4.7 Change of basis matrix

Given a p-dimensional subspace H of $\mathbb{R}^n$, suppose you have two bases for H: $\mathcal{B}$ and $\mathcal{C}$

$$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\} \qquad \mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_p\}$$

If $\vec{x}$ is in H: $\vec{x}$ has n entries.

$[\vec{x}]_{\mathcal{B}}$ has p entries.

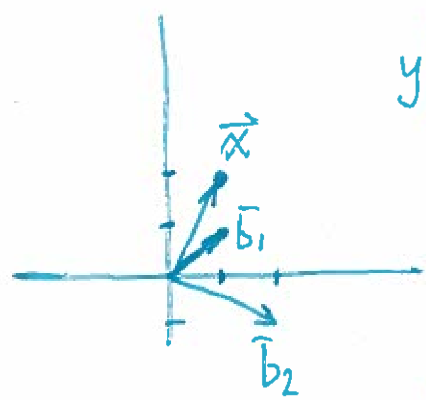
$[\vec{x}]_{\mathcal{C}}$ has p entries.

Ex $H = \mathbb{R}^2$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \end{bmatrix} \right\}$$

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $[\vec{x}]_{\mathcal{B}}$.



$$y\begin{bmatrix} 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & -1 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 5/3 \\ 0 & 1 & -1/3 \end{array}\right]$$

So $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix}$

In general $\vec{x} = [\beta][\vec{x}]_\beta$

An $n \times p$ matrix

If $H = \mathbb{R}^n$, then $p = n$. Then this is square

In this case call it $\underset{std. \leftarrow \beta}{P}$

$$\text{So} \quad \vec{x} = \underset{std. \leftarrow \beta}{P} \cdot [\vec{x}]_\beta$$

an $n \times n$ matrix (again $H = \mathbb{R}^n$)

Its columns are independent since they come from β. So this is invertible.

$$\left(\underset{std \leftarrow \beta}{P}\right)^{-1} \vec{x} = [\vec{x}]_\beta$$

$$\text{So} \quad \left(\underset{std \leftarrow \beta}{P}\right)^{-1} = \underset{\beta \leftarrow std}{P}$$

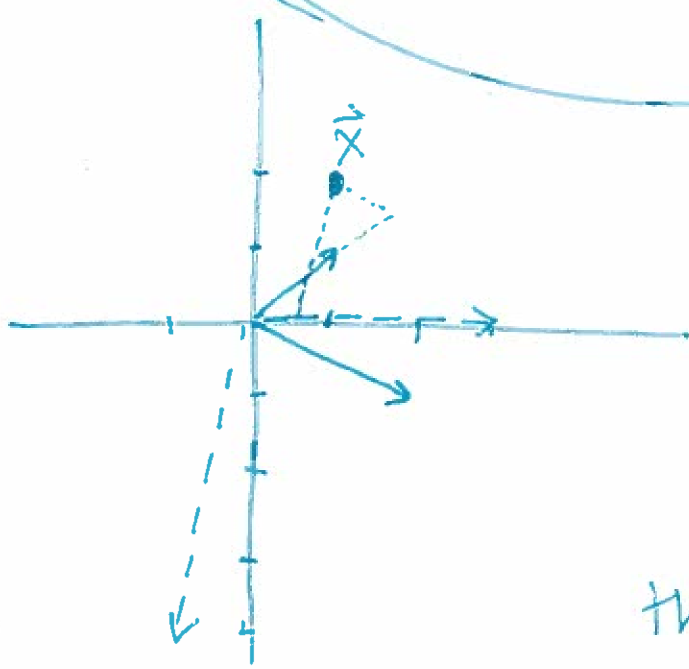
introduced another "change of basis matrix"

Returning to example, find $[\vec{x}]_{C}$ ....

$$\begin{bmatrix}1\\2\end{bmatrix} = \vec{x} = y\begin{bmatrix}3\\0\end{bmatrix} + z\begin{bmatrix}-1\\-4\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3 & -1 & 1\\0 & -4 & 2\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 0 & 1/6\\0 & 1 & -1/2\end{array}\right]$$

So $[\vec{x}]_C = \begin{bmatrix}1/6\\-1/2\end{bmatrix}$. Recall $[\vec{x}]_\beta = \begin{bmatrix}5/3\\-1/3\end{bmatrix}$



We want a "change of basis matrix",

$$\underset{C\leftarrow\beta}{P}$$

that works:

$$[\vec{x}]_C = \underset{C\leftarrow\beta}{P} \cdot [\vec{x}]_\beta$$

or

$$[\vec{x}]_\beta = \underset{\beta\leftarrow C}{P}[\vec{x}]_C$$

Think about $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_p$

$$[\vec{c}_1]_C = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} \quad \cdots \quad [\vec{c}_p]_C = \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

For example, $[\bar{c}_1]_\beta = \underset{\beta \leftarrow C}{P} [c_1]_C$ = First column of $\underset{\beta \leftarrow C}{P}$

$$[c_1]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Deduce

$$\underset{\beta \leftarrow C}{P} = \begin{bmatrix} | & | & & | \\ [\bar{c}_1]_\beta & [\bar{c}_2]_\beta & \cdots & [\bar{c}_p]_\beta \\ | & | & & | \end{bmatrix}$$

In our example: $[\bar{c}_1]_\beta$: $\begin{bmatrix} 3 \\ 0 \end{bmatrix} = y \underset{\bar{b}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + z \underset{\bar{b}_2}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$[\bar{c}_1]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $[\bar{c}_2]_\beta$ . . .

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 1 & -1 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right]$$

$$[\bar{c}_2]_\beta = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

So $\underset{\beta \leftarrow C}{P} = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$.

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$[\vec{x}]_\beta = \begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix}$$

$$[\vec{x}]_C = \begin{bmatrix} 1/6 \\ -1/2 \end{bmatrix}$$

We expect $[\vec{x}]_{\beta} = \underset{\beta \leftarrow C}{P} [\vec{x}]_C$

$$\begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix} \overset{?}{=} \underbrace{\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/6 \\ -1/2 \end{bmatrix}}$$

$$\begin{bmatrix} 1/6 + 3/2 \\ 1/6 - 1/2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix}$$

Shortcut: $\left[\beta \mid C \right]$

$$\left[\beta \mid C \right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c} I & \underset{\beta \leftarrow C}{P} \\ 0\,0\,0 \cdots 0 & 0\,0\,0 \cdots 0 \\ 0\,0\,0 \cdots 0 & 0\,0\,0 \cdots 0 \end{array} \right]$$

$$\left[\beta \mid C \right] \xrightarrow{\text{Sort of RREF}} \left[\begin{array}{c|c} \underset{C \leftarrow \beta}{P} & I \\ 0\,0\,0 \cdots 0 & 0\,0\,0 \cdots 0 \\ 0\,0\,0 \cdots 0 & 0\,0\,0 \cdots 0 \end{array} \right]$$

<u>Ex</u> H is a plane in $\mathbb{R}^3$.

One basis $\beta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$. $C = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} \right\}$

Find $\underset{\beta \leftarrow C}{P}$.

$$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 8 \\ 1 & 0 & 1 & 3 \end{array} \right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$$

So... if $[\vec{x}]_C = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, find $[\vec{x}]_\beta$.

$$[\vec{x}]_\beta = \underset{\beta \leftarrow C}{P} [\vec{x}]_C$$

$$= \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ -11 \end{bmatrix}$$

Ex $A = \begin{bmatrix} 1 & 3 & -1 & -8 \\ 2 & -1 & 5 & -2 \\ 1 & 0 & 2 & -2 \\ 3 & -1 & 7 & -4 \end{bmatrix}$ Consider Col A.

One basis for Col A

$A \xrightarrow{RREF} \begin{bmatrix} \boxed{1} & 0 & 2 & -2 \\ 0 & \boxed{1} & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\beta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}$

Note $\text{Col} A = \text{Row} A^T$

$A^T = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & -1 & 0 & -1 \\ -1 & 5 & 2 & 7 \\ -8 & -2 & -2 & -4 \end{bmatrix}$

$\xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1/7 & 1/7 \\ 0 & 1 & 3/7 & 10/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1/7 \\ 1/7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3/7 \\ 10/7 \end{bmatrix} \right\}$

both are bases for Col A

Find $\underset{\beta \leftarrow C}{P}$.

$$[\mathcal{B} \mid \mathcal{C}] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 1 & 0 & 1/7 & 3/7 \\ 3 & -1 & 1/7 & 10/7 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & 1/7 & 3/7 \\ 0 & 1 & 2/7 & -1/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} 1/7 & 3/7 \\ 2/7 & -1/7 \end{bmatrix}$.

Section 5.1

Ex $\begin{bmatrix} 35 & -42 \\ 20 & -23 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 35 \\ 25 \end{bmatrix}$

$[\ A\ ] \cdot \vec{x} = 5 \cdot \vec{x}$

For this A, $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ is a very special vector. $A \cdot \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ happens to be a scalar of $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$.

When this happens and $\vec{x} \neq \vec{0}$, $\vec{x}$ is an eigenvector.

The scaling factor is called an eigenvalue.

Whenever $A \cdot \vec{x} = \lambda \cdot \vec{x}$ with $\vec{x} \neq \vec{0}$, $\vec{x}$ is an eigenvector for A with eigenvalue λ.

Ex $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Is $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ an eigenvector for A?

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

So yes... with eigenvalue -4.

Is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector?

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

not parallel, no.

Ex Same A... is 5 an eigenvalue?

If so, $A\vec{x} = 5\vec{x}$ has a nontrivial solution.

$$A\vec{x} - 5\vec{x} = \vec{0}$$

$$(A - 5I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} -4 & 6 \\ 5 & -3 \end{bmatrix}\vec{x} = \vec{0}$$

is there a nontrivial sol?

No, for several reasons...

Therefore 5 is not an eigenvalue.

Ex Is 7 an eigenvalue?

$$A\vec{x} = 7\vec{x}$$

(does it have nontrivial sols?)

$$A\vec{x} - 7\vec{x} = \vec{0}$$

$$(A - 7I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\vec{x} = 0$$

dependent cols... there is a free column... so there are nontrivial sols... 7 is an eigenvalue.

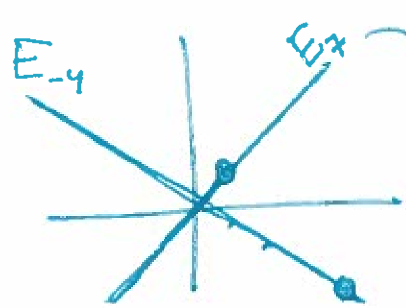
$$\left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \vec{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$$

so... $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 17 \\ 17 \end{bmatrix}$ etc. are eigenvectors with eigenvalue 7.

with $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

we found an eigenvector $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ with eigenvalue -4.

and an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue 7.



Eigenspace for $\lambda = 7$.

Ex Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an e'vector for $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$?

$$\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

So yes, with e'value 0.

Is 3 an eigenvalue for same matrix?

Does $A\vec{x} = 3\vec{x}$ have a nontrivial sol?

$$A\vec{x} - 3\vec{x} = \vec{0}$$

$$(A - 3I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 & 7 & 9 \\ -4 & -8 & 1 \\ 2 & 4 & 1 \end{bmatrix}\vec{x} = \vec{0}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

no nontriv. sol

$\Downarrow$

3 is not an e'value.

Ex Is 3 an eigenvalue for $\begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 3 & 3 \end{bmatrix}$?

Does $A\vec{x} = 3\vec{x}$ have a nontriv. sol?

$\Longrightarrow (A-3I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

free column

So... 3 is an eigenvalue..

General sol.. $\vec{x} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

This is an eigenvector!