If $\vec{u}, \vec{v} \in \mathbb{R}^n$, then the inner product of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Properties

1. commutative: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. distributive: $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{u} \cdot \vec{v}$
3. associative: $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. $\vec{u} \cdot \vec{u} \geq 0$, $\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = \vec{0}$

The length of a vector is

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

6.1 problems 2, 4, 6, 8

$$\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

2. $\vec{w} \cdot \vec{w} = (3)(3) + (-1)(-1) + (-5)(-5)$
$= 35$

4. $\frac{1}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{1}{5} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
$= \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$

6. $\frac{\vec{x} \cdot \vec{w}}{\vec{x} \cdot \vec{x}} \vec{x} = \frac{5}{49} \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$

8. $|| \vec{x} || = \sqrt{\vec{x} \cdot \vec{x}}$
$= \sqrt{49}$
$= 7$

Definitions

- A unit vector, $\vec{u}$, has the property that $||\vec{u}|| = 1$
- Normalizing a vector is finding the unit vector that points in the same

direction as the original vector.

10) Normalize $\begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix} = \vec{v}$

The unit vector is

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{[-6, 4, 3]^T}{\sqrt{61}}$$

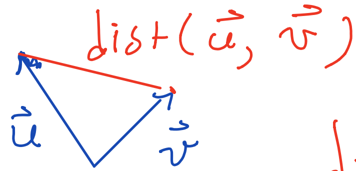
$$= \frac{1}{\sqrt{61}} \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}$$

12) normalize $\vec{w} = \begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$

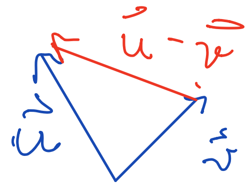
$3\vec{w}$ is codirectional with $\vec{w}$, so we can normalize $3\vec{w}$ and meet our objective

$$\frac{\begin{bmatrix} 8 \\ 6 \end{bmatrix}}{\left\| \begin{bmatrix} 8 \\ 6 \end{bmatrix} \right\|} = \frac{1}{10} \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Moving on ...



$$dist(\vec{u}, \vec{v}) = \| \vec{u} - \vec{v} \|$$



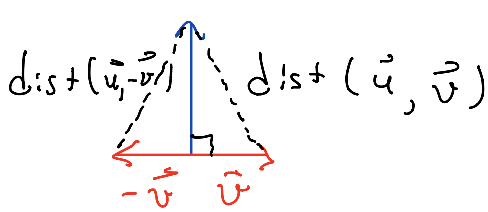
14) $\vec{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$

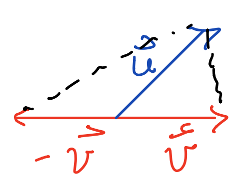
$$\begin{aligned} dist(\vec{u}, \vec{v}) &= \| \vec{u} - \vec{v} \| \\ &= \left\| \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix} \right\| \\ &= \sqrt{68} \end{aligned}$$

Theorem

Two vectors, $\vec{u}$ & $\vec{v}$, are perpendicular iff

$$dist(\vec{u}, \vec{v}) = dist(\vec{u}, -\vec{v})$$





From SAS

$dist(\vec{u}, -\vec{v}) = dist(\vec{u}, \vec{v})$

$dist(\vec{u}, -\vec{v}) > dist(\vec{u}, \vec{v})$

Proof

$$dist(\vec{u}, \vec{v}) = dist(\vec{u}, -\vec{v})$$

iff $\quad \|\vec{u} - \vec{v}\| = \|\vec{u} - (-\vec{v})\|$

iff $\quad \|\vec{u} - \vec{v}\|^2 = \|\vec{u} + \vec{v}\|^2$

iff $\quad (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$

iff $\quad \vec{u} \cdot (\vec{u} - \vec{v}) - \vec{v} \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot (\vec{u} + \vec{v}) + \vec{v} \cdot (\vec{u} + \vec{v})$

iff $\quad (\vec{u} - \vec{v}) \cdot \vec{u} - (\vec{u} - \vec{v}) \cdot \vec{v} = (\vec{u} + \vec{v}) \cdot \vec{u} + (\vec{u} + \vec{v}) \cdot \vec{v}$

iff $\quad \cancel{\vec{u} \cdot \vec{u}} - \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{v}} = \cancel{\vec{u} \cdot \vec{u}} + \vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{v}}$

iff $\quad -\vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{v}$

iff $\quad 0 = 4\vec{u} \cdot \vec{v}$

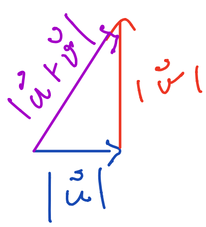
iff $\quad \vec{u} \cdot \vec{v} = 0$

iff $\vec{u} \perp \vec{v}$ (see ☆ below)

Definition: $\vec{u}$ and $\vec{v}$ are orthogonal iff $\vec{u} \cdot \vec{v} = 0$. In 2 & 3 dimensions, non-zero orthogal vectors are perpendicular.

P. Theorem! Theorem:

Two vectors are orthogonal iff $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$



$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|v\|^2$$

iff $(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v}$

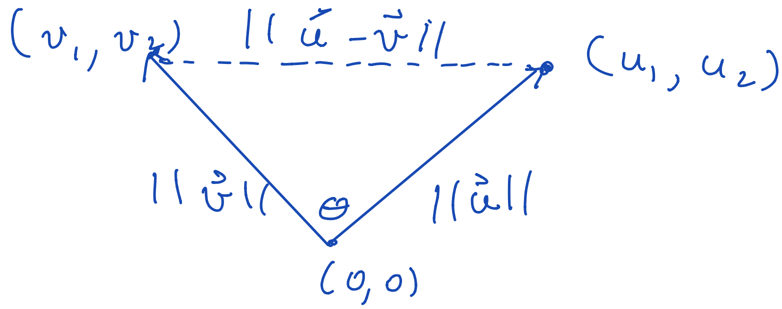
iff $\cancel{\vec{u}\cdot\vec{u}} + 2\vec{u}\cdot\vec{v} + \cancel{\vec{v}\cdot\vec{v}} = \cancel{\vec{u}\cdot\vec{u}} + \cancel{\vec{v}\cdot\vec{v}}$

iff $2\vec{u}\cdot\vec{v} = 0$

iff $\vec{u}\cdot\vec{v} = 0$

☆ Theorem: $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(\theta)$ where θ is the angle formed when $\vec{u}$ & $\vec{v}$ are drawn tail-to-tail

2-d



Law of cosines

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos(\theta)$$

$$(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2\|\vec{u}\|\|\vec{v}\| \cos(\theta)$$

$$\langle u_1 - v_1, u_2 - v_2 \rangle \cdot \langle u_1 - v_1, u_2 - v_2 \rangle$$

$$= \langle u_1, u_2 \rangle \cdot \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle \cdot \langle v_1, v_2 \rangle - 2\|\vec{u}\|\|\vec{v}\|\cos(\theta)$$

$$(u_1 - v_1)(u_1 - v_1) + (u_2 - v_2)(u_2 - v_2)$$

$$= u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\|\vec{u}\|\|\vec{v}\|\cos(\theta)$$

$$\cancel{u_1^2} - 2u_1v_1 + \cancel{v_1^2} + \cancel{u_2^2} - 2u_2v_2 + \cancel{v_2^2}$$

$$= \cancel{u_1^2} + \cancel{u_2^2} + \cancel{v_1^2} + \cancel{v_2^2} - 2\|\vec{u}\|\|\vec{v}\|\cos(\theta)$$

$$-2u_1v_1 - 2u_2v_2 = -2\|\vec{u}\|\|\vec{v}\|\cos(\theta)$$

$$\therefore \quad u_1v_1 + u_2v_2 = \|\vec{u}\|\|\vec{v}\|\cos(\theta)$$

$$\therefore \quad \vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(\theta).$$

note: The use of this is:

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|},$$

so $\theta = \cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$

Definition

Let W be a subspace of $\mathbb{R}^n$. The orthogonal compliment of W, $W^\perp$, is the set of vectors from $\mathbb{R}^n$ that are orthogal to every vector in W.

Theorem:

$W^\perp$ is a subspace of $\mathbb{R}^n$.

Suppose that $\vec{u}$ & $\vec{v} \in W^\perp$.

1) closure over addition.

we're given that $\vec{u} \cdot \vec{w} = 0$

and $\vec{v} \cdot \vec{w} = 0$ $\forall$ $\vec{w} \in W$.

$\forall$ for every

So $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

$= 0 + 0$

$= 0$ $\forall$ $\vec{w} \in W$

So $\vec{u} + \vec{v} \in W^\perp$

2) closure over scalar multiplication

$(c\vec{u}) \cdot \vec{w} = \vec{u} \cdot (c\vec{w})$

$= 0$ $\forall$ $\vec{w} \in W$

So $c\vec{u} \in W^\perp$ QED

Theorem

- $(\text{Row } A)^\perp = \text{Nul } A$
- $(\text{Col } A)^\perp = \text{Nul } A^T$

Proof

$((\text{Row } A)^\perp = \text{Nul } A)$ Let $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$ & $\vec{x} \in \text{Nul } A$

Then

$$A\vec{x} = \begin{bmatrix} R_1 \cdot \vec{x} \\ R_2 \cdot \vec{x} \\ \vdots \\ R_n \cdot \vec{x} \end{bmatrix}, \text{ so}$$

$$A\vec{x} = \vec{0} \implies R_i \cdot \vec{x} = 0 \ \forall i$$

QED

$$\text{Nul } A^T = [\text{Row } A^T]^\perp$$
$$= (\text{Col } A)^\perp$$

26 ish) $\vec{u} = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$

$W = \text{Span}(\vec{u})$

The orthogonal compliment is a plane through the origin that is perpendicular to $\vec{u}$.

Let's find a basis for that plane

$$\vec{u} \cdot \vec{x} = 0 \implies 5x_1 - 6x_2 + 7x_3 = 0$$
$$\implies x_1 = \frac{6}{5}x_2 - \frac{7}{5}x_3$$

So vectors in $W^\perp$ have form

$$\begin{bmatrix} \frac{6}{5}x_2 - \frac{7}{5}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 6/5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -7/5 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ A basis for $W^\perp$ is

$$\left\{ \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 5 \end{bmatrix} \right\}$$

6.2 // Definition

$\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is an <u>orthogonal</u> <u>set</u> iff $\vec{u}_i \cdot \vec{u}_j = 0$ $\forall\, i \neq j$.

<u>Theorem</u>

An orthogonal set of non-zero vectors is always linearly independent.

<u>Proof</u> Let the set be $\{\vec{u}_1, \dots, \vec{u}_n\}$.

$$\begin{aligned} 0 &= 0 \cdot \vec{u}_1 \\ &= (c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_n \vec{u}_n) \cdot \vec{u}_1 \\ &= (c_1 \vec{u}_1) \cdot \vec{u}_1 + (c_2 \vec{u}_2) \cdot \vec{u}_1 + \dots + (c_n u_n) \cdot \vec{u}_1 \end{aligned}$$

$= c_1(\vec{u}_1 \cdot \vec{u}_1) + c_2(\vec{u}_2 \cdot \vec{u}_1) + \cdots + c_n(\vec{u}_n \cdot \vec{u}_1)$

$= c_1(\vec{u}_1 \cdot \vec{u}_1) + 0 + 0 + \cdots + 0$

So, $0 = c_1(u_1 \cdot u_1)$

Since $\vec{u}_1 \neq \vec{0}$, $\vec{u}_1 \cdot \vec{u}_1 \neq 0$

$\therefore c_1 = 0$

Repeating the process by dotting on the right by $\vec{u}_2$, then $\vec{u}_3, \ldots$

establishes that every other $c_i = 0$

Theorem

An orthogonal set of n non-zero vectors from $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

Theorem

If $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is an orthogonal basis for $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$, then

$\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$

where $c_i = \dfrac{\vec{y} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i}$

proof

Suppose that $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$

then:

$$\vec{y}\cdot\vec{u}_1 = (c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n)\cdot\vec{u}_1$$

$$= c_1(\vec{u}_1\cdot\vec{u}_1) + c_2(\vec{u}_2\cdot\vec{u}_1) + \cdots + c_n(\vec{u}_n\cdot\vec{u}_1)$$

$$= c_1(\vec{u}_1\cdot\vec{u}_1) + 0 + 0 + \cdots + 0$$

So, $\vec{y}\cdot\vec{u}_1 = c_1(\vec{u}_1\cdot\vec{u}_1)$

$$\therefore \quad c_1 = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}$$

Repeat for $\vec{u}_2, \ldots, \vec{u}_n$

Sect 6.2, #10

$\vec{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$,

$\vec{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

$\vec{u}_1\cdot\vec{u}_2 = 6 - 6 + 0 = 0$

$\vec{u}_1\cdot\vec{u}_3 = 3 - 3 + 0 = 0$

$\vec{u}_2\cdot\vec{u}_3 = 2 + 2 - 4 = 0$

$\therefore$ $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

$$\therefore \vec{x} = \frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \frac{\vec{x}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\vec{u}_3$$

$$= \frac{24}{18}\begin{bmatrix}3\\-3\\0\end{bmatrix} + \frac{3}{9}\begin{bmatrix}2\\2\\-1\end{bmatrix} + \frac{6}{18}\begin{bmatrix}1\\1\\4\end{bmatrix}$$

$$= \frac{4}{3}\begin{bmatrix}3\\-3\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}2\\2\\-1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\1\\4\end{bmatrix}$$

Check
$$\begin{bmatrix}4 + \frac{2}{3} + \frac{1}{3}\\ -4 + \frac{2}{3} + \frac{1}{3}\\ 0 - \frac{1}{3} + \frac{4}{3}\end{bmatrix} = \begin{bmatrix}5\\-3\\1\end{bmatrix} \checkmark$$

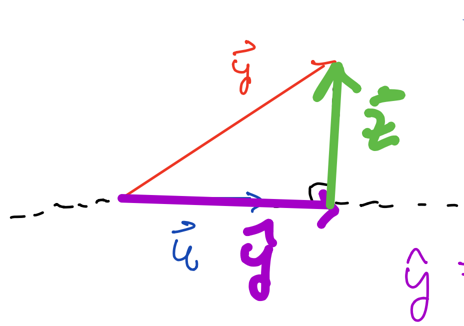
Definition and Theorem.

The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is:

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

Furthermore, there is a unique vector, $\vec{z}$, with the property that $\vec{z}$ is orthogonal to $\hat{y}$ and $\hat{y} + \vec{z} = \vec{u}$.

$\vec{z}$ is called the orthogonal component of $\vec{y}$ onto $\vec{u}$ and $\vec{z} = \vec{y} - \hat{y}$



$\hat{y} = \text{proj}_{\vec{u}}\vec{y}$

$\vec{z} = \text{orth}_{\vec{u}}\vec{y}$

6.2 #12) $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\text{proj}_{\vec{u}}\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \frac{-4}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

14) $\vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$$\text{proj}_{\vec{u}}\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \frac{20}{50}\cdot\begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

$$\text{orth}_{\vec{u}}\vec{y} = \vec{y} - \text{proj}_{\vec{u}}\vec{y}$$

$$= \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}.$$

Check $\quad proj_{\vec{w}_0}\vec{y} \cdot orth_{\vec{w}_0}\vec{y}$

$$= -\frac{56}{25} + \frac{56}{25}$$

$$= 0 \checkmark$$