11.3 Absolute Value Equations and Inequalities

An expression like $|x-5|$ has special meaning...

Imagine x is close to 5

... like 4.9 $\rightarrow$ result is a small positive number.

... like 5.1

Or if x is far from 5

... like 100 $\rightarrow$ result is a large positive number

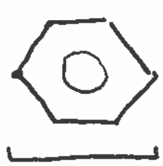
... like −100

$|x-5|$

gives you the distance between x and 5 on a number line

(not caring about x being larger (to the right) or smaller (to the left) of 5)

<u>Ex</u> A metal, machined, bolt is <u>supposed</u> to have a 10mm diameter



10 mm

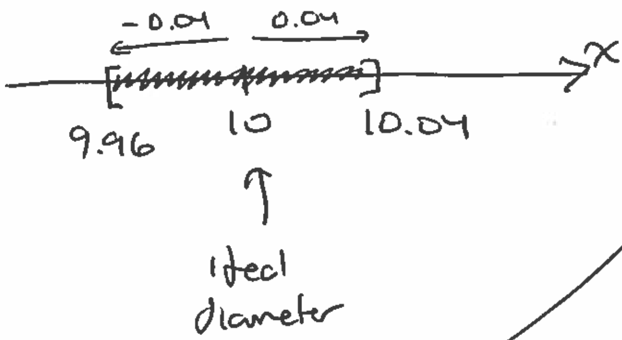
. It's acceptable to be within 0.04 mm of this.

So if x represents the diameter of one bolt...

$$|x-10| \leq 0.04$$ (an absolute value inequality)

(the diameter must be within 0.04 of 10.)

visually



The solution set is

$[9.96, 10.04]$

algebraically

$x-10$ could be positive, then $x-10 \le 0.04$

or it could be negative, then $-(x-10) \le 0.04$

$x \le 10.04$

$-x+10 \le 0.04$

$-x \le -9.96$

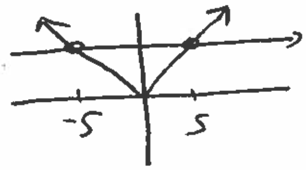
$x \ge 9.96$

and!

$$9.96 \le x \le 10.04$$

$\Longrightarrow$ the solution set is $[9.96, 10.04]$.

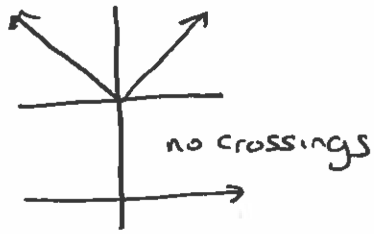
__Ex__ Solve $|x| = 5$.



Well, x could be 5 or -5.

The solution set is $\{5, -5\}$

__Ex__ Solve $|x| = -8$.



There are no solutions, since abs value cannot produce a negative output.

The solution set is $\{\ \}$.

__Ex__ Solve $|2x+5| = 9$

Realize: $2x+5$ could be pos... $\Longrightarrow$ $2x+5 = 9$

or $2x+5$ could be neg... $\Longrightarrow$ $-(2x+5) = 9$

$$\begin{aligned} 2x+5 &= 9 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} -(2x+5) &= 9 \\ -2x-5 &= 9 \\ -2x &= 14 \\ x &= -7 \end{aligned}$$

Tells us that $2, -7$ are the only <u>possibilities</u> for solutions.

Check: $x=2$: $|2\cdot 2+5| \overset{?}{=} 9$

$|4+5| \overset{?}{=} 9$

$|9| \overset{\checkmark}{=} 9$

$x=-7$: $|2(-7)+5| \overset{?}{=} 9$

$|-14+5| \overset{?}{=} 9$

$|-9| \overset{\checkmark}{=} 9$

So the solution set is $\{2, -7\}$

Ex Solve $|23-8x| = 17$

$$23-8x = 17 \quad \text{or} \quad -(23-8x) = 17$$

$$-8x = -6 \qquad\qquad -23+8x = 17$$

$$x = \frac{-6}{-8} = \frac{3}{4} \qquad\qquad 8x = 40$$

$$x = 5$$

Check: $|23-8(\tfrac{3}{4})| \stackrel{?}{=} 17$ $\qquad$ $|23-8(5)| \stackrel{?}{=} 17$

$|23-6| \stackrel{?}{=} 17$ $\qquad$ $|23-40| \stackrel{?}{=} 17$

$|17| \stackrel{\checkmark}{=} 17$ $\qquad$ $|-17| \stackrel{\checkmark}{=} 17$

Solution set is $\left\{\frac{3}{4}, 5\right\}$

Ex Solve $|4-9x| = 0$

$$4-9x = 0 \quad \text{or} \quad -(4-9x) = 0$$

$$-9x = -4 \qquad\qquad -4+9x = 0$$

$$x = \frac{4}{9} \qquad\qquad 9x = 4$$

$$x = \frac{4}{9}$$

Same!

Check: $|4-9(\tfrac{4}{9})| \stackrel{?}{=} 0$

$|4-4| \stackrel{?}{=} 0$

$|0| \stackrel{\checkmark}{=} 0$

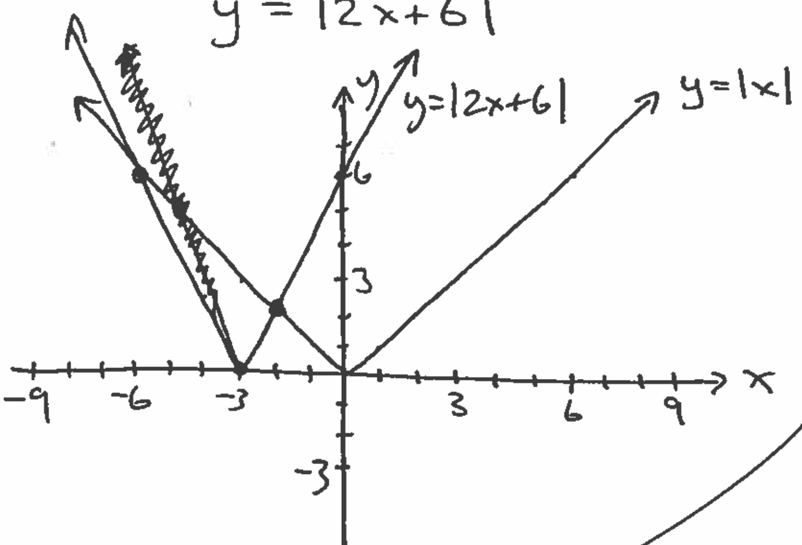
The solution set is $\left\{\frac{4}{9}\right\}$.

Solve $|x| = |2x+6|$. (Now abs. value expression on both sides!)

graphical

$y = |x|$

$y = |2x+6|$



Intersection Points

$(-2,2)$ & ~~$(-5,5)$~~ $(-6,6)$

Solutions are

-2 and ~~-5~~ -6

Sol set is $\{-2, \cancel{-5}\}$. -6

algebraically

x could be pos, $|x| = x$

x could be neg, $|x| = -x$

$2x+6$ could be pos,

$|2x+6| = 2x+6$

or negative...

$|2x+6| = -(2x+6)$

$$|x| = |2x+6|$$

$x = 2x+6$

$-x = 6$

$x = -6$

$-x = 2x+6$

$-3x = 6$

$x = -2$

~~$x = -(2x+6)$~~

redundant, same as second.

~~$-x = -(2x+6)$~~

redundant, same as first.

Check -6 $|-6| \stackrel{?}{=} |2(-6)+6|$

$6 \stackrel{?}{=} |-6|$

✓

Check -2

$|-2| \stackrel{?}{=} |2(-2)+6|$

$2 \stackrel{?}{=} |2|$ ✓

Sol set is $\{-6, -2\}$.

Ex Solve $\left|\frac{1}{2}x+1\right| = \left|\frac{1}{3}x+2\right|$

$\frac{1}{2}x+1 = \frac{1}{3}x+2$ $\left\{ \frac{1}{2}x+1 = -\left(\frac{1}{3}x+2\right) \right\}$ $\cancel{-\left(\frac{1}{2}x+1\right) = \frac{1}{3}x+2}$

redundant

$\cancel{-\left(\frac{1}{2}x+1\right) = -\left(\frac{1}{3}x+2\right)}$

redundant

$$\frac{1}{2}x+1-\frac{1}{3}x = 2$$
$$\frac{1}{2}x-\frac{1}{3}x = 1$$
$$\frac{3}{6}x - \frac{2}{6}x = 1$$
$$\frac{1}{6}x = 1$$
$$x = 6$$

$$\frac{1}{2}x+1 = -\frac{1}{3}x-2$$
$$\frac{1}{2}x+1+\frac{1}{3}x = -2$$
$$\frac{1}{2}x+\frac{1}{3}x = -3$$
$$\frac{3}{6}x+\frac{2}{6}x = -3$$
$$\frac{5}{6}x = -3$$
$$\frac{\cancel{6}}{\cancel{5}}\left(\frac{\cancel{5}}{\cancel{6}}x\right) = \frac{6}{5}(-3)$$
$$x = \frac{-18}{5}$$

Check:

$$\left|\frac{1}{2}(6)+1\right| \stackrel{?}{=} \left|\frac{1}{3}(6)+2\right|$$
$$|3+1| \stackrel{?}{=} |2+2|$$
$$|4| \stackrel{\checkmark}{=} |4|$$

$$\left|\frac{1}{2}\left(\frac{-18}{5}\right)+1\right| \stackrel{?}{=} \left|\frac{1}{3}\left(\frac{-18}{5}\right)+2\right|$$
$$\left|\frac{-9}{5}+1\right| \stackrel{?}{=} \left|\frac{-6}{5}+2\right|$$
$$\left|\frac{-9}{5}+\frac{5}{5}\right| \stackrel{?}{=} \left|\frac{-6}{5}+\frac{10}{5}\right|$$
$$\left|\frac{-4}{5}\right| \stackrel{?}{=} \left|\frac{4}{5}\right| \checkmark$$

The sol. set is

$$\left\{6, -\frac{18}{5}\right\}.$$

Solve $\quad |2x+8| = |2x-7|$

$2x+8 = 2x-7$
$\quad -2x \qquad -2x$

$8 = -7$

No solutions here!

$\left\{ \begin{array}{l} 2x+8 = -(2x-7) \\ 2x+8 = -2x+7 \\ 4x = -1 \\ x = -\frac{1}{4} \end{array} \right\}$

$\cancel{-(2x+8) = 2x-7}$ redundant

$\cancel{-(2x+8) = -(2x-7)}$ redundant

Check

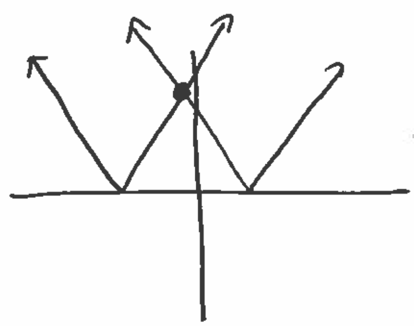
$|2(-\frac{1}{4})+8| \stackrel{?}{=} |2(-\frac{1}{4})-7|$

$|-\frac{1}{2}+8| \stackrel{?}{=} |-\frac{1}{2}-7|$

$|7.5| \stackrel{\checkmark}{=} |-7.5|$

Solution set is
$\{-\frac{1}{4}\}$

(visually



only one crossing because of parallel lines)

Ex Solve $\quad |x-1| = |1-x|$

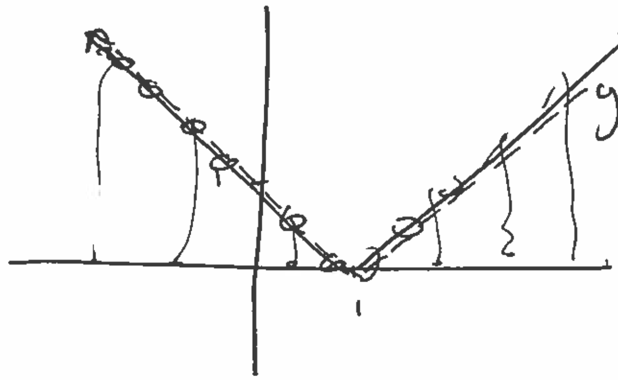
$x-1 = 1-x$
$2x-1 = 1$
$2x = 2$
$x = 1$
(Checks ✓)

$\left\{ \begin{array}{l} x-1 = -(1-x) \\ x-1 = -1+x \end{array} \right\}$

$-1 = -1$
(true, no matter what x is!)
true for all real numbers x . . .

$\cancel{-(x-1) = 1-x}$ redundant

$\cancel{-(x-1) = -(1-x)}$ redundant



these graphs overlap.

for all x, $|x-1| = |1-x|$.

So the solution set is $(-\infty, \infty)$.

Absolute value inequalities

Now solve $|2x+5| \leq 9$

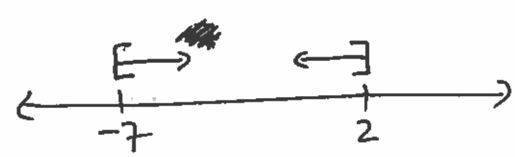
Either $2x+5$ is positive and we solve $2x+5 \leq 9$

Or $2x+5$ is negative and we solve $-(2x+5) \leq 9$

$$2x+5 \leq 9$$
$$2x \leq 4$$
$$x \leq 2$$

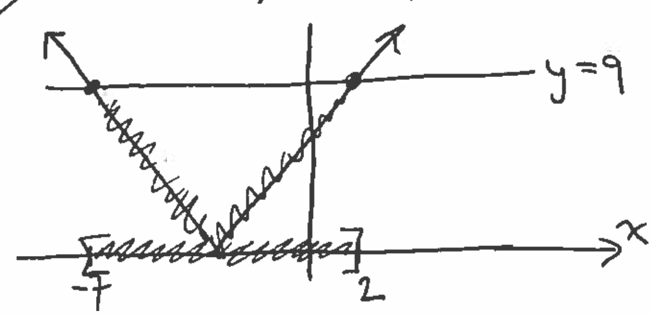
$$-(2x+5) \leq 9$$
$$-2x-5 \leq 9$$
$$-2x \leq 14$$
$$x \geq -7$$

"or equal to"



Solution set is $[-7, 2]$.

Graphically check



$|2x+5| \leq 9$

"V" under flat line

Ex Solve $\left|\frac{2}{3}x+1\right| < 3$

$$\overbrace{\frac{2}{3}x+1}^{\text{if this is posite}} < 3$$

$$\frac{2}{3}x < 2$$

$$x < \frac{3}{2}\cdot 2$$

$$x < 3$$

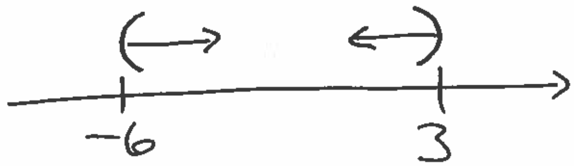
$$-\Big(\overbrace{\tfrac{2}{3}x+1}^{\text{if this is negative}}\Big) < 3$$

$$-\frac{2}{3}x - 1 < 3$$

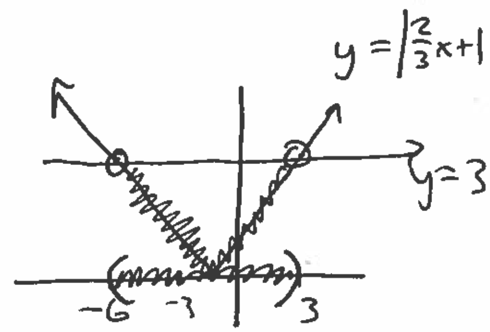
$$-\frac{2}{3}x < 4$$

$$x > \left(-\frac{3}{2}\right)\cdot 4$$

$$x > -6$$



The solution set is $(-6,3)$.



Last two exercises were about being close to some number ($\leq$, $<$)

$\left|\frac{2}{3}x+1\right| < 3$

"V" under flat line

Ex Solve $|x-1| > 3$

Now we're looking to be far away from some number.

either $\overbrace{x-1}^{\text{if this is pos.}} > 3$

$$x > 4$$

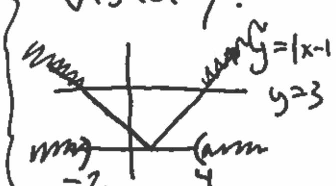
$$-(\overbrace{x-1}^{\text{if this is neg}}) > 3$$

$$-x+1 > 3$$

$$-x > 2$$

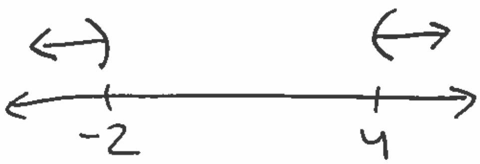
$$x < -2$$

Visually.



$|x-1| > 3$

"V" above flat line



Solution set is $(-\infty,-2)\cup(4,\infty)$.