6.1 Inner Product

(Dot product is a special case of "inner product")

A way to multiply two vectors resulting in a scalar.

A inner product between $\vec{u}$ and $\vec{v}$ is written $\langle \vec{u}, \vec{v} \rangle$.

Rules:
- $\langle \vec{u}+\vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$
- $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- $\langle c\vec{u}, \vec{v} \rangle = \langle \vec{u}, c\vec{v} \rangle = c \cdot \langle \vec{u}, \vec{v} \rangle$
- $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = \vec{0}$.

One special case is the dot product between two vectors in $\mathbb{R}^n$.

$$\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 v_2 + \cdots + u_n \cdot v_n$$

Ex
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix} = 1 \cdot 3 + 2(-4) = 3 - 8 = -5$$

Note:
$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = [u_1 \ u_2 \ \cdots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \vec{u}^T \vec{v}$$

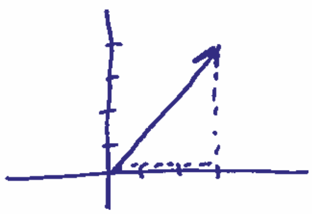
Length in $\mathbb{R}^n$

define length of $\vec{v}$ to be...

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

Ex $\left\|\begin{bmatrix}3\\4\end{bmatrix}\right\| = \sqrt{\begin{bmatrix}3\\4\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix}} = \sqrt{9+16} = \sqrt{25}$

$= 5$



Ex Suppose $H = \text{Span}\left\{\begin{bmatrix}4\\7\end{bmatrix}\right\}$.

Well $\mathcal{B} = \left\{\begin{bmatrix}4\\7\end{bmatrix}\right\}$ is a basis for H...

Instead, find a basis $\mathcal{C}$ with unit-length vectors...

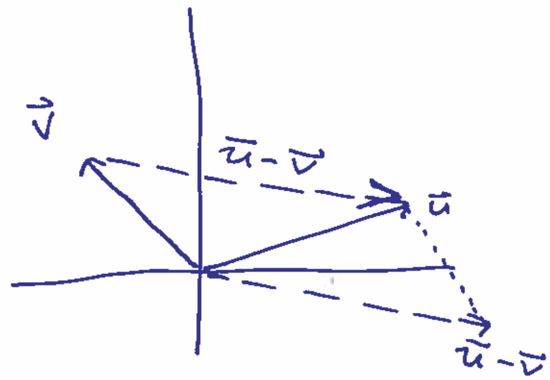
$$\left\|\begin{bmatrix}4\\7\end{bmatrix}\right\| = \sqrt{\begin{bmatrix}4\\7\end{bmatrix}\cdot\begin{bmatrix}4\\7\end{bmatrix}} = \sqrt{16+49} = \sqrt{65}$$

Take $\left\{\begin{bmatrix}4/\sqrt{65}\\7/\sqrt{65}\end{bmatrix}\right\}$.

Replacing $\vec{v}$ with $\dfrac{\vec{v}}{\|\vec{v}\|}$ gives a parallel vector, one unit long.

Define distance between $\vec{u}$ and $\vec{v}$ is:

$$\|\vec{u} - \vec{v}\|$$



Ex $\quad \text{dist}\left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}\right)$

$$= \left\| \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\| = \sqrt{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}$$

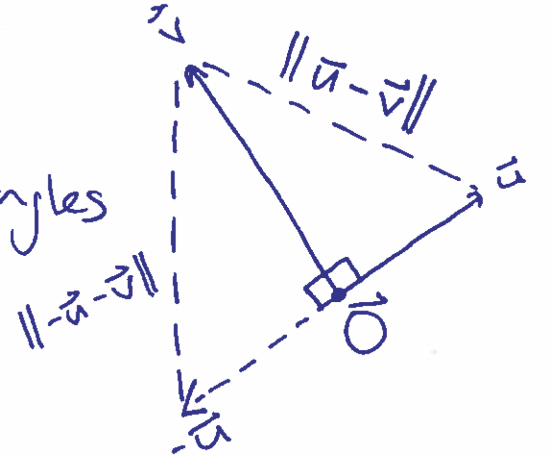
$$= \sqrt{9 + 4 + 1}$$

$$= \sqrt{14}$$

Orthogonal vectors

"right angles..."

Visualize $\vec{u}$, $\vec{v}$ at right angles



Picture shows

$$\|\vec{u} - \vec{v}\| = \|-\vec{u} - \vec{v}\|$$

$$\sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})} = \sqrt{(-\vec{u} - \vec{v}) \cdot (-\vec{u} - \vec{v})}$$

$$\cancel{\vec{u} \cdot \vec{u}} - 2\vec{u} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{v}} = \cancel{\vec{u} \cdot \vec{u}} + 2\vec{u} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{v}}$$

$$-2\vec{u}\cdot\vec{v} = 2\vec{u}\cdot\vec{v} \implies \vec{u}\cdot\vec{v} = 0$$

When $\vec{u}, \vec{v}$ are ~~of~~ perpendicular, $\vec{u}\cdot\vec{v} = 0$.

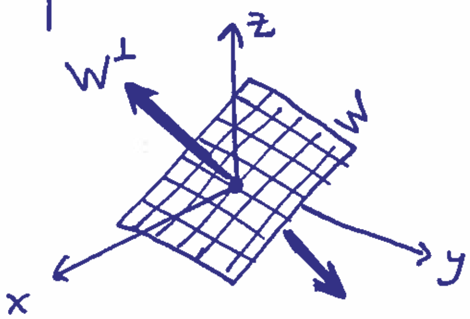
We could define "orthogonal" to mean $\vec{u}\cdot\vec{v} = 0$.

Ex Are $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ orthogonal?

Check $\vec{u}\cdot\vec{v} \stackrel{?}{=} 0$

$-6+2-3 \neq 0 \implies$ these are not orthogonal.

Given a subspace W of $\mathbb{R}^n$.....



Define $W^\perp$

"W perp",

"the orthogonal complement to W"

... is $\{\vec{v} \mid \vec{v} \text{ is } \perp \text{ to all vectors in } W\}$

Is $W^\perp$ a subspace of $\mathbb{R}^n$?

W is a subspace...

- Is $\vec{0}$ in $W^\perp$?

  Well, let $\vec{w}$ be in W.

  $\vec{0} \cdot \vec{w} = 0$.

  So $\vec{0}$ is $\perp$ to $\vec{w}$.

  So $\vec{0}$ is in $W^\perp$.

- Take $\vec{x}$ and $\vec{y}$ in $W^\perp$. Let $\vec{w}$ be any vector in W.

  $$(\vec{x}+\vec{y})\cdot\vec{w} = \underbrace{\vec{x}\cdot\vec{w}} + \vec{y}\cdot\vec{w}$$
  $$= 0 + 0 = 0$$

  So $\vec{x}+\vec{y}$ is in $W^\perp$ as well.

- Take $\vec{x}$ in $W^\perp$, c in $\mathbb{R}$, $\vec{w}$ in W.

  $$(c\vec{x})\cdot\vec{w} = c\cdot(\vec{x}\cdot\vec{w})$$
  $$= c\cdot 0 = 0$$

  So $c\cdot\vec{x}$ is in $W^\perp$ too.

So an orthogonal complement is a subspace.

Suppose A is an $m \times n$ matrix...

and $\vec{x}$ is in Nul A....

$$\Longrightarrow \quad A \cdot \vec{x} = \vec{0}$$

$$\begin{bmatrix} \text{— row 1 —} \\ \text{— row 2 —} \\ \vdots \\ \text{— row } m \text{ —} \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \vec{0} \quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} (\text{row } 1)^T \cdot \vec{x} \\ (\text{row } 2)^T \cdot \vec{x} \\ \vdots \\ (\text{row } m)^T \cdot \vec{x} \end{matrix}$$

This shows that any $\vec{x}$ in Nul A is orthogonal to each row of A...

So each $\vec{x}$ in Nul A is orthogonal to any $\vec{y}$ in Row(A).

$$\Longrightarrow \quad \text{Nul}(A) = \left(\text{Row}(A)\right)^{\perp}$$

and

$$\left(\text{Nul}(A)\right)^{\perp} = \text{Row}(A) = \text{Col}(A^T)$$

That is... $\text{Nul}(A)^{\perp} = \text{Col}(A^T)$

Ex $\quad A = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 1 & 6 \\ 0 & 1 & 2 \end{bmatrix} \qquad \mathbb{R}^3 \xrightarrow{A\cdot} \mathbb{R}^3$

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 3 & 8 & 0 \\ 2 & 1 & 6 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free

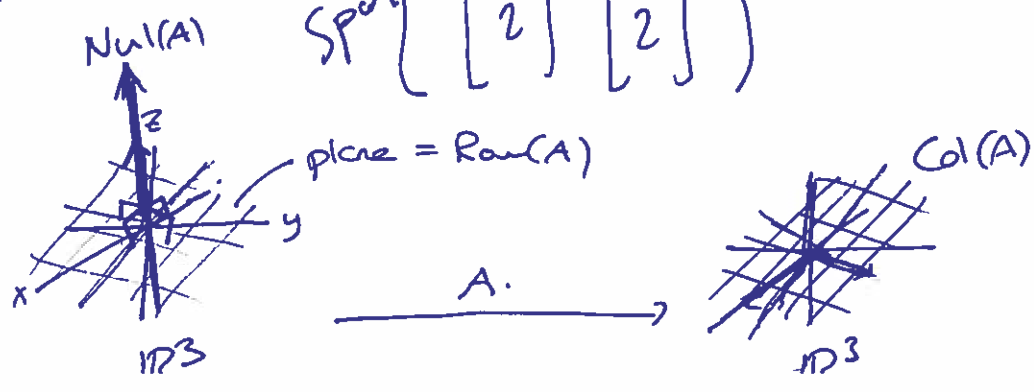
$x_2 = -2x_3$

$x_1 = -2x_3$

$$\vec{x} = \begin{bmatrix} -2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} \qquad \text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

are supposed to be $\perp$ complements.

Lastly $\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$



Fact If $S = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$ is a set of nonzero vectors, all orthogonal to each other....

then these vectors are independent.

Suppose $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p = \vec{0}$.

Inner product with $\vec{u}_i$

$$c_1\vec{u}_i\cdot\vec{u}_1 + c_2\vec{u}_i\cdot\vec{u}_2 + \cdots + c_p\vec{u}_i\cdot\vec{u}_p = \vec{u}_i\cdot\vec{0}$$

$$0 + 0 \cdots + c_i\underbrace{\vec{u}_i\cdot\vec{u}_i}_{\text{positive number}} + \cdots + 0 = 0$$

$$\longrightarrow c_i = \frac{0}{\vec{u}_i\cdot\vec{u}_i} = 0$$

So each c_i would have to be 0.

So that proves $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_p\}$ is l.n. indep.

So when you have $\{\vec{u}_1, \ldots \vec{u}_p\}$ mutually orthogonal, all non-zero, ... then you have a basis for $\text{Span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$.

You have an "orthogonal basis".

It's relatively easy to find coordinates for an orthogonal basis compared to regular basis.

Consider $\beta = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$ to be an orthogonal basis. Given $\vec{y}$ in $\text{Span}\{\vec{b}_1, \ldots, \vec{b}_p\}$, we often need

$$[\vec{y}]_\beta = \begin{bmatrix} y_{\beta 1} \\ y_{\beta 2} \\ \vdots \\ y_{\beta p} \end{bmatrix} \begin{matrix} ? \\ ? \\ \\ ? \end{matrix}$$

$$\vec{y} = y_{\beta 1} \cdot \vec{b}_1 + y_{\beta 2} \cdot \vec{b}_2 + \cdots + y_{\beta p} \vec{b}_p$$

$$\vec{b}_1 \cdot \vec{y} = y_{\beta 1} \vec{b}_1 \cdot \vec{b}_1 + 0 + \cdots + 0$$ since $\vec{b}_1$ is $\perp$ to other $\vec{b}_i$

So $y_{\beta 1} = \dfrac{\vec{b}_1 \cdot \vec{y}}{\vec{b}_1 \cdot \vec{b}_1}$ $\quad \boxed{y_{\beta i} = \dfrac{\vec{y} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i}}$

Ex $H = \text{Span}\underbrace{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} \right\}}$

a basis for H ✓

an orthogonal basis for H ✓

Let $\vec{y} = \begin{bmatrix} 3 \\ 11 \\ 15 \end{bmatrix}$. $\vec{y}$ is in H.

Find $[\vec{y}]_\beta$. Last page $[\vec{y}]_\beta = \begin{bmatrix} \dfrac{\vec{y}\cdot\vec{b}_1}{\vec{b}_1\cdot\vec{b}_1} \\ \\ \dfrac{\vec{y}\cdot\vec{b}_2}{\vec{b}_2\cdot\vec{b}_2} \end{bmatrix}$

$\vec{y}\cdot\vec{b}_1 = 3+22+45 = 70$

$\vec{b}_1\cdot\vec{b}_1 = 1+4+9 = 14$

$\vec{y}\cdot\vec{b}_2 = 12-22+0 = -10$

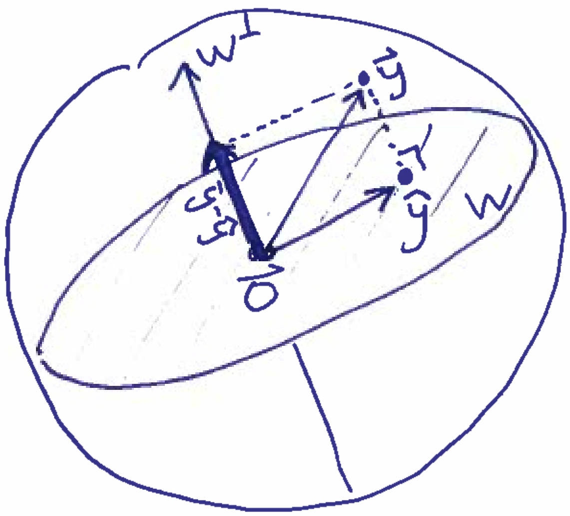
$\vec{b}_2\cdot\vec{b}_2 = 16+4+0 = 20$

$= \begin{bmatrix} 70/14 \\ \\ -10/20 \end{bmatrix} = \begin{bmatrix} 5 \\ -1/2 \end{bmatrix}$

So $\vec{y} = 5\vec{b}_1 + \left(\frac{-1}{2}\right)\vec{b}_2$.

Found without row reduction.

Given a subspace W of $\mathbb{R}^n$



Given $\vec{y}$ in $\mathbb{R}^n$....

there's <u>one</u> vector in W

that is as close as possible to $\vec{y}$.

Call this vector $\hat{y}$,

"orthogonal projection of $\vec{y}$ onto W."

We may have a basis for W...

if we have an orthogonal basis $\beta = \{\vec{b}_1, \vec{b}_2 \dots, \vec{b}_p\}$,

then we're interested in $[\hat{y}]_\beta$.

$$\hat{y} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_p\vec{b}_p$$

$$\vec{y} = (\vec{y} - \hat{y}) + \hat{y}$$

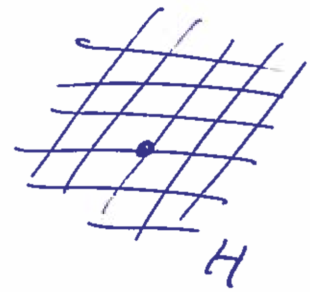
$$\vec{y} = (\vec{y} - \hat{y}) + c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_p\vec{b}_p$$

Now dot with $\vec{b}_1$

$$\vec{y}\cdot\vec{b}_1 = \underbrace{(\vec{y} - \hat{y})\cdot\vec{b}_1}_{0} + c_1\vec{b}_1\cdot\vec{b}_1 + 0 + \cdots 0$$

So $c_1 = \dfrac{\vec{y}\cdot\vec{b}_1}{\vec{b}_1\cdot\vec{b}_1}$ $\Bigg\}$ $c_i = \dfrac{\vec{y}\cdot\vec{b}_i}{\vec{b}_i\cdot\vec{b}_i}$

Ex $\quad H = \text{Span}\underbrace{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} \right\}}_{\text{an orthogonal basis}}$



Consider $\vec{y} = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$. Is $\vec{y}$ in H?

Probably not...

Where is $\hat{y}$? (The vector in H, close as possible to $\vec{y}$.)

Find. $[\hat{y}]_\beta = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

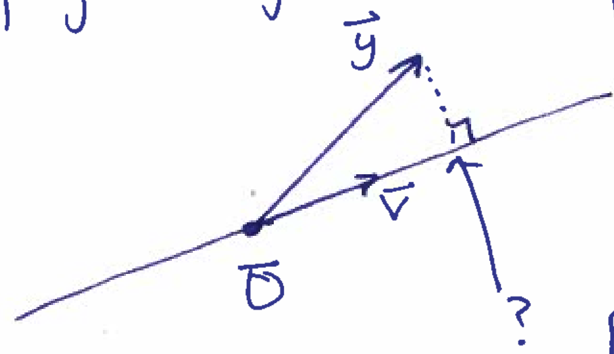
$$c_1 = \frac{\vec{y} \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1} = \frac{1+16+6}{1+4+9}$$

$$c_2 = \frac{\vec{y} \cdot \vec{b}_2}{\vec{b}_2 \cdot \vec{b}_2} = \frac{4+-16+0}{16+4+0}.$$

$$= \begin{bmatrix} 23/14 \\ -12/20 \end{bmatrix}$$

$$\hat{y} = \frac{23}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{12}{20}\begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sim \\ \sim \\ \sim \end{bmatrix}$$

If you have a vector $\vec{y}$, and there's another vector $\vec{v}$, what does it look like to project $\vec{y}$ onto $\mathrm{Span}\{\vec{v}\}$?



$\mathrm{proj}_{\vec{v}}(\vec{y})$ "the projection of $\vec{y}$ onto $\vec{v}$."

Fact: $$\mathrm{proj}_{\vec{v}}(\vec{y}) = \frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\cdot\vec{v}$$

Memorize this!

Ex $$\mathrm{proj}_{\begin{bmatrix}1\\2\\-2\end{bmatrix}}\left(\begin{bmatrix}2\\3\\2\end{bmatrix}\right) = \frac{\begin{bmatrix}2\\3\\2\end{bmatrix}\cdot\begin{bmatrix}1\\2\\-2\end{bmatrix}}{\begin{bmatrix}1\\2\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\2\\-2\end{bmatrix}}\begin{bmatrix}1\\2\\-2\end{bmatrix} =$$

$$= \frac{4}{9}\begin{bmatrix}1\\2\\-2\end{bmatrix} = \begin{bmatrix}4/9\\8/9\\-8/9\end{bmatrix}$$

Given W a subspace of $\mathbb{R}^3$, with a basis $\left\{ \underbrace{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}_1}, \underbrace{\begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix}}_{\vec{b}_2} \right\}$. Given $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(a) Find $\text{proj}_{\vec{b}_1}(\vec{y})$. $= \cdots = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

(b) Find $\text{proj}_{\vec{b}_2}(\vec{y}) = \cdots = \begin{bmatrix} -5/53 \\ -30/53 \\ 20/53 \end{bmatrix}$

(c) Find $\text{proj}_W(\vec{y}) =$ add previous together $= \begin{bmatrix} 101/53 \\ 23/53 \\ 126/53 \end{bmatrix}$

6.4 Gram-Schmidt orthogonalization process

.... You have a subspace of interest ....

... you have a basis, obtained in a natural way...

... typically these bases are not automatically orthogonal.

Start with:

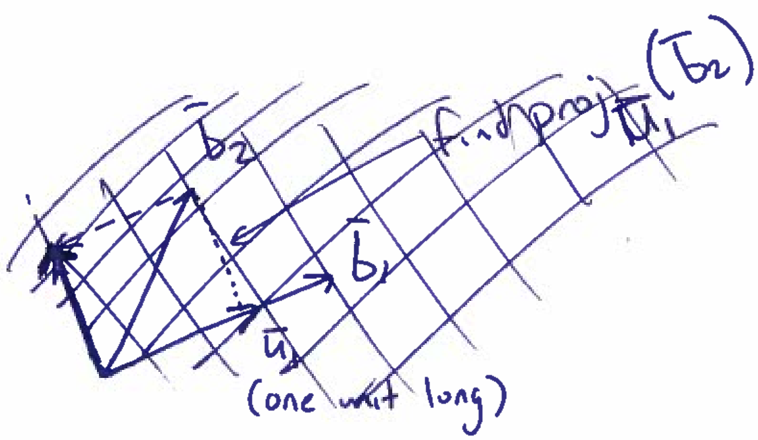
$$\{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_p\} \xrightarrow{\text{gram-schmidt}} \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_p\}$$

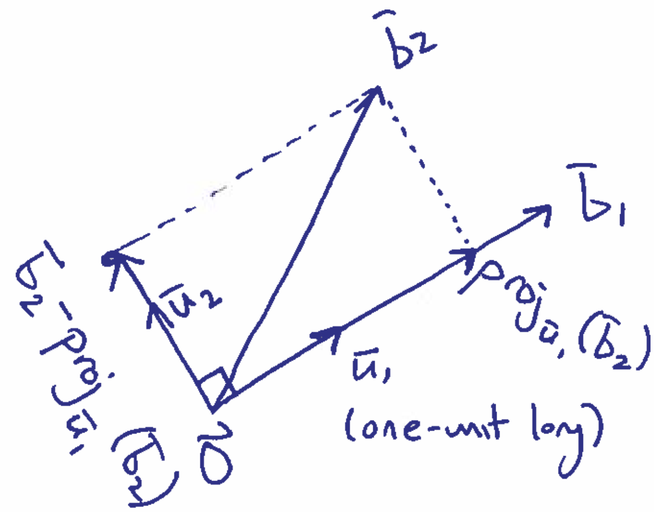
where all $\bar{u}_i$ have $\|\vec{u}_i\| = 1$, and they are mutually $\perp$

~~Visual~~

Visually in a dim 2 subspace ...

$$\{\vec{b}_1, \vec{b}_2\}$$





Ex $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & 3 \\ 3 & 1 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Consider Col(A).

First, a basis...

$$\mathcal{B} = \left\{ \underset{\bar{b}_1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \underset{\bar{b}_2}{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}} \right\}$$

not $\perp$ to each other

also, not unit length.

- Find $\bar{u}_1$. $\bar{u}_1 = \dfrac{\bar{b}_1}{\|\bar{b}_1\|}$

$\|b_1\| = \sqrt{1+4+9} = \sqrt{14}$

$\bar{u}_1 = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\mathcal{C} = \left\{ \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{matrix} ? \\ \vdots \\ \bar{u}_2 \\ ? \end{matrix} \right\}$$

- Find $\bar{u}_2$ $\quad \bar{b}_2 - \text{proj}_{\bar{u}_1}(\bar{b}_2)$ ← will be $\perp$ to $\bar{u}_1$

then rescale...

$$\cancel{\bar{u}_2} = \begin{bmatrix}-2\\-1\\1\end{bmatrix} - \frac{\begin{bmatrix}-2\\-1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}}{\frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}}\cdot\frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$= \begin{bmatrix}-2\\-1\\1\end{bmatrix} - \frac{-1}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-27/14\\-6/7\\17/14\end{bmatrix}$$

factoring out fractions.. $\quad = \frac{1}{14}\begin{bmatrix}-27\\-12\\17\end{bmatrix}$

to get $\bar{u}_2$, need to scale this down.

want a vector parallel to $\frac{1}{14}\begin{bmatrix}-27\\-12\\17\end{bmatrix}$ but 1 unit long...

... same as a vector parallel to $\begin{bmatrix}-27\\-12\\17\end{bmatrix}$ but 1 unit long...

$$\dots \frac{\begin{bmatrix}-27\\-12\\17\end{bmatrix}}{\left\|\begin{bmatrix}-27\\-12\\17\end{bmatrix}\right\|} = \frac{\begin{bmatrix}-27\\-12\\17\end{bmatrix}}{\sqrt{729+144+289}} = \frac{1}{\sqrt{1162}}\begin{bmatrix}-27\\-12\\17\end{bmatrix}$$

So $\left\{ \underbrace{\frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\bar{u}_1}, \underbrace{\frac{1}{\sqrt{1162}}\begin{bmatrix}-27\\-12\\17\end{bmatrix}}_{\bar{u}_2} \right\}$ is an orthonormal ~~orthogonal~~ basis for Col (A).

<u>Ex</u> $A = \begin{bmatrix}1&0&0&1\\1&1&0&1\\1&1&1&0\\1&1&1&0\end{bmatrix}$ Find an orthonormal basis for Col (A).

$A \xrightarrow{\text{RREF}} \begin{bmatrix}1&0&0&1\\0&1&0&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}$

So $\left\{ \overset{\bar{b}_1}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}, \overset{\bar{b}_2}{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}}, \overset{\bar{b}_3}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}} \right\}$

is a basis . . .

• Find $\vec{u}_1$ . . . $\frac{\bar{b}_1}{\|\bar{b}_1\|} = \frac{1}{\sqrt{1+1+1+1}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

• Find $\bar{u}_2$ $\bar{b}_2 - \text{proj}_{\bar{u}_1}(\bar{b}_2)$

$$\begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\cdot\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Find a unit length parallel vector

$$\frac{\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\|} = \frac{1}{\sqrt{12}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \bar{u}_2$$

$$\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{matrix} ? \\ \bar{u}_3 \\ ? \end{matrix} \right\}$$

- Find $\bar{u}_3$ ... $\bar{b}_3 - \text{proj}_{\bar{u}_1}(\bar{b}_3) - \text{proj}_{\bar{u}_2}(\bar{b}_3)$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{12}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\frac{1}{\sqrt{12}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{12}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \cdot \frac{1}{\sqrt{12}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

Find a unit-length version...

$$\frac{\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

So... $\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

Consider

$$A = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$$

A's 1st column • A's 2nd column

A's 1st column • A's 1st column

Think about $A^T \cdot A = \begin{bmatrix} \boxed{1} & \boxed{0} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So $A^T = A^{-1}$.

when $A^T = A^{-1}$ (when columns of A form an orthonormal basis for $\mathbb{R}^n$)

A is a "orthogonal" matrix.

If A is orthogonal, take $\vec{x}$ in $\mathbb{R}^n$

$$\|A\vec{x}\| = \sqrt{(A\vec{x})\cdot(A\vec{x})}$$
$$= \sqrt{(A\vec{x})^T(A\vec{x})}$$
$$= \sqrt{\vec{x}^T A^T A \vec{x}}$$
$$= \sqrt{\vec{x}^T \vec{x}}$$
$$= \sqrt{\vec{x}\cdot\vec{x}}$$
$$\|A\vec{x}\| = \|\vec{x}\|$$

An orthogonal matrix preserves length.

Same argument shows $(A\vec{x})\cdot(A\vec{y}) = \cdots = \vec{x}\cdot\vec{y}$

implies angle between $A\vec{x}$ and $A\vec{y}$ equals angle between $\vec{x}$ and $\vec{y}$.