frac{6x^2-1}{(x-5)(3x-1)}$$

i) $\dfrac{3}{(x-1)(x-2)} + \dfrac{4x}{(x+1)(x-2)}$

denoms,
factored: $(x-1)(x-2)$, $(x+1)(x-2)$
LCM: $(x-1)(x-2)(x+1)$

$$= \frac{3}{(x-1)(x-2)} \cdot \frac{(x+1)}{(x+1)} + \frac{4x}{(x+1)(x-2)} \cdot \frac{(x-1)}{(x-1)}$$

$$= \frac{3x+3}{(x-1)(x-2)(x+1)} + \frac{4x^2-4x}{(x+1)(x-2)(x-1)}$$

$$= \frac{4x^2-x+3}{(x-1)(x-2)(x+1)}$$

j) $\dfrac{3}{x^2-2x+1} - \dfrac{1}{x^2-3x+2}$

denoms: x^2-2x+1, x^2-3x+2
factored: $(x-1)^2$, $(x-1)(x-2)$
LCM: $(x-1)^2(x-2)$

$$= \frac{3}{(x-1)^2} \cdot \frac{(x-2)}{(x-2)} - \frac{1}{(x-1)(x-2)} \cdot \frac{(x-1)}{(x-1)}$$

$$= \frac{3x-6}{(x-1)^2(x-2)} - \frac{x-1}{(x-1)^2(x-2)}$$

$$= \frac{2x-5}{(x-1)^2(x-2)}$$

k) $-\dfrac{1}{x-5} - 2$

$$= \frac{-1}{x-5} - \frac{2}{1}$$

$$= \frac{-1}{x-5} - \frac{2}{1} \cdot \frac{(x-5)}{(x-5)}$$

$$= \frac{-1}{x-5} - \frac{2x-10}{x-5}$$

$$= \frac{-2x+9}{x-5}$$

l) $\dfrac{3}{x+1} + 2$

$$= \frac{3}{x+1} + \frac{2}{1}$$

$$= \frac{3}{x+1} + \frac{2}{1} \cdot \frac{(x+1)}{(x+1)}$$

$$= \frac{3}{x+1} + \frac{2x+2}{x+1}$$

$$= \frac{2x+5}{x+1}$$

2. You drove to Salem, 45 miles away, averaging a speed of x miles per hour. On the way back, your average speed was 5 miles per hour *slower*. Let $T(x)$ be the total amount of time spent driving. Write a simplified formula for $T(x)$.

$d=45,\ s=x,\ t=\frac{45}{x}$

$P \rightleftarrows S$

$d=45,\ s=x-5,\ t=\frac{45}{x-5}$

$$T(x) = \frac{45}{x} + \frac{45}{x-5}$$
$$= \frac{45}{x}\cdot\frac{(x-5)}{(x-5)} + \frac{45}{(x-5)}\cdot\frac{x}{x}$$
$$= \frac{45x-225}{x(x-5)} + \frac{45x}{x(x-5)}$$
$$= \frac{90x-225}{x(x-5)}$$
$$= \frac{45(2x-5)}{x(x-5)}$$

Alternative: factor out 45

$$\frac{45}{x} + \frac{45}{x-5} = 45\left(\frac{1}{x} + \frac{1}{x-5}\right)$$
$$= 45\left(\frac{1}{x}\cdot\frac{(x-5)}{(x-5)} + \frac{1}{(x-5)}\cdot\frac{x}{x}\right)$$
$$= 45\left(\frac{x-5}{x(x-5)} + \frac{x}{x(x-5)}\right)$$
$$= 45\left(\frac{2x-5}{x(x-5)}\right)$$

3. You and a partner will paint the siding of a house this summer. This house has 3000 square feet of siding. If you work alone, you could paint it all in x hours. If your partner works alone, they take 10 more hours than you do.

   a) What is the rate (in square feet per hour) that you paint at?

   $$\frac{3000}{x}$$

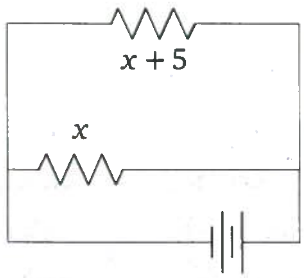
   b) What is the rate (in square feet per hour) that your partner paints at?

   $$\frac{3000}{x+10}$$

   c) If you work together, it makes sense to add your rates. Write a simplified expression for the rate that you paint the house when you work as a team.

$$\frac{3000}{x} + \frac{3000}{x+10}$$
$$= \frac{3000}{x}\cdot\frac{(x+10)}{(x+10)} + \frac{3000}{(x+10)}\cdot\frac{(x)}{(x)}$$
$$= \frac{3000x+30{,}000}{x(x+10)} + \frac{3000x}{x(x+10)}$$
$$= \frac{6000x+30{,}000}{x(x+10)} = \frac{6000(x+5)}{x(x+10)}$$

4. Here is an electrical circuit where a battery is wired to power two lights in parallel. The lights are represented by the resistors. One has resistance x ohms, and the other's resistance is 5 ohms more.



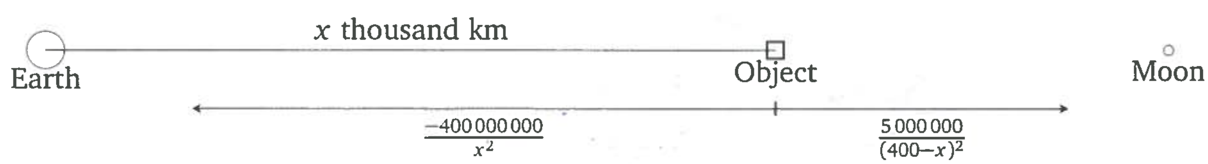
With parallel circuits, the overall resistance R satisfies this equation:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Find $\frac{1}{R}$ for this circuit in terms of x.

$$\frac{1}{R} = \frac{1}{x} + \frac{1}{x+5}$$
$$= \frac{1}{x}\cdot\frac{(x+5)}{(x+5)} + \frac{1}{(x+5)}\cdot\frac{(x)}{(x)}$$
$$= \frac{x+5}{x(x+5)} + \frac{x}{x(x+5)}$$
$$= \frac{2x+5}{x(x+5)}$$

5. Suppose a 1 kilogram object is in between the Earth and the Moon, x thousand kilometers from the center of the Earth. Gravity from Earth pulls the object to the left in the diagram, while gravity from the moon pulls the object to the right. (The numbers have been rounded a lot.)



The two forces are illustrated in the diagram. Add them together to get the net gravitational force on the object.

$$\frac{-400{,}000{,}000}{x^2} + \frac{5000000}{(400-x)^2} = 5{,}000{,}000\left(\frac{-80}{x^2} + \frac{1}{(400-x)^2}\right)$$
$$= 5{,}000{,}000\left(\frac{-80}{x^2}\cdot\frac{(400-x)^2}{(400-x)^2} + \frac{1}{(400-x)^2}\cdot\frac{x^2}{x^2}\right)$$
$$= 5{,}000{,}000\left(\frac{-80(160000 - 800x + x^2)}{x^2(400-x)^2} + \frac{x^2}{x^2(400-x)^2}\right)$$
$$= 5{,}000{,}000\left(\frac{-1{,}280{,}000 + 64{,}000x - 79x^2}{x^2(400-x)^2}\right)$$