3.1 & 3.2 cont'd

Ex Use row reduction to find $\begin{vmatrix} 2 & 1 & 0 & -2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & -1 & 1 \end{vmatrix}$.

$$\xrightarrow[\text{negates det.}]{R_1 \leftrightarrow R_2} \begin{bmatrix} \boxed{1} & 1 & 2 & 1 \\ 2 & 1 & 0 & -2 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow[\text{no effect on det}]{cR_1 + R_j} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & -4 \\ 0 & -1 & -5 & 1 \\ 0 & 5 & 3 & 3 \end{bmatrix} \xrightarrow[\text{no effect on det}]{cR_2 + R_j}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & 4 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & -17 & -17 \end{bmatrix} \xrightarrow{-17R_3 + R_4} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & 4 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & -102 \end{bmatrix}$$

this has det -102.

Tracing back, $\det A = 102$.

Try: $\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ using row reduction.

$$\xrightarrow{\text{replacement}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} \xrightarrow{\text{replacement}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

det is 2...

has det 2 as well, since these actions didn't affect determinant.

Another way:

$$\det\begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix} = \underbrace{\det\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}}_{\text{this is equal to } \det\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}.} + \underbrace{\det\begin{bmatrix} 0 & 3 & 1 \\ -2 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}}_{\text{equal to } (-2)\cdot\det\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}} + \underbrace{\det\begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}}_{\text{this is equal to } \det\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

except here we have to introduce a minus sign.

$$\det\begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix} = +1\cdot\det\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} - (-2)\cdot\det\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} + 1\cdot\det\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 1\cdot 4 - 2\cdot 0 + 2(3\cdot 4 - 1\cdot 0) + (3\cdot 2 - 1\cdot 1)$$

$$= 4 + 24 + 5$$

$$= 33 \checkmark$$

obey a $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Expanding a determinant by a column (or by a row).

Pick a column (or a row)

$$\det\begin{bmatrix} & * & \\ & * & \\ & \vdots & \\ & * & \end{bmatrix} = \sum$$

determinants of $(n-1)$ by $(n-1)$ matrices, scaled by an entry from the chosen column, signed according to $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & & \ddots & \\ \vdots & & & \end{bmatrix}$

Ex

$$\det\begin{bmatrix} -2 & 6 & -5 & 0 \\ 1 & 0 & 8 & 0 \\ -2 & 4 & 1 & 3 \\ 3 & 0 & -2 & 0 \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

(use expansion...)

lots of 0's $\longrightarrow$ good column to expand on

now here... $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$$= -0\cdot\det(\quad) + 0\cdot\det(\quad) - 3\cdot\det\begin{bmatrix} -2 & 6 & -5 \\ 1 & 0 & 8 \\ 3 & 0 & -2 \end{bmatrix} + 0\cdot\det(\quad)$$

$$= -3\left(-6\cdot\det\begin{bmatrix} 1 & 8 \\ 3 & -2 \end{bmatrix} + 0\cdot\det(\quad) - 0\cdot\det(\quad)\right)$$

$$= 18\left(1\cdot(-2) - 8(3)\right) = 18(-26) = -468$$

You Try: $\det\begin{bmatrix} 3 & 0 & 4 & 0 \\ 1 & 2 & -2 & 1 \\ 2 & 0 & 0 & 5 \\ 1 & 1 & 0 & -1 \end{bmatrix}$ using expansion...

expanding on first row

$$= +3 \cdot \det\begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 5 \\ 1 & 0 & -1 \end{bmatrix} + 4 \cdot \det\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 5 \\ 1 & 1 & -1 \end{bmatrix}$$

now 2nd row... now 2nd column...

$$= 3\left(-5 \cdot \det\begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}\right) + 4\left(-2 \det\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - 1 \cdot \det\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}\right)$$

$$= -15\,(0 - (-2)) \qquad + 4\,(-2(-2) - (5-2))$$

$$= -30 \qquad + 4\,(4-3)$$

$$= -30 \qquad + 4 \qquad = -26$$

Wrong!
(det is actually 14)

With 3x3 matrices

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\cdot\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{bmatrix}$$

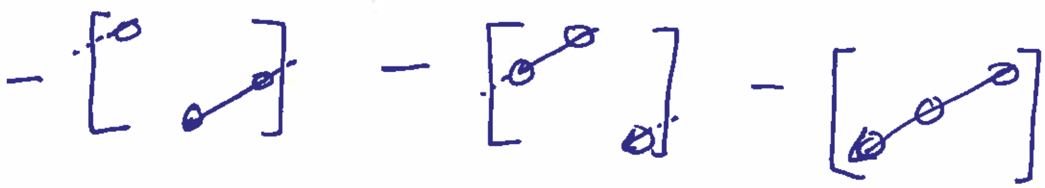
$$+ a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{bmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}$$

$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$



<u>Ex</u>

$$\det\begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 4\end{bmatrix} = 1\cdot1\cdot4 + 3\cdot2\cdot1 + 1\cdot(-2)(0)$$

$$-1\cdot2\cdot0 - 3(-2)(4) - 1\cdot1\cdot1$$

$$= 4 + 6 + 0 - 0 + 24 - 1$$

$$= 10 + 23$$

$$= 33$$

Ex $\det \begin{bmatrix} 0 & 8 & 1 & 0 \\ 0 & 0 & 2 & 5 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 3 & 1 \end{bmatrix}$ is really a sum of products of 4 entries... where each product uses each row once, each column once...

$1 \cdot 8 \cdot 2 \cdot 1$

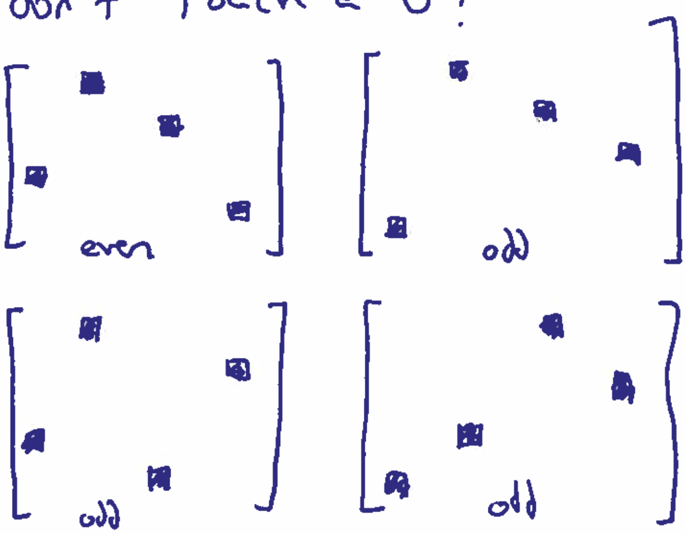
this term gets a + sign attached automatically

two row-swaps to bring this dot pattern back to main diagonal..

"even" dot pattern $\Longrightarrow$ gets + sign

("odd" dot pattern gets − sign).

Which dot patterns don't touch a 0?



$8 \cdot 2 \cdot 1 \cdot 1 - 8 \cdot 2 \cdot 2 \cdot 1$

$- 8 \cdot 5 \cdot 1 \cdot 3 - 1 \cdot 5 \cdot 1 \cdot 1$

$= 16 - 32 - 120 - 5$

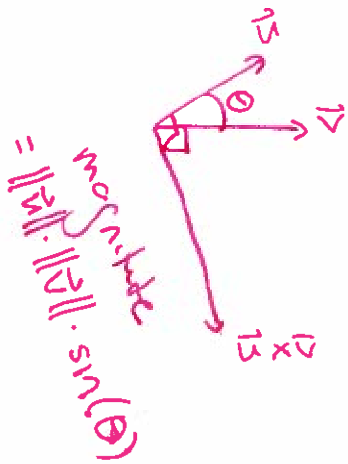
$= -141$

no other dot patterns contribute to det!

Applications

Given $\vec{u}, \vec{v}$ in $\mathbb{R}^3 \ldots$

Define $\vec{u} \times \vec{v}$



$\vec{u} \times \vec{v}$ respects R.H.R.

Given $\vec{u} = \langle 1, 3, -2 \rangle$, $\vec{v} = \langle 4, 1, 8 \rangle$

How to calculate $\vec{u} \times \vec{v}$?

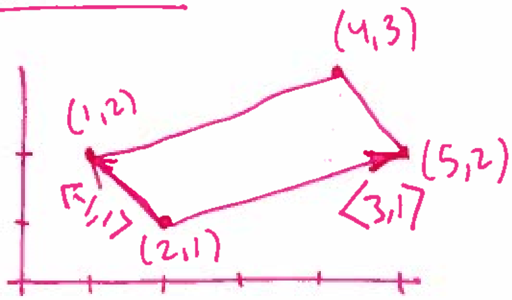
Set up $\det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ 4 & 1 & 8 \end{bmatrix} \begin{array}{l} \\ \leftarrow \vec{u} \\ \leftarrow \vec{v} \end{array}$

$$= 24\hat{\imath} - 8\hat{\jmath} + \hat{k} - (-2\hat{\imath}) - 8\hat{\jmath} - 12\hat{k}$$

$$= 26\hat{\imath} - 16\hat{\jmath} - 11\hat{k} = \langle 26, -16, -11 \rangle$$

Parallelogram Area

Ex



Area = ?

Find "side vectors"

$$\left| \det \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \right| = |3 - (-1)| = 4$$

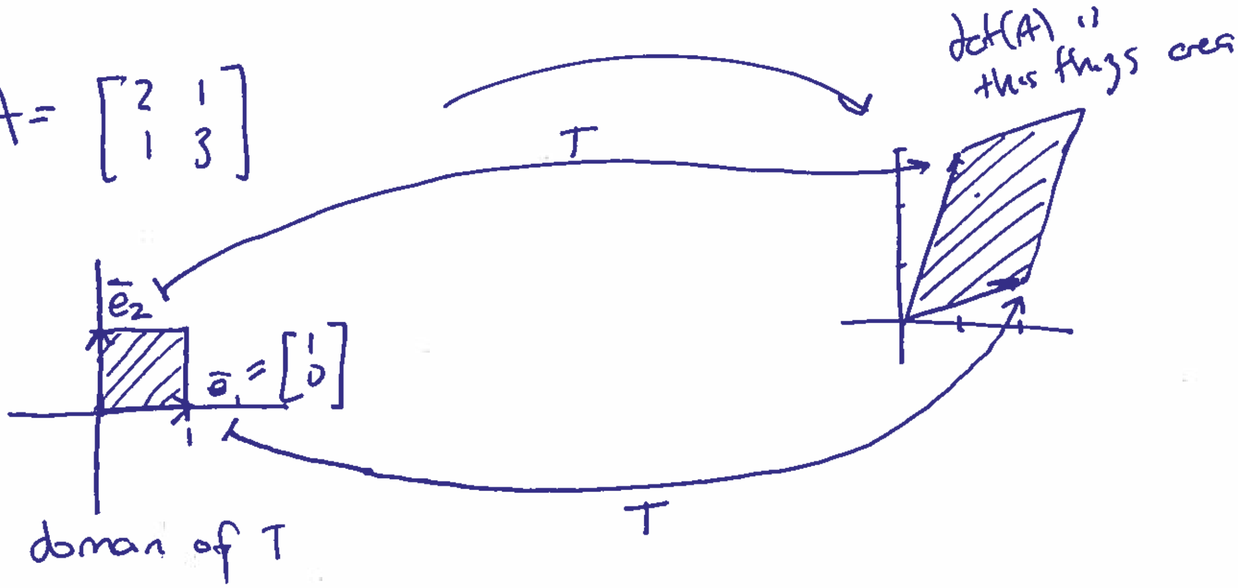
With $T(\vec{x}) := A \cdot \vec{x}$

Square, $n \times n$ matrix...

<u>Ex</u> Columns of A form a parallelepiped whose "volume" is det (A).

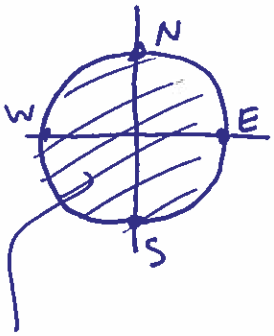
$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

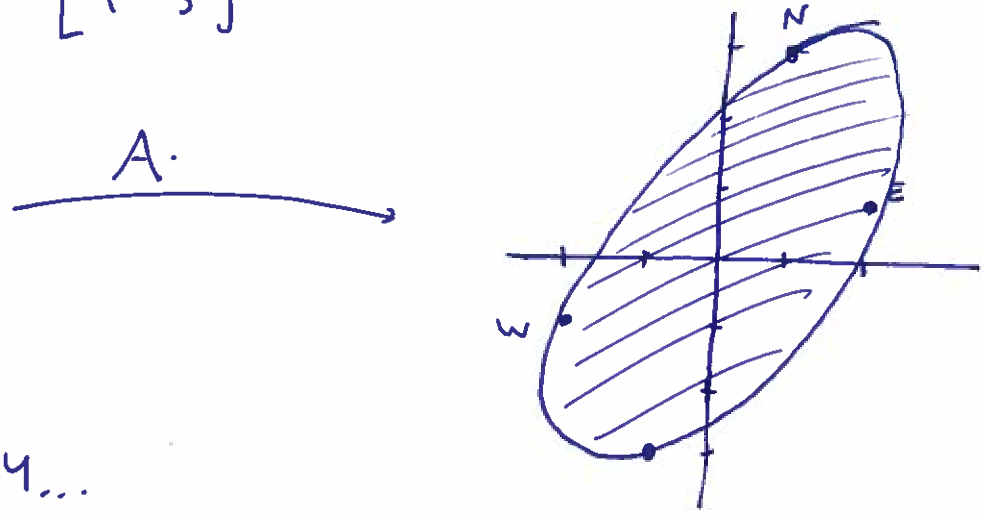


Interpretation of det A: When you apply T_A to some shape in $\mathbb{R}^n$, ~~&~~ areas/volumes/hypervolumes are scaled by det A.

We have $\det\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix} = 5$



area is $\pi \approx 3.14\ldots$



we're saying this has area 5π
$\approx 15.7\ldots$

<u>More Facts</u>

- $\det(A \cdot B) = \det(A) \cdot \det(B)$

$$T_{A\cdot B} = T_A \circ T_B$$

scales volume by det B

scales volume by det A

so this scales volume by det A · det B

- $\det(A \cdot B) = \det(BA)$
- I.M.T. item (s): A invertible $\iff \det(A) \neq 0$.
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$ $\left[\begin{array}{l} A \cdot A^{-1} = I \\ \det(A \cdot A^{-1}) = 1 \\ \det(A) \cdot \det(A^{-1}) = 1 \end{array}\right]$
- $\det(A^T) = \det(A)$
- $\det(k \cdot A) = k^n \cdot \det(A)$

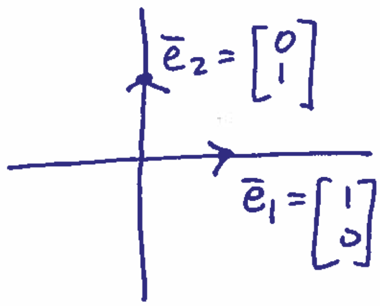
Ex $\det\begin{bmatrix} 200 & 900 \\ 300 & 100 \end{bmatrix} = 100^2 \cdot \det\begin{bmatrix} 2 & 9 \\ 3 & 1 \end{bmatrix}$

$= -25 \cdot 100^2$

$= -250{,}000$

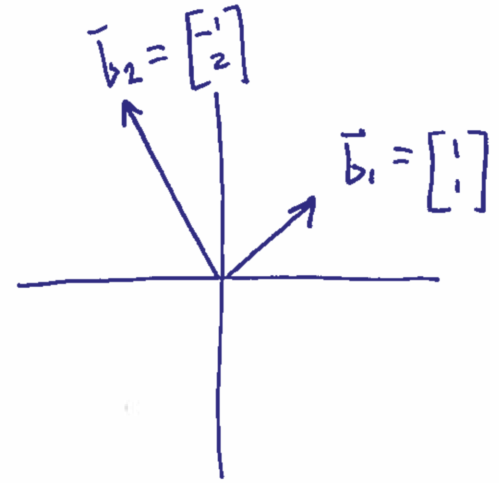
4.4 Coordinate Systems

Consider $H = \mathbb{R}^2$.



Standard basis for $\mathbb{R}^2$:

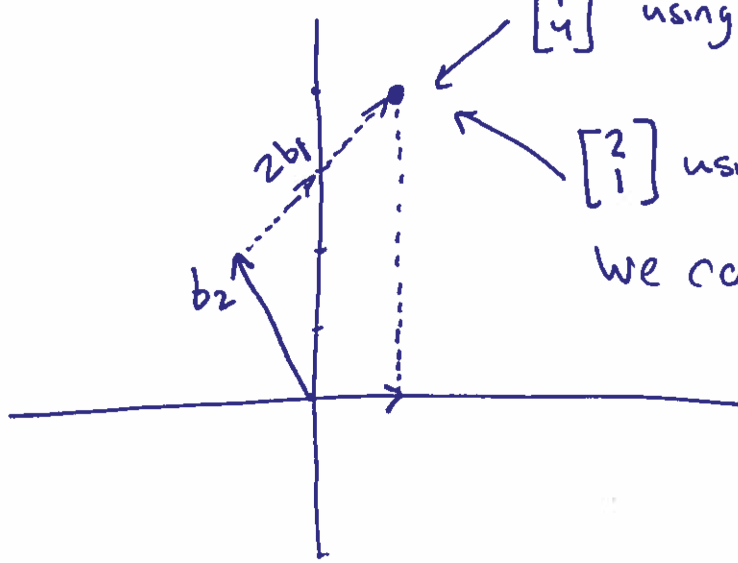
$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$



$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$$

is a fine alternative basis for $\mathbb{R}^2$.

To get to $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$



$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ using the standard coordinate system.

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ using $\mathcal{B}$ coordinates.

We can write $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{B}}$

a coordinate vector using some non-standard basis.

Ex Let $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ contained in $\mathbb{R}^3$.

This is a plane through the origin.

Note: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for H.

So we can talk about going places in H using this basis for coordinates...

Ex Consider $\begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$. (regular, $\mathbb{R}^3$ coordinates.)

It may or may not be in H...

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \overset{?}{=} \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

There is a solution...

So $\vec{v}$ is in H...

We can write $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$ or $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

optional $\mathcal{B}$

Given H, a subspace of $\mathbb{R}^n$ of dimension p

$\Downarrow$

All bases of H have p vectors in them.

Take one basis $\mathcal{B}$ for H..

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_p\}$

Can create coordinate vectors for any $\vec{v}$ in H.

Write $\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_p\vec{b}_p$

Then write $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$. and you have a $\mathcal{B}$-coord. vector for $\vec{v}$.

Ex Take $\mathbb{R}^3$ with basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Take $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$.

Find $[\vec{x}]_{\mathcal{B}}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Conversely, suppose $[\vec{x}]_\beta = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$. Find $\vec{x}$.

Well... $5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + -2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}$.

Another way to put this $\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$

In general $\vec{x} = [\beta][\vec{x}]_\beta$

in $\mathbb{R}^n$