2.8 continued $\longrightarrow$ 2.9

Recall H is a subspace of $\mathbb{R}^n$:

(H contains $\vec{0}$,
additive closure: for $\vec{u}, \vec{v}$ in H, $\vec{u}+\vec{v}$ is in H
scalar multiplicative closure: for $\vec{u}$ in H, for c in $\mathbb{R}$
$c\cdot\vec{u}$ is in H)

<u>Definition</u> A <u>basis</u> $\mathcal{B}$ of a subspace H,

is a list of vectors in H that

- linearly independent
- spans H

<u>Ex</u> The standard basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$.

(mentally check that its is independent and spans $\mathbb{R}^3$.)

<u>Ex</u> Consider $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$. Is this a basis for $\mathbb{R}^3$?

Yes: $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ so these vectors are independent.

Do these span $\mathbb{R}^3$? Yes.. pivot in every row by thm 4 means these span $\mathbb{R}^3$.

Ex Is $\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}3\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\8\end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

No. With 4 vectors in $\mathbb{R}^3$, they are dependent.
We'd have at most 3 pivots in the 3×4 matrix.
So not all columns are pivot columns.
The set is not a basis.

Ex Is $\left\{ \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}3\\-2\\4\end{bmatrix}, \begin{bmatrix}-2\\4\\-6\end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Are the vectors independent? Do they span $\mathbb{R}^3$?

$$\begin{bmatrix}2 & 3 & -2\\0 & -2 & 4\\1 & 4 & -6\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 2\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$$

No! Dependency!

Ex In $\mathbb{R}^3$, $x+y+z=0$ describes a plane.
passes through origin, perpendicular to $\begin{bmatrix}1\\1\\1\end{bmatrix}$.
Call this plane H, and H is a subspace of $\mathbb{R}^3$.

Consider $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}-3\\2\\1\end{bmatrix} \right\}$. Is $\mathcal{B}$ a basis for H?

Are they independent? $\begin{bmatrix}1 & -3\\1 & 2\\-2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$ Yes, a pivot in every column.

Do these vectors span H? A generic vector in H is $\begin{bmatrix} a \\ b \\ -a-b \end{bmatrix}$. We're wondering if

$$x_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ -a-b \end{bmatrix}$$

always has a solution no matter what a, b are.

$$\left[\begin{array}{cc|c} 1 & -3 & a \\ 1 & 2 & b \\ -2 & 1 & -a-b \end{array}\right] \xrightarrow[2R_1+R_3]{-R_1+R_2} \left[\begin{array}{cc|c} 1 & -3 & a \\ 0 & 5 & -a+b \\ 0 & -5 & a-b \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{cc|c} 1 & -3 & a \\ 0 & 5 & -a+b \\ 0 & 0 & 0 \end{array}\right]$$

the system is consistent.

the two vectors do span H and so $\mathcal{B}$ is a basis for H.

Ex $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \end{bmatrix}$. Find a basis for Col A. Find a basis for Nul A.

Take Col A = Span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix} \right\}$ and replace the list of 3 vectors with a list of independent vectors.

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} \boxed{1} & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

So Col A = Span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. So $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ spans Col A and is indep, So $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ is a basis for Col A.

Now for Nul A, which is a subspace of $\mathbb{R}^3$.

$$\begin{bmatrix}1 & 3 & -2\\2 & 6 & -4\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 3 & -2\\0 & 0 & 0\end{bmatrix}$$

Recall $\text{Nul}(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$. If $\vec{x}$ is in Nul A, $A\cdot\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c}1 & 3 & -2 & 0\\2 & 6 & -4 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 3 & -2 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

x_3 free

x_2 free

$x_1 = -3x_2 + 2x_3$

$$\vec{x} = \begin{bmatrix}-3x_2 + 2x_3\\x_2\\x_3\end{bmatrix} \Longleftarrow$$

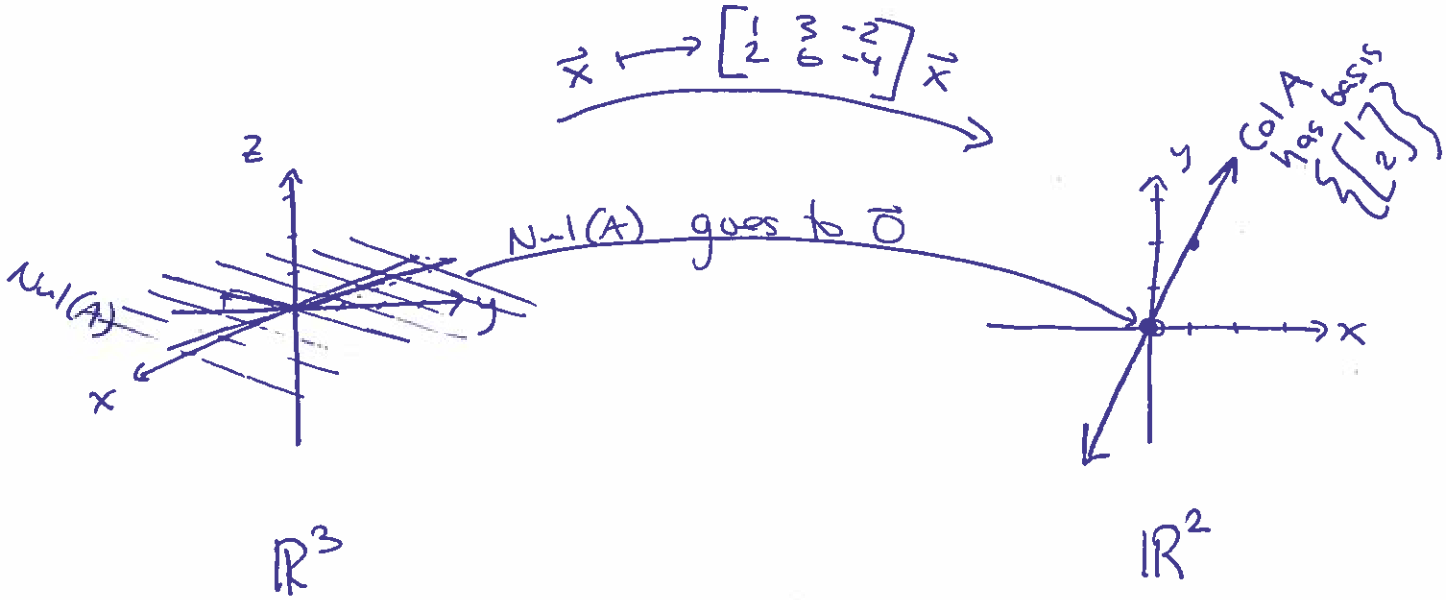
$$= x_2\begin{bmatrix}-3\\1\\0\end{bmatrix} + x_3\begin{bmatrix}2\\0\\1\end{bmatrix} \longrightarrow \text{Nul}A = \text{Span}\left\{\begin{bmatrix}-3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$$

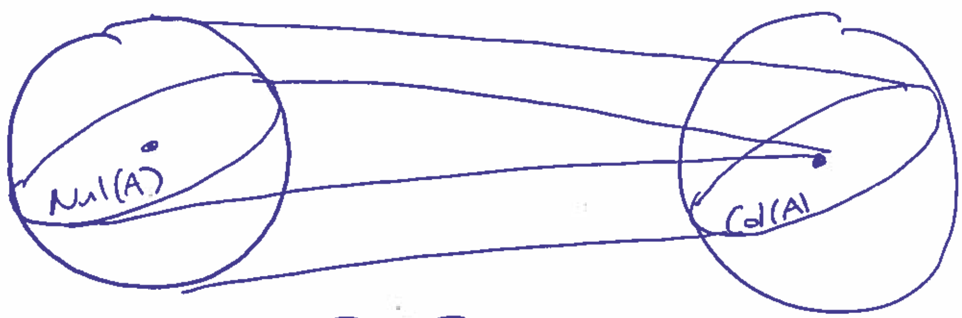
Is $\left\{\begin{bmatrix}-3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$ a basis for Nul A?

- spans Nul A
- are they independent? Yes

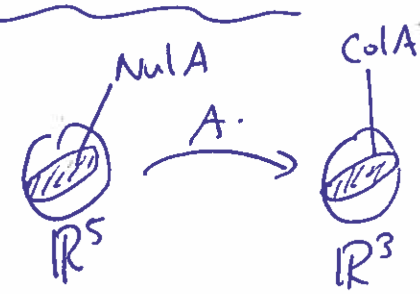
$$\begin{bmatrix}-3 & 2\\1 & 0\\0 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix} \Rightarrow$$

Yes, this forms a basis.





Ex $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$



Find a basis for Col A
and a basis for Nul A

$$\begin{bmatrix} \boxed{\begin{matrix}-3\\1\\2\end{matrix}} & \begin{matrix}6\\-2\\-4\end{matrix} & \boxed{\begin{matrix}-1\\2\\5\end{matrix}} & \begin{matrix}1\\3\\8\end{matrix} & \begin{matrix}-7\\-1\\-4\end{matrix} \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & -2 & 0 & -1 & 3 \\ 0 & 0 & \boxed{1} & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Col A is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}.$

$\vec{x}$ in Nul(A) ... $\vec{x}$ is a soln to $A\vec{x}=\vec{0}$ ...

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_5 free

x_4 free,

$x_3 = -2x_4 + 2x_5$

x_2 free

$x_1 = 2x_2 + x_4 - 3x_5$

$$\vec{x} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

indep? Check ✓

they do span Nul(A) by this work

So $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A.

Ex $A = \begin{bmatrix} 2 & -3 & 1 & -4 \\ 1 & 1 & 3 & 3 \\ -2 & 2 & -2 & 2 \\ -1 & 2 & 0 & 3 \end{bmatrix}$

Find a basis for Col A, and a basis for Nul A.

$A \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So Col A has a basis:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

$\vec{x}$ in Nul(A) $\Rightarrow$ $\vec{x} = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix}$

So Nul(A) has a basis

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We've defined what it means for $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_p\}$ to be a basis of H.

H has <u>lots</u> of different bases...

We'll establish that if $\mathcal{B}$ is one basis for H and $\mathcal{C}$ is another basis for H, then $\mathcal{B}$ and $\mathcal{C}$ have the same number of vectors in them.

The <u>dimension</u> of a subspace H is how many vectors go into a basis.

Proving this. We have a subspace H of $\mathbb{R}^n$

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_p\}$ $\quad$ $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_q\}$

(want to conclude $p = q$.)

Assume $p \neq q$. Assume $q > p$.

Each $\vec{c}_i$ is in H... and $\mathcal{B}$ span H...

$$\vec{c}_i = x_{1i}\vec{b}_1 + x_{2i}\vec{b}_2 + \cdots + x_{pi}\vec{b}_p$$

$$\begin{bmatrix} | & | & & | \\ \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_q \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \bar{b}_1 & \bar{b}_2 & \cdots & \bar{b}_p \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{1q} \\ x_{21} & x_{22} & \cdots & x_{2q} \\ \vdots & \vdots & & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pq} \end{bmatrix}$$

$$C = B \cdot X$$

Assumed $q > p$

X has more columns than rows. So there is a column that's not a pivot column.

X's columns are dependent.

$\Longrightarrow$ there is a nontrivial $\vec{v}$ such that $X\vec{v} = \vec{0}$.

$$C\vec{v} = BX\vec{v}$$

$$C\vec{v} = \vec{0}$$

$\vec{v}$ is nontrivial... C's columns are dependent. Violates our premise that C's columns make a basis.

So q is not $> p$. Similarly, p is not $> q$.

So $p = q$. Having bases all be the same size justifies our def. for <u>dimension</u> of H.

Recall: $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

Col(A) has a basis

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

The dimension of Col A is 2

$$\dim \operatorname{Col} A = 2$$

Vocab: <u>rank</u> := dim Col

you'd say

$\operatorname{rank}(A) = 2$

Nul(A) has a basis

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of Nul A is 3

$$\dim \operatorname{Nul} A = 3$$

Vocab: <u>nullity</u> := dim Nul

you'd say

$\operatorname{nullity}(A) = 3$

$\operatorname{null}(A) = 3$

<u>Theorem:</u> With an $m \times n$ matrix

Rank-Nullity Theorem.

$n = \#\text{pivot cols} + \#\text{not pivot cols}$

$n = \dim \operatorname{Col} A + \#\text{free variables}$

$n = \operatorname{rank} A + \dim \operatorname{Nul} A$

$n = \operatorname{rank} A + \operatorname{nullity} A$

Ex $\quad A$ is a $\overset{m\ \times\ n}{21 \times 10}$ matrix,

and the range of A is 7-dimensional.

What is the dimension of $\mathrm{Nul}(A)$?

Col(A)

$$n = \mathrm{rank}(A) + \mathrm{nullity}(A)$$

$$10 = 7 + \mathrm{nullity}(A)$$

So... it's 3.

Invertible Matrix Theorem (for a square A)

(a) A is invertible

⋮

(l) A^T is invertible

(m) columns of A form a basis for $\mathbb{R}^n$

(n) $\mathrm{Col}(A) = \mathbb{R}^n$

(o) $\dim \mathrm{Col}(A) = n$

(p) $\mathrm{rank}(A) = n$

(q) $\mathrm{Nul}(A) = \{\vec{0}\}$

(r) $\mathrm{nullity}(A) = 0$

Concludes 2.8, 2.9

3.1 ~ 3.2 Determinant of a Matrix.

(will be continued on Monday)

Given a <u>square</u> matrix A, "the determinant of A" is supposed be a single number that captures something important about A.

Notaton: $\det(A)$ or $|A|$

- How to calculate $\det(A)$?
- What is the meaning of $\det(A)$?

<u>"Volume interpretaton"</u> A is $n \times n$ matrix, its columns live in $\mathbb{R}^n$. Use those columns to make a... parallelogram, parallelipiped, or higher dim. analogy. $\det(A)$ is the "volume" of this shape.

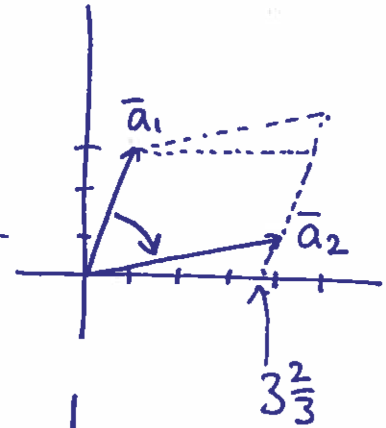
area in 2D

volume in 3D

hypervolume in higher D

also... it might be negative depending on the orientation of columns.

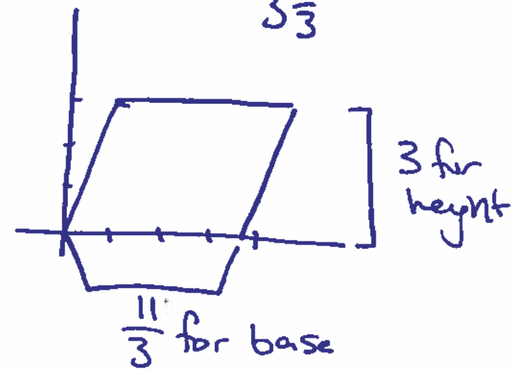
Ex Find $\det\left(\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}\right)$



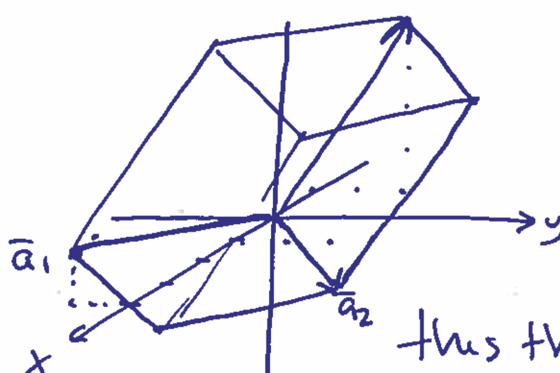
this orientation: ent with a negative value.

$\det(A) = -11$

Area is $\frac{11}{3} \cdot 3 = 11$



Ex $\det\left(\begin{bmatrix} 4 & 1 & -1 \\ -1 & 2 & 2 \\ 1 & -1 & 4 \end{bmatrix}\right)$



this thing's volume is the determinant...

Right Hand Rule

$\Longrightarrow$ this has positive orientation

Fact If A's columns are dependent, what kind of parallelepiped do we get?

The shape would be "flat". The "volume" will be 0 since the columns don't span $\mathbb{R}^n$ (they are stuck in a hyperplane.)

Ex $\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = ?$

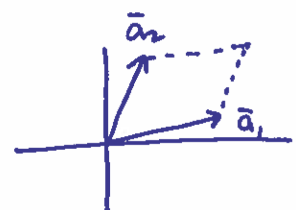
0 area

$\implies \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) = 0$

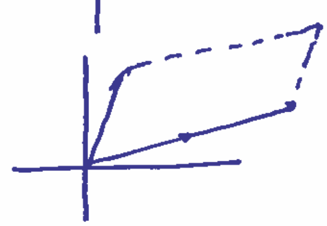
Ex $\det\left(\begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0$

Fact $\det\left(\begin{bmatrix} | & | & | & & | \\ c\bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_n \\ | & | & | & & | \end{bmatrix}\right) = c \cdot \det(A)$

In $\mathbb{R}^2$, $A = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 \end{bmatrix}$



Consider $\begin{bmatrix} 2\bar{a}_1 & \bar{a}_2 \end{bmatrix}$

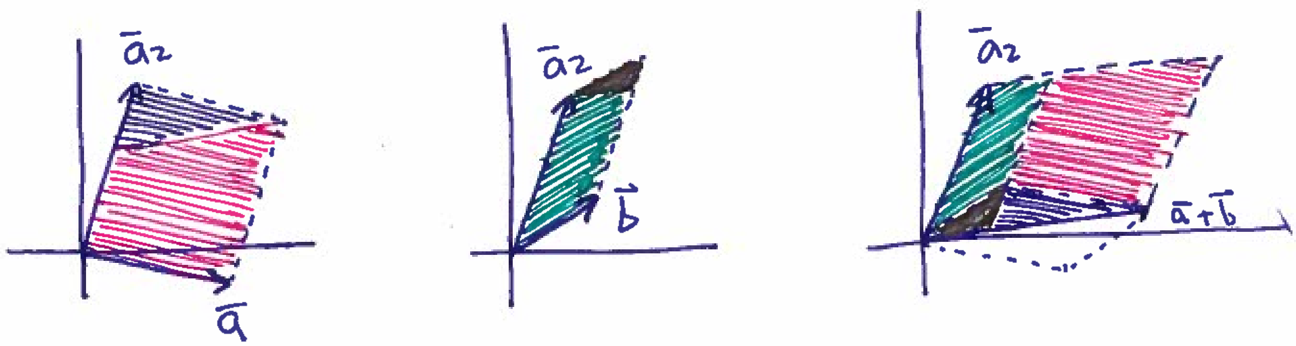


stretching *one* column by c, scales the determinant by c too!

Fact $\det\left(\begin{bmatrix}\vec{a} & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_n\end{bmatrix}\right)$
$+ \det\left(\begin{bmatrix}\vec{b} & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_n\end{bmatrix}\right)$
$= \det\left(\begin{bmatrix}\bar{a}+\vec{b} & \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_n\end{bmatrix}\right)$

$\det\left(\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & 7\\ 3 & 2 & 1\end{bmatrix}\right)$
$+ \det\left(\begin{bmatrix}4 & 1 & 1\\ 1 & 1 & 7\\ 0 & 2 & 1\end{bmatrix}\right)$
$= \det\left(\begin{bmatrix}5 & 1 & 1\\ 3 & 1 & 7\\ 3 & 2 & 1\end{bmatrix}\right)$

$\det\left(\begin{bmatrix}\vec{a} & \bar{a}_2\end{bmatrix}\right)$ $\det\left(\begin{bmatrix}\vec{b} & \bar{a}_2\end{bmatrix}\right)$ $\det\left(\begin{bmatrix}\bar{a}+\vec{b} & \bar{a}_2\end{bmatrix}\right)$



Let's use these features to find a formula

for $\det\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = \det\left(\begin{bmatrix}a\cdot 1 & b\\ a\cdot\frac{c}{a} & d\end{bmatrix}\right) = a\cdot\det\left(\begin{bmatrix}1 & b\\ \frac{c}{a} & d\end{bmatrix}\right)$

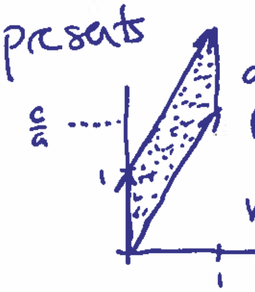
assume $a \neq 0$

dependent columns, so this adds 0.

$= a\cdot\left(\det\begin{bmatrix}1 & b\\ \frac{c}{a} & d\end{bmatrix} + \det\begin{bmatrix}1 & -b\cdot 1\\ \frac{c}{a} & -b\cdot\frac{c}{a}\end{bmatrix}\right)$

$= a\cdot\det\left(\begin{bmatrix}1 & 0\\ \frac{c}{a} & d-b\frac{c}{a}\end{bmatrix}\right) = a\left(d-b\frac{c}{a}\right)\cdot\underbrace{\det\left(\begin{bmatrix}1 & 0\\ \frac{c}{a} & 1\end{bmatrix}\right)}$

represents



a 1x1 parallelogram has area 1

assume $d-b\frac{c}{a} \neq 0$

So $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a\left(d - b\frac{c}{a}\right)$ $\left(\begin{array}{l}\text{under} \\ \text{the assump.} \\ a \neq 0 \\ d - b\frac{c}{a} \neq 0\end{array}\right)$

$$= ad - bc$$

So.. $\det\left(\begin{bmatrix} 11 & 3 \\ 2 & -4 \end{bmatrix}\right) = 11(-4) - 3(2)$

$$= -44 - 6 = -50$$

Actually... can now calculate parallelogram area quickly!

<u>More observations</u>

- $\det\left(\begin{bmatrix} ta & b \\ tc & d \end{bmatrix}\right) = t \cdot \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  scaling a column, scales det.

- In 2×2 case: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc = \det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right)$

  $\det A = \det(A^t)$ (also works for $n\times n$ matrices)

$$t \cdot R_1 + R_2 \longrightarrow R_2$$

- $\det\left(\begin{bmatrix} a & b \\ ta+c & tb+d \end{bmatrix}\right) = a(tb+d) - b(ta+c)$

$$= \cancel{abt} + ad \cancel{- abt} - bc$$

$$= ad - bc = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

- $\det\left(\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right) = c\cdot b - d\cdot a$

$$= -(ad-bc) = -\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

switching two rows negates determinant.

These ~~fo~~ observations hold for $n\times n$ matrices.

Use this to calculate $\det\left(\begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}\right)$

same determinant since replacement doesn't affect det.

$$\begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix} \xrightarrow[-R_1+R_3]{2R_1+R_2} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 7 & 4 \\ 0 & -3 & 3 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_3 \to R_3} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 7 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

det is 33 $\Longleftarrow$ det was 33 $\Longleftarrow$ det was -11 $\Longleftarrow$

$$\xrightarrow[-3R_3+R_1]{-7R_3+R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 11 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 11 \end{bmatrix} \xrightarrow{\frac{1}{11}R_3 \to R_3} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Longleftarrow$ det was -11 $\Longleftarrow$ det was 11 $\Longleftarrow$ det is 1

$$\xrightarrow{\text{"replacements"}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

det is 1

Find $\det\left(\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -2 \\ 3 & 4 & 5 \end{bmatrix}\right)$.

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -2 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[\text{negate det}]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & 1 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[\text{no change in det}]{\substack{-2R_1+R_2 \\ -3R_1+R_2}} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -4 & 5 \\ 0 & -2 & 11 \end{bmatrix}$$

$$\xrightarrow[\text{no change in det}]{-2R_3+R_2} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & -17 \\ 0 & -2 & 11 \end{bmatrix} \xrightarrow[\text{negate det}]{R_2 \leftrightarrow R_3} \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 11 \\ 0 & 0 & -17 \end{bmatrix}}$$

this upper triangular matrix has det $(1)(-2)(-17) = 34$

$34 \Longleftarrow -34 \Longleftarrow$

$\downarrow$

my determinant!