Finding a matrix's inverse

Ex $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 6 & 1 & -3 \end{bmatrix}$ Find A^{-1}.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 6 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 6 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-6R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -3 & 1 & -2 & 0 \\ 0 & 1 & -15 & 0 & -6 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -15 & 0 & -6 & 1 \\ 0 & 3 & -3 & 1 & -2 & 0 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -15 & 0 & -6 & 1 \\ 0 & 0 & 42 & 1 & 16 & -3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -15 & 0 & -6 & 1 \\ 0 & 0 & 1 & 1/42 & 8/21 & -1/14 \end{array}\right] \xrightarrow[-2R_3+R_1]{15R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/21 & 5/21 & 1/7 \\ 0 & 1 & 0 & 15/42 & -2/7 & 8/7 \\ 0 & 0 & 1 & 1/42 & 8/21 & -1/14 \end{array}\right]$$

A^{-1}. (but could be arithmetic errors; didn't take my time.)

2.3 The Invertible Matrix Theorem

For an $n \times n$ matrix...

"A is an invertible matrix" means

there is some other matrix "A^{-1}"

where $A \cdot A^{-1} = I$ and $A^{-1} \cdot A = I$.

This section lists 11 other things about A that are equivalent.

(a) A is invertible.

(b) A row reduces to I.

(c) A has n pivot positions.

(d) The equation $A\vec{x} = \vec{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

(f) The transformation T_A (defined by $T_A(\vec{x}) = A\vec{x}$) is one-to-one.

(g) The equation $A\vec{x} = \vec{b}$ has at least one solution for all vectors $\vec{b}$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The transformation T_A is onto.

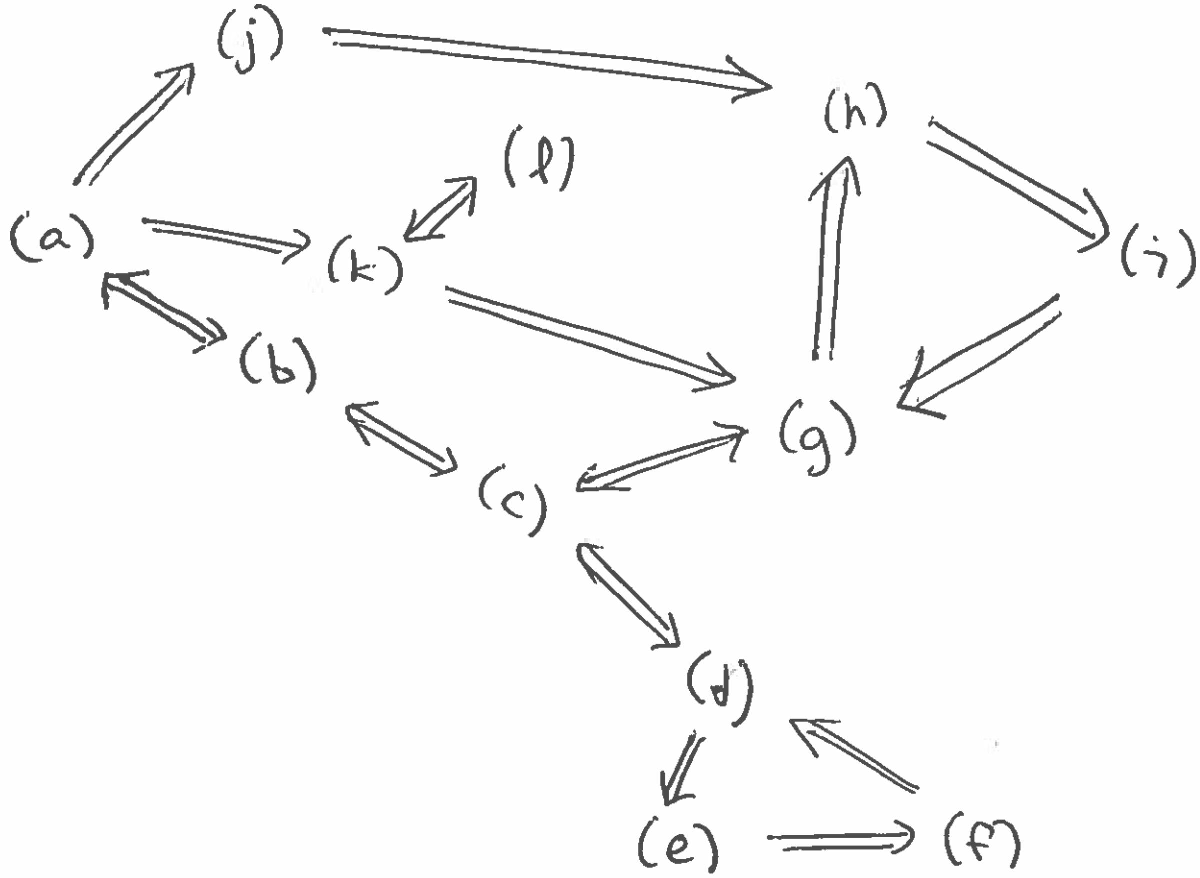
(j) There exists a matrix C with $A \cdot C = I$.

(k) There exists a matrix D with $D \cdot A = I$

"left-inverse"

"right-inverse"

(l) A^t is invertible



(a) A is invertible.

(There exists "A^{-1}" with

$A \cdot A^{-1} = I$ and

$A^{-1} \cdot A = I$)

In 2.2 we showed that the row reductions that change A to I, change I to A^{-1}.

(b) A row reduces to I

since A is $n \times n$...

by def of pivots

(c) A has n pivot positions.

(d) "$A\vec{x} = \vec{0}$" has only the trivial solution.

$[A \mid \vec{0}]$ there's no free ~~var~~ columns so the system has 1 sol.

(d) "$A\vec{x}=\vec{0}$" only has trivial sol

If A's nontrivial columns are dependent, there are $c_1, c_2, \ldots c_n$ where

$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = \vec{0} \longleftrightarrow [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$

nontrivial

a nontrivial sol to $A\vec{x}=\vec{0}$.

(e) The columns of A are linearly independent

If T_A is <u>not</u> one-to-one, there are $\vec{u}\neq\vec{v}$, with $T_A(\vec{u}) = T_A(\vec{v})$

$\Rightarrow T_A(\vec{u}-\vec{v}) = \vec{0}$.

$\Rightarrow A\cdot(\vec{u}-\vec{v}) = \vec{0} \Rightarrow (u_1-v_1)\vec{a}_1 + (u_2-v_2)\vec{a}_2 + \cdots = \vec{0}$

by item (e), $u_1=v_1$, $u_2=v_2$, ...

contradiction.

(f) T_A is one-to-one.

Assuming f... $T_A(\vec{x}) = \vec{0}$ forces $\vec{x}$ to be $\vec{0}$.

$A\cdot\vec{x}=\vec{0}$ only has $\vec{x}=\vec{0}$ as a solution.

So item (d) is true.

(d)

(c) A has n pivots.

$\Updownarrow$ we only have n rows by Theorem 4

(g) The equation "$A\vec{x} = \vec{b}$" is consistent for all $\vec{b}$ in $\mathbb{R}^n$.

$\Downarrow$ there is an $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ solving $A\vec{x} = \vec{b}$,

(h) The columns of A span $\mathbb{R}^n$.

(For any $\vec{b}$ in $\mathbb{R}^n$, there is some $c_1, c_2, \dots c_n$ with $c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n = \vec{b}$)

Take $\vec{b}$ in $\mathbb{R}^n$. Assuming h, $\exists$ $c_1, c_2, \dots c_n$ where $c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n = \vec{b}$. $\Longrightarrow$ $A \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{b}$ $\Longrightarrow$ $T_A\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = \vec{b}$.

$\Downarrow$

(i) T_A is onto.

(For every $\vec{b}$ in $\mathbb{R}^n$, there is $\vec{x}$ with $T_A(\vec{x}) = \vec{b}$.)

Assuming (i), no matter what $\vec{b}$ is, $T_A(\vec{x}) = \vec{b}$ has a solution. So no matter what $\vec{b}$ is $A \cdot \vec{x} = \vec{b}$ has a sol.

$\Downarrow$

(g)

(a) A is invertible.

(there is an "A^{-1}" with $A \cdot A^{-1} = I$ and $A^{-1} \cdot A = I$)

let C be A^{-1}

let $D = A^{-1}$

(j) There is a matrix C with $A \cdot C = I$.

Suppose $\vec{b}$ is in $\mathbb{R}^n$. Then $(AC)\vec{b} = I\vec{b} = \vec{b}$

$\Longrightarrow A(C\vec{b}) = \vec{b}$

take vector $C\vec{b}$'s entries as weights for columns of A and...

(h) The columns of A span $\mathbb{R}^n$.

(for any $\vec{b}$ in $\mathbb{R}^n$ there is a $\vec{x}$ with $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n a_n = \vec{b}$)

(k) There exists D with $DA = I$

(g) "$A\vec{x} = \vec{b}$" has at least one solution $\vec{x}$ no matter what $\vec{b}$ is.

~~Consider $\vec{b}$ in $\mathbb{R}^n$.~~

~~Then $(DA)\vec{b} = I\vec{b} = \vec{b}$~~

$\vec{x} = D\vec{b}$ is certainly one solution to $A\vec{x} = \vec{b}$.

Consider $A\vec{x} = \vec{b}$.

$\Longrightarrow DA\vec{x} = D\vec{b}$

$\Longrightarrow I\vec{x} = D\vec{b}$

$\vec{x} = D\vec{b}$

(ℓ) A^T is invertible..

A^T invertible

$\Downarrow$

$(A^T)(A^T)^{-1} = I$ (now apply transpose to both sides..)

$((A^T)^{-1})^T \cdot \underbrace{\cancel{(A^T)^T}}_{A} = I$

(take D to be $((A^T)^{-1})^T$)

$\Updownarrow$

(k) ~~A~~ there is a matrix D where $DA = I$.

transpose both sides

$A^T \cdot \underbrace{D^T} = I$

So A^T has a right inverse..

by item (j) A^T is invertible..

So (ℓ).

Ex Is $A = \begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$ invertible?

well $A^t = \begin{bmatrix} 5 & -3 & 8 \\ 0 & -7 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ is in REF, and it has 3 pivots. So A^t is invertible by item (c)

So by (l), A is invertible.

Ex Is $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$ invertible?

start row reducing

$\longrightarrow \longrightarrow \begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

don't waste effort on full RREF

we can see A has 4 pivot positions. By IMT (c), A is invertible

Ex Is $\begin{bmatrix} 2 & 1 & 3 & -4 \\ 1 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 \\ 3 & 4 & 7 & -6 \end{bmatrix}$ invertible?

we can see that $\bar{a}_4 = -2\bar{a}_1$

The columns of A are <u>not</u> linearly indep.

By IMT (e), A is not invertible.

Ex Is $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 4 & 7 & 4 \\ 0 & 1 & -2 & 4 \end{bmatrix}$ invertible?

Well, A^T has dependent columns. So by IMT, A^T is not invertible. So by IMT, A is not invertible.

Ex For an $n \times n$ matrix A, If A's columns are linearly independent, do the columns of A^2 span $\mathbb{R}^n$?

A's columns lin indep $\xrightarrow{\text{IMT}}$ A is invertible.

$\Longrightarrow A \cdot A^{-1} = I$

$\Longrightarrow A A A^{-1} = A I$

$\Longrightarrow A A A^{-1} A^{-1} = A I A^{-1}$

$\Longrightarrow A^2 (A^{-1})^2 = I \xrightarrow{\text{by IMT}} A^2$ is invertible.

$\Downarrow$ IMT

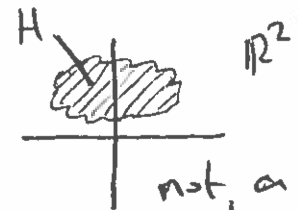
Columns of A^2 span $\mathbb{R}^n$.

2.8 Subspaces of $\mathbb{R}^n$

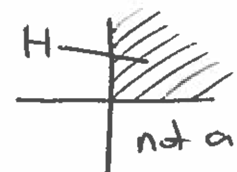
$\mathbb{R}^n$ ← set of all $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

A subspace H of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ where

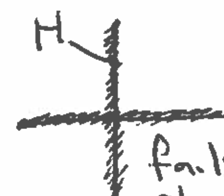
- $\vec{0}$ is in H
- for $\vec{u}, \vec{v}$ are in H, it needs to be the case that $\vec{u}+\vec{v} \in H$.
- for $\vec{u} \in H$, $c \in \mathbb{R}$, it needs to be the case that $c \cdot \vec{u}$ is in H.



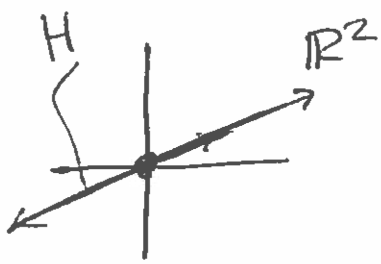
not a subspace, no $\vec{0}$ vector



not a subspace because neg scaling exits H.

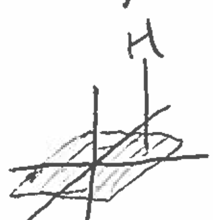


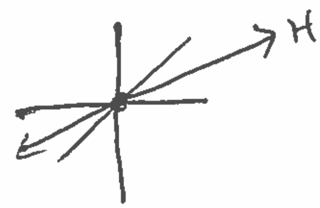
fails to always keep $\vec{u}+\vec{v}$ in H.



In $\mathbb{R}^2$, any line through $\vec{0}$ is a subspace

subspaces | not subspaces





In $\mathbb{R}^3$ any plane or line passing through $\vec{0}$ is a subspace.

<u>Fact</u> Any span of a set of vectors is a subspace.

If $H = \text{Span}\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p\}$

$$= \left\{ c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_p\bar{v}_p \;\middle|\; c_1, c_2, \dots, c_p \in \mathbb{R} \right\}$$

- Is $\vec{0}$ in H? Yes $0\bar{v}_1 + 0v_2 + \dots + 0\vec{v}_p \in H$

$$\Longrightarrow \vec{0} \in H$$

- If $\vec{a}, \vec{b}$ are in H...

$\vec{b} = b_1\bar{v}_1 + b_2\vec{v}_2 + \dots + b\vec{v}_p$

$\vec{a} = a_1\bar{v}_1 + a_2\bar{v}_2 + \dots + a_p\bar{v}_p$

Consider $\vec{a} + \vec{b} = (a_1+b_1)\bar{v}_1 + (a_2+b_2)\bar{v}_2 + \dots + (a_p+b_p)\bar{v}_p$

So $\vec{a}+\vec{b}$ is in H.

- If $\vec{a}$ is in H, c in $\mathbb{R}$

Then $c \cdot \vec{a} = (ca_1)\bar{v}_1 + (ca_2)\bar{v}_2 + \dots + (ca_p)\bar{v}_p$

$\Longrightarrow c\vec{a}$ is in H

Ex In $\mathbb{R}^n$, why is any line through $\vec{0}$ automatically a subspace?

Because it is $\text{Span}\{\vec{v}\}$ ← pick some vector on line other than $\vec{0}$.

$\left.\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}\right\}n.$

Ex Is $\{\vec{0}\}$ a subspace of $\mathbb{R}^n$?

Well $\{\vec{0}\} = \text{Span}\{\vec{0}\}$. So... yes.

We call $\{\vec{0}\}$ a <u>trivial</u> subspace.

Ex Is $\mathbb{R}^n$ a subspace of $\mathbb{R}^n$? Yes. Trivially (the other trivial subspace).

Given an $m \times n$ matrix A, the <u>column space</u> of A, $\text{Col}\,A$, is defined as the span of A's columns. Note $\text{Col}\,A$ is automatically a subspace of $\mathbb{R}^m$.

Ex $A = \begin{bmatrix}1 & 3 & 5\\2 & 4 & 6\end{bmatrix}$, $\text{Col}\,A = \text{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\right\}$

Ex $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Well, Col A is

$\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Col A is a subspace of $\mathbb{R}^3$.
(any span is a subspace)

Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in Col A?

Do scalars c_1, c_2, c_3 exist with

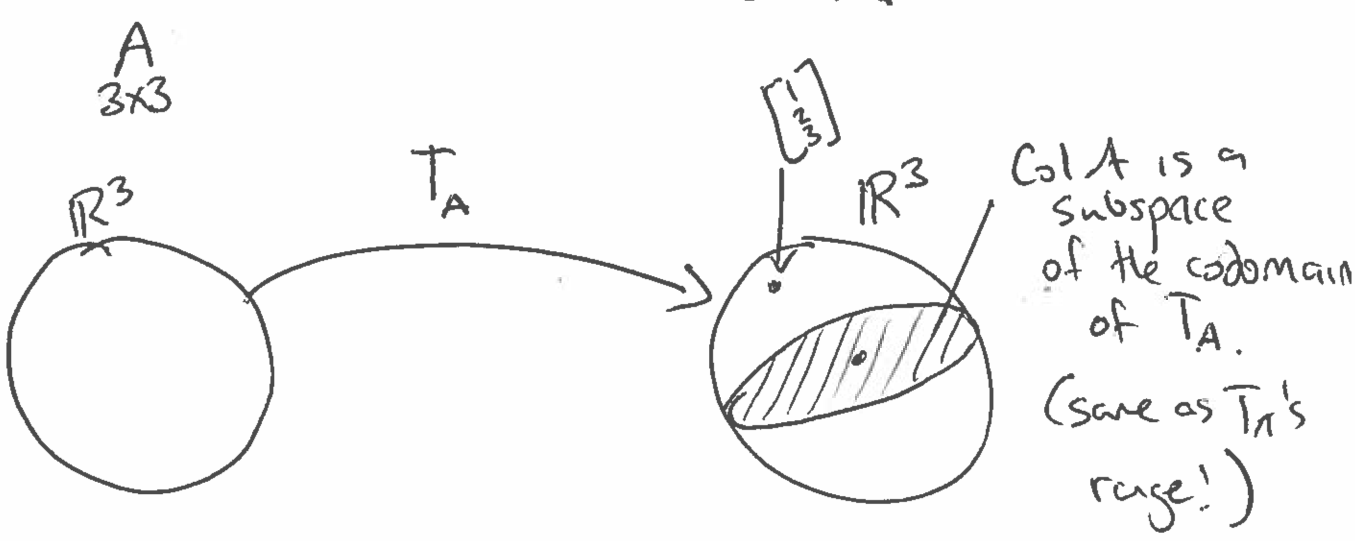
$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} ?$$

So... $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right]$

$\xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right]$

No solution for c_1, c_2, c_3.

So... no. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not in Col A.



Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, write Col A as a span of <u>linearly independent</u> vectors.

Trivial: $\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

but these are <u>de</u>pendent.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-\bar{a}_1 + \bar{a}_2 = \bar{a}_3$

$-\vec{a}_1 + \vec{a}_2 = \vec{a}_3$

wrote this for columns of A's RREF...

still holds here!

$\bar{a}_3$ is redundant. The non-pivot column is redundant. So $\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Ex $A = \begin{bmatrix} 1 & 5 & 9 & 4 & 2 \\ 2 & 6 & 10 & 4 & 0 \\ 3 & 7 & 11 & 4 & 1 \\ 4 & 8 & 12 & 4 & 3 \end{bmatrix}$. Write Col A as a span of lin ind vectors.

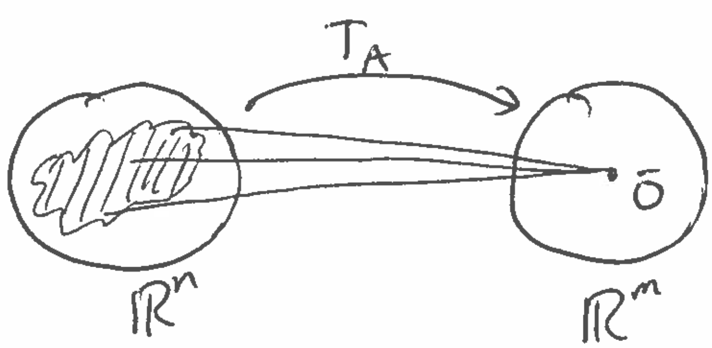
$\xrightarrow{RREF}$ $1^{st}, 2^{nd}, 5^{th}$ are pivot cols

So $\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

The Null Space of A, Nul A, is defined as

$m \times n$ domain of T_A

the set of all vectors $\vec{x}$ in $\mathbb{R}^n$ such that $A \cdot \vec{x} = \vec{0}$.



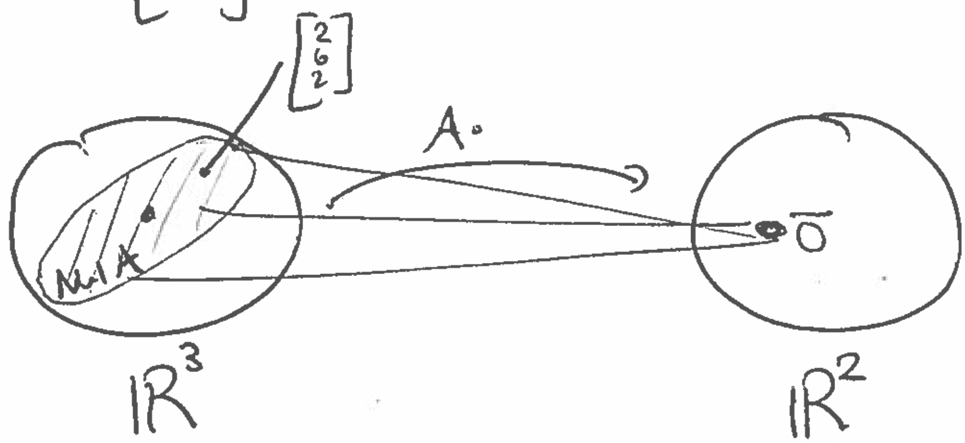
Is Nul A a subspace of $\mathbb{R}^n$?

- $\vec{0}$ in Nul A? ✓
- If $\vec{u}, \vec{v}$ are in Nul A, then $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$. Then $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0}$. So $\vec{u} + \vec{v}$ is in Nul A.
- if $\vec{u}$ is in Nul A, $A(c \cdot \vec{u}) = c \cdot A(\vec{u}) = c\vec{0} = \vec{0}$. So $c\vec{u}$ is in Nul A.

$\underline{\text{Ex}}$ $A = \begin{bmatrix} 2 & 1 & -5 \\ -3 & 1 & 0 \end{bmatrix}$. Is $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$ in Nul A?

Check $A\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Yes, $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$ is in Nul A.



$\underline{\text{Ex}}$ $A = \begin{bmatrix} 2 & 1 & -5 \\ -3 & 1 & 0 \end{bmatrix}$. Write Nul A as the span of <u>linearly independent</u> vectors.

Imagine $\vec{x}$ is in Nul A. So... $A\vec{x} = \vec{0}$

So $\left[\begin{array}{ccc|c} 2 & 1 & -5 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right]$ to solve for $\bar{x}$.

$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$ So sol set is $\left\{ \begin{bmatrix} x_3 \\ 3x_3 \\ x_3 \end{bmatrix} \right\}$

x_3 free

$x_2 = 3x_3$

$x_1 = x_3$

So solution set is $\left\{ x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \text{ where } x_3 \in \mathbb{R} \right\}$

So $\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$. (Done.)

Ex $A = \begin{bmatrix} 2 & 3 & 1 & -5 \\ 1 & 7 & 0 & 1 \end{bmatrix}$. Write Nul A as span of lin. indep vectors.

... the sol set to $A\vec{x} = \vec{0}$...

$$\left[\begin{array}{cccc|c} 2 & 3 & 1 & -5 & 0 \\ 1 & 7 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\cancel{\text{RREF}}} \left[\begin{array}{cccc|c} 1 & 7 & 0 & 1 & 0 \\ 0 & -11 & 1 & -7 & 0 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 7/11 & -39/11 & 0 \\ 0 & 1 & -1/11 & 7/11 & 0 \end{array}\right]$$

x_4 free

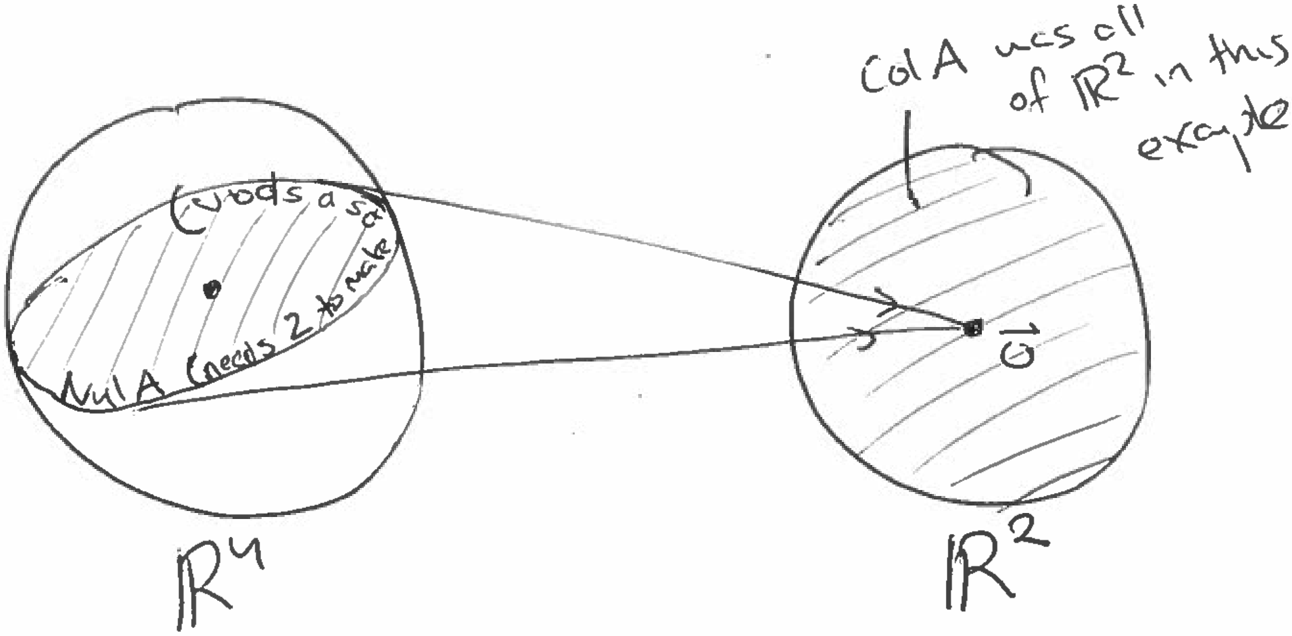
x_3 free

$x_2 = \frac{1}{11}x_3 - \frac{7}{11}x_4$

$x_1 = -\frac{7}{11}x_3 + \frac{39}{11}x_4$

$$\vec{x} = \begin{bmatrix} -\frac{7}{11}x_3 + \frac{39}{11}x_4 \\ \frac{1}{11}x_3 - \frac{7}{11}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -7/11 \\ 1/11 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 39/11 \\ -7/11 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -7/11 \\ 1/11 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 39/11 \\ -7/11 \\ 0 \\ 1 \end{bmatrix} \right\}$$



Col A was all of $\mathbb{R}^2$ in this example
Nul A (needs 2 to make a set)
$\vec{0}$
$\mathbb{R}^4$
$\mathbb{R}^2$