1.9 The Matrix of a Linear Transformation

Two Ideas with Linear Transformations

two conditions $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ and $T(c\vec{u}) = cT(\vec{u})$.

A matrix can be used to define a linear transformation

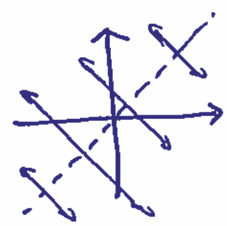
<u>Ex</u> $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Define $T_A(\vec{x}) = A\vec{x}$

Or matrix-free definitions for a linear trans.

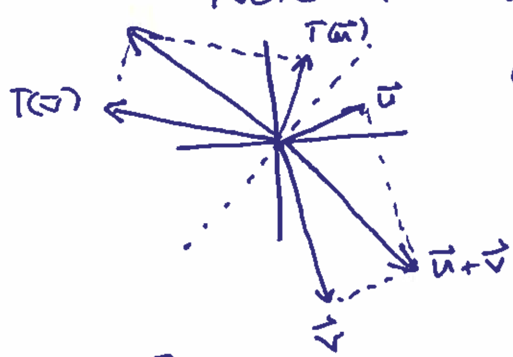
<u>Ex</u>: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

T reflects over the line $y = x$.



Ultimately these ideas are the same.

Note: this <u>is</u> a linear trans.



confirms one feature of linearity.

In $\mathbb{R}^n$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ are the <u>standard unit vectors</u> or <u>standard coordinate vectors</u>

$\vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_n$

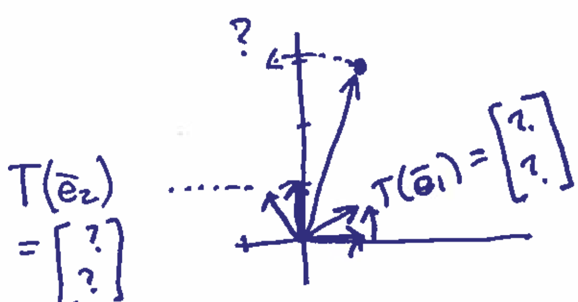
Take $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, a linear transformation

Well
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= T\left(x\bar{e}_1 + y\bar{e}_2\right)$$
$$= T(x\bar{e}_1) + T(y\bar{e}_2)$$
$$= x \cdot T(\bar{e}_1) + y \cdot T(\bar{e}_2)$$
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} | & | \\ T(\bar{e}_1) & T(\bar{e}_2) \\ | & | \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

We can always study a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ by building a certain matrix, built from what T does to the $\bar{e}_i$ vectors.

Ex Suppose $T:$ ~~rota~~ $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, rotates by $\frac{\pi}{6}$ radians about the origin. (This is linear; granted this may not be obvious.)

What does T do to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$?



$\cos(\pi/6) = \sqrt{3}/2$

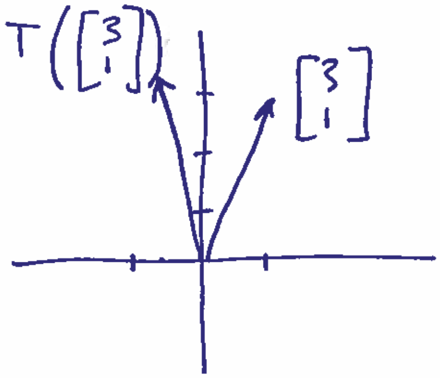
$\sin(\pi/6) = 1/2$

Well, let's find T's matrix, A_T

$$A_T = \begin{bmatrix} | & | \\ T(\bar{e}_1) & T(\bar{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

(We have to work out $T(\bar{e}_1)$, $T(\bar{e}_2)$, easier than $T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$.

Now $T\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = A\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}$



$$= \begin{bmatrix}\sqrt{3}/2\\ 1/2\end{bmatrix} + 3\begin{bmatrix}-1/2\\ -\sqrt{3}/2\end{bmatrix}$$

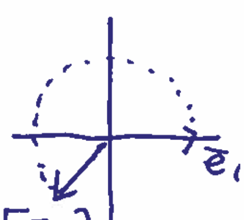
$$= \begin{bmatrix}\sqrt{3}/2 - 3/2\\ \frac{1}{2} + \frac{3}{2}\sqrt{2}\end{bmatrix} \approx \begin{bmatrix}-0.63\\ 3.09\end{bmatrix}$$

You try: T rotates about the origin of $\mathbb{R}^2$ by $5\pi/4$ counterclockwise.

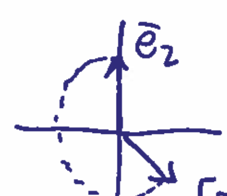
Find A_T, the matrix for T.

$$A_T = \begin{bmatrix} | & | \\ T(\bar{e}_1) & T(\bar{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix}-\sqrt{2}/2 & \sqrt{2}/2\\ -\sqrt{2}/2 & -\sqrt{2}/2\end{bmatrix}$$

think through what the verbal description says to do to $\bar{e}_1$



$\begin{bmatrix}-\sqrt{2}/2\\ -\sqrt{2}/2\end{bmatrix}$



$\begin{bmatrix}\sqrt{2}/2\\ -\sqrt{2}/2\end{bmatrix}$

So.. for example.. to rotate $\begin{bmatrix}5\\3\end{bmatrix}$ this way...

$$\begin{bmatrix}-\sqrt{2}/2 & \sqrt{2}/2\\ -\sqrt{2}/2 & -\sqrt{2}/2\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}-\sqrt{2}\\ -4\sqrt{2}\end{bmatrix}$$

Ex Suppose $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-2y \\ 5x-y \\ 8x+y \end{bmatrix}$. Find A_T.

domain $\mathbb{R}^2$

Last class

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix} + y\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 5 & -1 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

A_T

Today

two since domain is $\mathbb{R}^2$.

$$A_T = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) \end{bmatrix}$$

do what you can do to find these...

$$T(\bar{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}$$

$$T(\bar{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

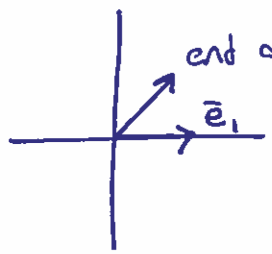
So $A_T = \begin{bmatrix} 1 & -2 \\ 5 & -1 \\ 8 & 1 \end{bmatrix}$.

Ex $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

reflect over the x-axis,

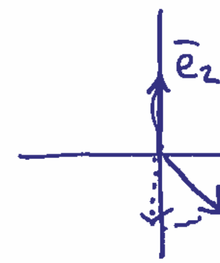
then rotate c.c.w by $\pi/4$ radians.

Find A_T. $\quad A_T = \begin{bmatrix} | & | \\ T(\bar{e}_1) & T(\bar{e}_2) \\ | & | \end{bmatrix}$



reflection leaves it alone...

then rotate...

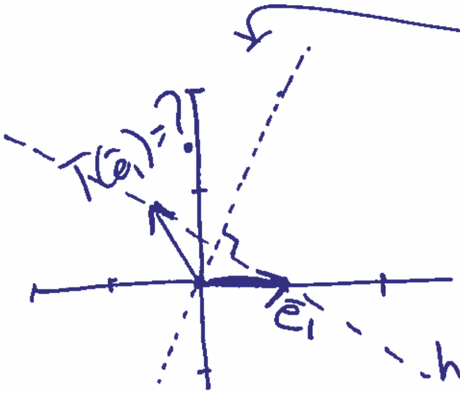


reflection makes it point south

So... $\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$.

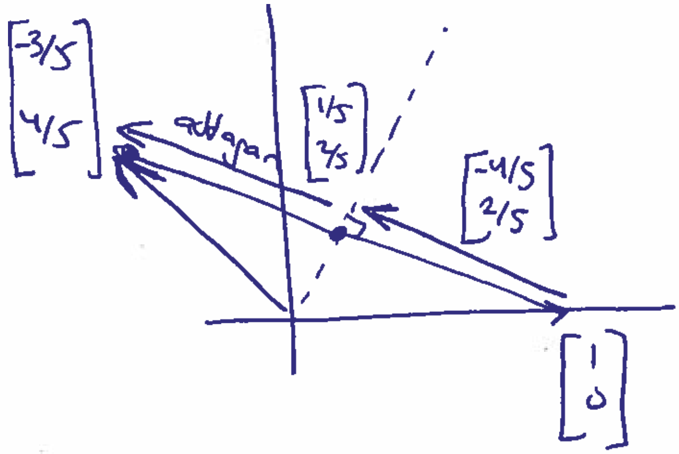
Ex T reflects over the line $y = 2x$.

Find A_T. $\quad A_T = \begin{bmatrix} | & | \\ T(\bar{e}_1) & T(\bar{e}_2) \\ | & | \end{bmatrix}$



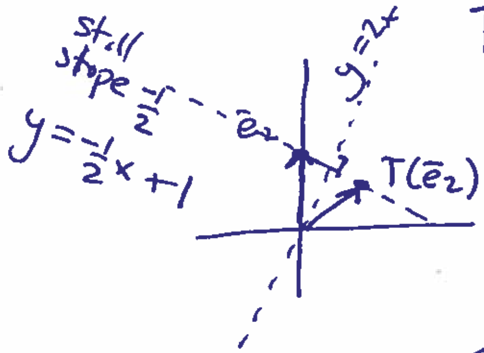
Find cross: $\begin{cases} y = 2x \\ y = -\frac{1}{2}(x-1) \end{cases} \xrightarrow{\text{solve}} \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$

has slope $-\frac{1}{2}$ (neg reciprocal of 2)



So... $A_T = \begin{bmatrix} -3/5 & ? \\ 4/5 & ? \end{bmatrix}$

$T(\bar{e}_2)$



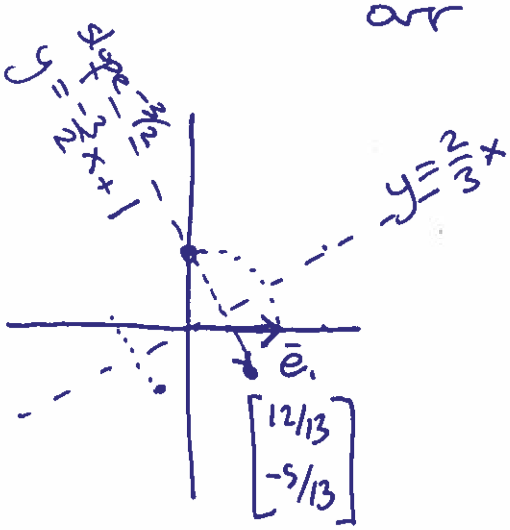
$= \cdots = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

Same kind of geometry work.

So $A_T = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$

(reflecting over $y = 2x$)

<u>You try</u>: T rotates by $90°$ ccw, then reflects over line $y = \frac{2}{3}x$. Find T's matrix, A_T.



$A_T = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) \end{bmatrix}$

$= \cdots$

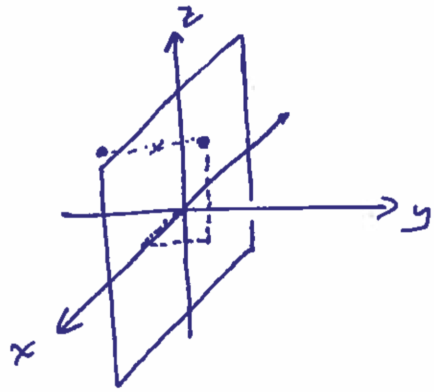
$= \begin{bmatrix} 12/13 & -5/13 \\ -5/13 & -12/13 \end{bmatrix}$

Cross $\begin{cases} y = \frac{2}{3}x \\ y = -\frac{3}{2}x + 1 \end{cases} \Longrightarrow \begin{bmatrix} 6/13 \\ 4/13 \end{bmatrix}$

Ex $T: \mathbb{R}^3 \to \mathbb{R}^3$, T is linear,

domain $\mathbb{R}^3$

. T reflects vectors through the xz-plane.



Find A_T.

$$\begin{bmatrix} | & | & | \\ T(\bar{e}_1) & T(\bar{e}_2) & T(\bar{e}_3) \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} \bar{e}_1 & -\bar{e}_2 & \bar{e}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ for reflecting through xz-plane.

<u>**Def**</u>: A transformation is <u>onto</u> if every $\vec{b}$ in $\mathbb{R}^m$ has an $\vec{x}$ in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$.

$\mathbb{R}^m$: codomain

$\mathbb{R}^n$: domain

A transformation is <u>one-to-one</u> if whenever $T(\vec{x}) = T(\vec{y})$, it means $\vec{x} = \vec{y}$.

"onto" and "one-to-one" are adjectives to apply to a transformation... A linear transf. can be written as a matrix transf. So we can study T's onto-ness and one-to-one-ness using T's matrix.

Ex $T: \mathbb{R}^2 \to \mathbb{R}^3$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ 5x + 7y \\ x + 3y \end{bmatrix}$$

(This is linear.)

Is T onto? Is T one-to-one?

$$A_T = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

"Does $T(\vec{x}) = \vec{b}$ always have a solution no matter what $\vec{b}$ is?"

"Does $A_T\vec{x} = \vec{b}$ always have a sol. no matter what $\vec{b}$ is?"

No. One row is missing a pivot position (Theorem 4.)

"Is ~~it~~ it possible to have $\vec{x} \neq \vec{y}$, but $T(\vec{x}) = T(\vec{y})$?"

(No $\Rightarrow$ one-to-one)

"Is it possible to have $\vec{x} - \vec{y} \neq \vec{0}$, but $\cancel{T(\vec{x}) - T(\vec{y})} = \vec{0}$?"

$T(\vec{x} - \vec{y}) = \vec{0}$

"Is it possible to have more than one solution to $T(\vec{u}) = \vec{0}$?"

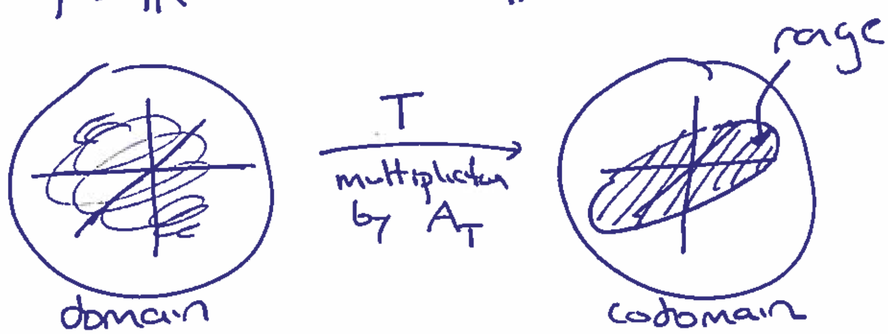
"Does $A_T \cdot \vec{u} = \vec{0}$ have non-trivial solutions?" No... because every column has a pivot.

Yes, T is one-to-one. $\Longleftarrow$

<u>Ex</u> Define $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y - z \\ 3x + 9z - 2y \\ -(x + 2z) \end{bmatrix}$

Is T onto? Is T one-to-one?

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$



$$A_T = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) & T(\bar{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 9 \\ -1 & 0 & -2 \end{bmatrix}$$

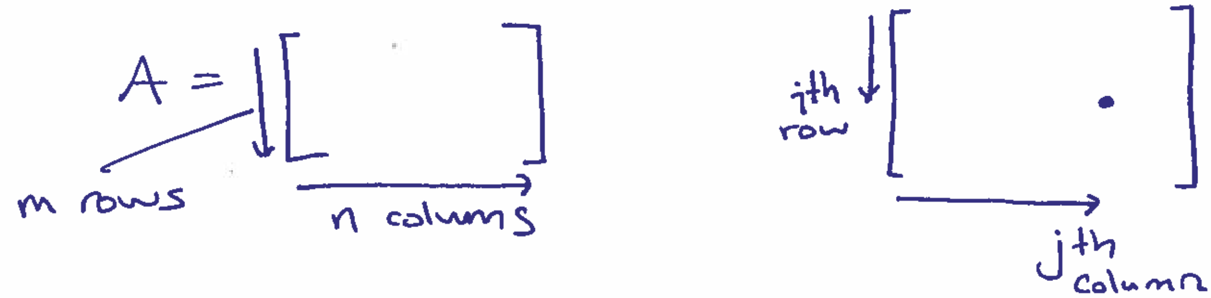
$$\xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

→ no pivot in this row $\Longrightarrow$ not onto

↓ no pivot in this column $\Longrightarrow$ not one-to-one

2.1 Matrix Operations

If A is an $m \times n$ matrix...



The (i,j)th entry of A.

Or... a_{ij}

Or... $(A)_{ij}$

Sometimes the a_{ij} are defined first...

We can write $[a_{ij}]$ to a matrix built with those values

A <u>diagonal matrix</u> has all a_{ij} with $i \neq j$, equal to 0.

<u>Ex</u> $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

A <u>square matrix</u> has $m = n$.

The <u>zero matrix</u> has $a_{ij} = 0$ for <u>all</u> i, j.

An <u>identity matrix</u> is a square matrix,

where $a_{ij} = \delta_{ij}$

Kronecker delta: $\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Note $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

vectors are unchanged with this

Write I for an identity matrix... use context to know the value of n.

We can add/subtract matrices:

Ex $\begin{bmatrix} 3 & 1 & 0 \\ 4 & 8 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 1 & -3 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 & -3 \\ 2 & 9 & 3 \end{bmatrix}$

(requires $m \times n$ for both to match)

$$A \quad + \quad B \quad = \quad (A+B)$$

In other words $(A+B)_{ij} = A_{ij} + B_{ij}$

Note $\begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 1 & 8 \end{bmatrix} + \begin{bmatrix} 5 & 8 \\ 2 & 9 \end{bmatrix}$ is undefined.

We can scale matrices

Ex

$$\underbrace{30}_{c} \cdot \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 8 & 1 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 30 & 60 & 30 \\ 240 & 30 & 90 \end{bmatrix}}_{cA}$$

$$(cA)_{ij} = c \cdot A_{ij}$$

We can make linear combinations

$$2\begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} - 3\begin{bmatrix} 1 & 8 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -18 \\ 14 & -3 \end{bmatrix}$$

Facts:

$$A + B = B + A$$

$$(A+B)+C = A + (B+C)$$

$$A + O = A$$

$$c(A+B) = cA + cB$$

$$(c+d) \cdot A = cA + dA$$

$$c(dA) = (c \cdot d)A$$

Multiplying Matrices

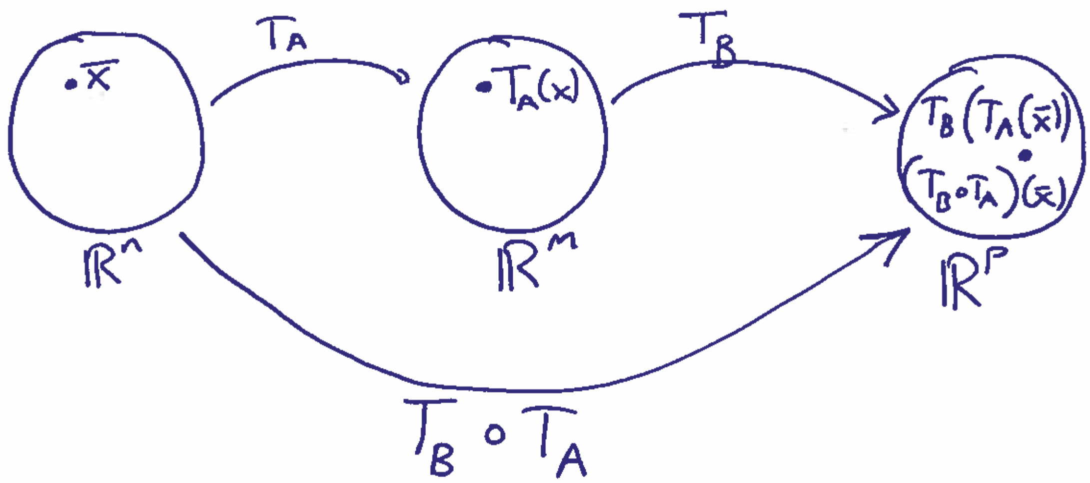
Given A, B matrices, does $A \cdot B$ make any sense?

Suppose you have $\underset{m \times n}{A}$, ...

gives us $T_A : \mathbb{R}^n \to \mathbb{R}^m$

Suppose you have $\underset{p \times m}{B}$

gives us $T_B : \mathbb{R}^m \to \mathbb{R}^p$



We define BA to be the matrix for $T_B \circ T_A$.

$p \times m$ $\quad$ $m \times n$

had to match!

Ex $A = \begin{bmatrix}1&3\\2&2\\3&1\end{bmatrix}$ $\quad B = \begin{bmatrix}1&0&3\\2&1&-2\end{bmatrix}$

Find BA. $\quad$ BA is The matrix for $\underbrace{T_B \circ T_A .}$

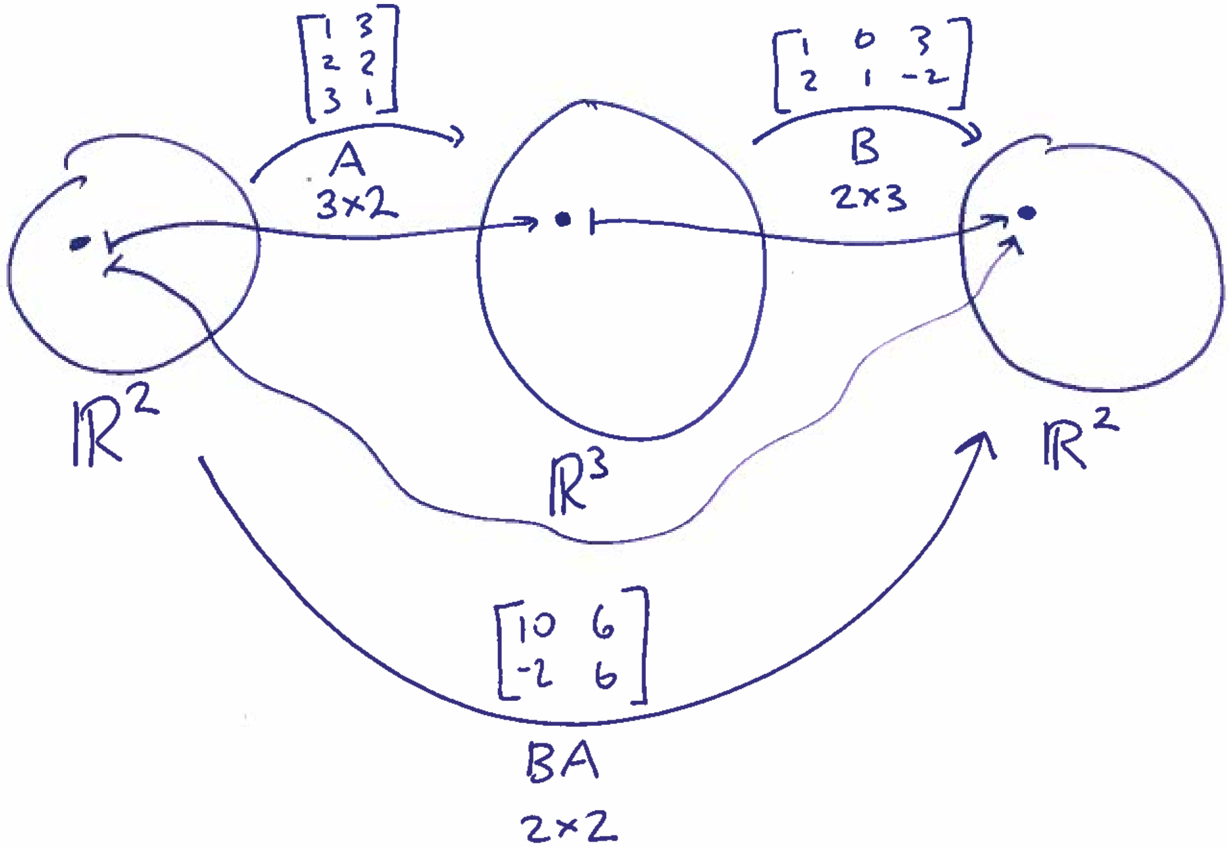
To build a matrix for a transf.

$$\begin{bmatrix} (T_B \circ T_A)(\bar{e}_1) & (T_B \circ T_A)(\bar{e}_2) & \cdots & (T_B \circ T_A)(\bar{e}_m) \end{bmatrix}$$

$$\begin{aligned}
&(T_B \circ T_A)\,\bar{e}_1 \\
&= T_B\big(T_A(\bar{e}_1)\big) \\
&= T_B\left(T_A\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\right) \\
&= T_B\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) \\
&= \begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} + 3\begin{bmatrix}3\\-2\end{bmatrix} \\
&= \begin{bmatrix}10\\-2\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&(T_B \circ T_A)\,\bar{e}_2 \\
&= T_B\big(T_A(\bar{e}_2)\big) \\
&= T_B\left(\begin{matrix}\text{second column}\\ \text{of matrix } A\end{matrix}\right) \\
&= T_B\left(\begin{bmatrix}3\\2\\1\end{bmatrix}\right) \\
&= 3\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}6\\6\end{bmatrix}
\end{aligned}$$

So $BA = \begin{bmatrix}1&0&3\\2&1&-2\end{bmatrix}\begin{bmatrix}1&3\\2&2\\3&1\end{bmatrix} = \begin{bmatrix}10&6\\-2&6\end{bmatrix}$



Shortcut (1)

To calculate $\overset{p\times m}{B}\,\overset{m\times n}{A}$, that is, $B\begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix}$,

just apply B to each $\bar{a}_i$: $\begin{bmatrix} B\bar{a}_1 & B\bar{a}_2 & \cdots & B\bar{a}_n \end{bmatrix}$

Ex

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} = \left[\underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\checkmark} \quad \underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}_{\checkmark}\right]$$

$$= \begin{bmatrix} 10 & 6 \\ -2 & 6 \end{bmatrix}$$

Shortcut (2) To find $[BA]_{ij}$, dot product the ith row of B with the jth column of A.

Ex
$$\underset{2\times 3}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 10 & 6 \\ -2 & 6 \end{bmatrix}}$$

1,1 pos.

1st row • 1st col

$$\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= 1\cdot 1 + 0\cdot 2 + 3\cdot 3 = 10$$

1,2 pos.

1st row 2nd col

$$\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= 1\cdot 3 + 0\cdot 2 + 3\cdot 1$$
$$= 6$$

You try: $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix} = ?$ $\begin{bmatrix} 1 & -8 \\ 16 & -7 \end{bmatrix}$

Then: $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 \\ 1 & 4 & 1 \end{bmatrix} = ?$ $\begin{bmatrix} 2 & 12 & 5 \\ 3 & 20 & 9 \end{bmatrix}$

Warning: $A\cdot B$ and $B\cdot A$ are generally not equal!