2.1 continued

<u>Facts</u>:

$$A(BC) = (AB)C$$
$$A(B+C) = AB + AC$$
$$(B+C)A = BA + CA$$
$$r(AB) = (rA)B = A(rB)$$

A is $m \times n$

$$I_m \cdot A = A$$
$$A \cdot I_n = A$$

<u>Warnings</u>: $A \cdot B$ is <u>not</u> generally equal to BA

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

but

$$\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$$

not at all the same

Also $(A+B)^2 \neq A^2 + 2AB + B^2$

Instead $(A+B)^2 = A^2 + AB + BA + B^2$

Transpose

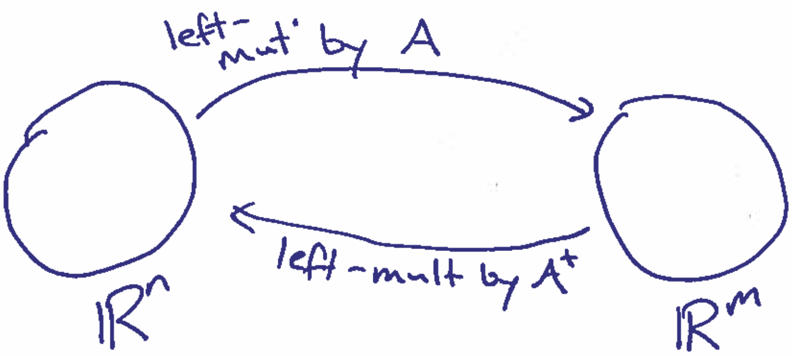
Turns rows into columns
and columns into rows.

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \qquad A^t = \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$$

not
an exponent.
it means
"transpose A"

$$(A^t)_{ij} = (A)_{ji}$$

when A is an $m \times n$ matrix,
A^t is an $n \times m$ matrix.



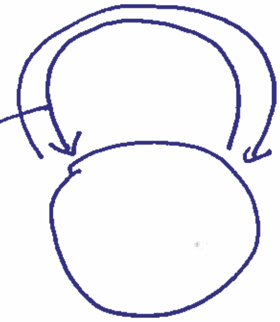
$$((AB)^t)_{ij}$$
$= (AB)_{ji}$
= dot product of jth row of A with ith column of B
= dot product of jth column of A^t with ith row of B^t.
$= (B^t \cdot A^t)_{ij}$

Facts

$$(A^t)^t = A$$
$$(A+B)^t = A^t + B^t$$
$$(rA)^t = rA^t$$
$$(A \cdot B)^t = B^t \cdot A^t$$

2.2 The Inverse of a Matrix

(we know matrices $\longleftrightarrow$ linear transformations



square matrix A, and T_A.

$\mathbb{R}^n$

there ought to be an inverse transformation if T_A was one-to-one and onto...)

Consider its matrix... this is what we would call A^{-1}

not an exponent. A^{-1} undoes mult. by A. We want $A^{-1}(A\vec{x}) = \vec{x}$ and $A(A^{-1}\vec{x}) = \vec{x}$.

square matrix

This leads to: $(A^{-1} \cdot A)\vec{x} = \vec{x}$
and $(A \cdot A^{-1})\vec{x} = \vec{x}$.
(for all vectors $\vec{x}$)

So we want $A^{-1} \cdot A = I$
and $A \cdot A^{-1} = I$ (alternative way to define A^{-1}...)

<u>Def:</u> Given a square matrix A,
If there is a matrix B such that $A \cdot B = I$ and $B \cdot A = I$, then
- A is <u>invertible</u>.
- We can write A^{-1} for B.

Ex $A = \begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix}$

Well, $AB = \begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix}$

$$= \left[\; 5\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -9 \\ 5 \end{bmatrix} \qquad 9\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -9 \\ 5 \end{bmatrix} \;\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

And $BA = \begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 5\cdot 2 + 9(-1) & 5(-9) + 9(5) \\ 1\cdot 2 + 2(-1) & 1(-9) + 2(5) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So A is invertible and $A^{-1} = \begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix}$

With 2×2 matrices, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, formula for A^{-1},

swapped positions

when $ad - bc \neq 0$, $\quad A^{-1} = \dfrac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

negated

In a 2×2 matrix, what if $ad - bc = 0$?

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

$$\Downarrow$$

$$ad = bc$$

$$\Downarrow$$

$$\begin{bmatrix} ad \\ cd \end{bmatrix} = \begin{bmatrix} bc \\ cd \end{bmatrix} \Longrightarrow d\begin{bmatrix} a \\ c \end{bmatrix} = c\begin{bmatrix} b \\ d \end{bmatrix} \Longrightarrow \begin{bmatrix} a \\ c \end{bmatrix} \text{ is parallel to } \begin{bmatrix} b \\ d \end{bmatrix}.$$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ... has linearly dependent columns....

So T_A is not one-to-one.

So T_A has no inverse transformation.

So there is no A^{-1}. So A is not invertible.

If A^{-1} exists (if A is invertible), then given an equation $A\vec{x} = \vec{b} \Longrightarrow$

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$
$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$
$$I\,\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}b$$

Ex $\begin{cases} 2x + 3y = 5 \\ -4x + y = 2 \end{cases}$

$$\begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -1 \\ 24 \end{bmatrix}$$

So $x = -1/14$ and $y = 24/14$

Ex Use matrix inversion to solve:

$$\begin{cases} 3x + 8y = 4 \\ -x + 4y = 2 \end{cases}$$

⟨You Try⟩

$$\begin{bmatrix} 3 & 8 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -1 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & -8 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

$x = 0$
$y = 1/2$

Facts If A is invertible:

- $(A^{-1})^{-1} = A$
- If B is also invertible, then AB is invertible.
  (argument: $\underbrace{(AB)(B^{-1} \cdot A^{-1})}_{}$ $= ((AB) \cdot B^{-1})A^{-1}$
  $= (A(BB^{-1})A^{-1})$
  $= (AIA^{-1})$
  $= (AA^{-1}) = I$

  so this is $(AB)^{-1}$

  Similarly, $(B^{-1}A^{-1})(AB) = I$.)

  and $(AB)^{-1} = B^{-1} \cdot A^{-1}$

- A^t is invertible, and $(A^t)^{-1} = (A^{-1})^t$

Elementary Row Operations:

- interchange rows
- Scale a row
- replace a row with some multiple of another row added to that row.

Ex $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ same!

Try Doing the same row operation to I.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ this is an elementary matrix. Result of some elementary row operation applied to I.

Now observe:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Fact: elementary matrix $\cdot$ A = result of doing same row op. on A.

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= A \qquad = E_{1,2} \cdot A \qquad = E_{-3;1,2} \cdot E_{1,2} \cdot A \qquad = E \cdot E \cdot E \cdot A \qquad = EEEEA$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1/5 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Here in this example,

$$\underbrace{E\,E\,E\,E}_{\text{four different elementary matrices}}A = I$$

$$\underbrace{(\overbrace{E_4E_3E_2E_1}^{\text{each are invertible.}})}_{\text{this is left-inverse of } A.}A = I$$

So

$$\begin{aligned}
(E_4E_3E_2E_1)A &= I \\
E_3E_2E_1A &= E_4^{-1}I \\
E_2E_1A &= E_3^{-1}E_4^{-1}I \\
E_1A &= E_2^{-1}E_3^{-1}E_4^{-1}I \\
A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}I \\
A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} \\
AE_4 &= E_1^{-1}E_2^{-1}E_3^{-1} \\
AE_4E_3 &= E_1^{-1}E_2^{-1} \\
AE_4E_3E_2 &= E_1^{-1} \\
A(E_4E_3E_2E_1) &= I
\end{aligned}$$

Because A could RREF to I, that meant

$(E_p \cdots E_2E_1)A = I$ and $A(E_p \cdots E_2E_1) = I$,

and that means A is invertible... and its inverse is $\underline{\underline{E_pE_{p-1}\cdots E_2E_1}}$.

This tells us the same sequence of row operations that converts A to I, will convert I to A^{-1}.

Ex Use the above to find $\begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}^{-1}$.

$[\ A \mid I\ ]$

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 0 & 2 & -3/2 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 2 & -3/2 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3/4 & 1/2 \end{array}\right]$$

Conclusion $\begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -3/4 & 1/2 \end{bmatrix}$

Use this to find $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}^{-1}$.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-4R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \xrightarrow{3R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\xrightarrow{-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

we got I here...

this is A^{-1}!

we start with square matrices...

if we don't get I, A has a free column, and one of A's rows doesn't have a pivot...

So multiplying by A is not one-to-one, nor is it onto...

So this action has no inverse. So A^{-1} doesn't exist.