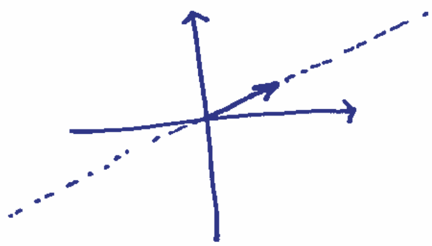
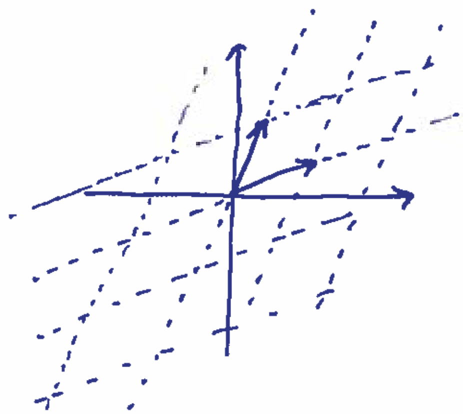
1.7 Linear Independence

Consider $\mathbb{R}^2$



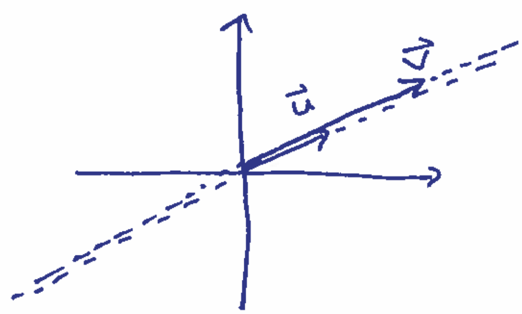
with a vector $\vec{u}$.
$\text{Span}\{\vec{u}\}$ is a line.

Now with another vector $\vec{v}$



$\text{Span}\{\vec{u},\vec{v}\}$ is the whole $\mathbb{R}^2$.

It <u>could have</u> looked like



$\text{Span}\{\vec{u},\vec{v}\} = \text{Span}\{\vec{u}\}$
vector $\vec{v}$ is redundant

then $\text{Span}\{\vec{u},\vec{v}\}$ could be just a line...

A vector in a set $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p\}$ is <u>redundant</u> if $\text{Span}\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p\} = \text{Span}\left\{\begin{matrix}\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p \\ \text{with that one vector removed}\end{matrix}\right\}$

Note: Don't just say some vector "is redundant". Say it's redundant in context of a certain collection $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p\}$. "Redundant" is non-standard.

Ex $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\bar{v}_1 \qquad \bar{v}_2 \qquad \bar{v}_3$

I'd like to say $\bar{v}_2$ is redundant...

Since $\bar{v}_2 = 2\bar{v}_1$, $\bar{v}_2$ doesn't seem to bring anything new to the span...

Need to be able to say

$$\mathrm{Span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \mathrm{Span}\{\bar{v}_1, \bar{v}_3\}$$

$\parallel \leftarrow$ use def of Spn

$$= \{c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$$

$$= \{c_1\bar{v}_1 + c_2 \cdot 2\bar{v}_1 + c_3\bar{v}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$$

$$= \{(c_1 + 2c_2)\bar{v}_1 + c_3\bar{v}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$$

$$= \{c_1\bar{v}_1 + c_3\bar{v}_3 \mid c_1, c_3 \in \mathbb{R}\} = \mathrm{Span}\{\bar{v}_1, \bar{v}_3\}$$

Note we could have settled on $\bar{v}_1$ as "redundant".

But not $\bar{v}_3$.

Ultimately this fact: $\vec{v}_2 = 2\cdot\vec{v}_1$ is what let us declare $\vec{v}_2$ "redundant".

$$\underbrace{-2\vec{v}_1 + \vec{v}_2 + 0\vec{v}_3}_{} = \vec{0}.$$

this expression is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ (with weights $-2, 1, 0$).

which equals the $\vec{0}$ vector.

Having a redundant vector $\iff$ there is a nontrivial linear combination of $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ that equals $\vec{0}$.

Ex $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

Is there a redundant vector?

$$-\vec{v}_1 + 2\vec{v}_2 + (-1)\vec{v}_3 = \vec{0}$$

Note

$$\underbrace{\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3}_{} = \vec{0}$$

a nontrivial lin. comb. of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ that equals $\vec{0}$.

$\vec{v}_1 = 2\vec{v}_2 - \vec{v}_3$

$\vec{v}_3 = 2\vec{v}_2 - \vec{v}_1$

$\vec{v}_2 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_3$

So $\vec{v}_1$ is redundant. (or $\vec{v}_3$ ... or $\vec{v}_2$)

$\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{Span}\{\vec{v}_2, \vec{v}_3\} \Longleftarrow$

To fully prove this...

$$\text{Span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \{c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$$

$$= \{c_1(2\bar{v}_2 - \bar{v}_3) + c_2\bar{v}_2 + c_3\bar{v}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$$

$$= \{(2c_1 + c_2)\bar{v}_2 + (-c_1 + c_3)\bar{v}_3 \mid c_1, c_2, c_3 \in \mathbb{R}\}$$

$$= \{c_1\bar{v}_2 + c_3\bar{v}_3 \mid c_1, c_3 \in \mathbb{R}\}$$

<u>Def</u> A set of vectors $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p\}$ is <u>linearly dependent</u> if there is a nontrivial linear combination of these vectors that equals $\vec{0}$.

this adjective is to be applied to <u>sets</u> of vectors.

<u>Ex</u> $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

"redundant" is an adjective to apply to an individual vector (in context of a set..)

We saw

$$\underbrace{2\bar{v}_1 - \bar{v}_2 + 0\bar{v}_3}_{\text{nontrivial L.C. that equals } \vec{0}.} = \vec{0}_n$$

So $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is linearly ~~a~~<u>de</u>pendent

Ex $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

We saw $\bar{v}_1 - 2\bar{v}_2 + \bar{v}_3 = \vec{0}$

So this set is linearly dependent.

A set $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p\}$ is <u>linearly independent</u> if it's not linearly dependent...

linearly <u>dependent</u>	linearly <u>in</u>dependent.
there <u>is</u> a nontrivial collection of weights $c_1, c_2, \ldots, c_p$ such that $c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_p\bar{v}_p = \vec{0}$	whenever you manage to have $c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_p\bar{v}_p = \vec{0}$, the $c_1, c_2, \ldots, c_p$ must all be 0.

Ex Is $\left\{ \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix} \right\}$ linearly independent?

This asks: $c_1 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Is $c_1, c_2, c_3 = 0$ the only solution?

Set up $\begin{bmatrix} 3 & -2 & -2 \\ 1 & 0 & 4 \\ -2 & 1 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$

there's a free variable

$\Longrightarrow$ there's a nontrivial sol. to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$.

$\Longrightarrow$ there's a nontrivial lin combo that equals $\vec{0}$.

$\Longrightarrow$ So by def, $\left\{ \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix} \right\}$ is linearly dependent.

$\begin{cases} c_1 = -4c_3 \\ c_2 = -7c_3 \\ c_3 \text{ free} \end{cases}$

general solution

let $c_3 = 1$
$c_2 = -7$
$c_1 = -4$

a specific solution

$\Longrightarrow -4\vec{v}_1 - 7\vec{v}_2 + \vec{v}_3 = \vec{0}$

Since we could solve for $\vec{v}_1$, or $\vec{v}_2$, or $\vec{v}_3$, we may view any of them as "redundant".

Ex

Is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}$ linearly independent?

What if $c_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?

Are there nontrivial solutions?

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & -2 \\ 3 & 4 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow[-3R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 4 & -5 \end{bmatrix} \xrightarrow{-4R_2+R_3} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -4 \\ 0 & 0 & \boxed{11} \end{bmatrix}$$

pivot in every column of A.

So no nontrivial solution to our equation. So it must be that $c_1 = 0$, $c_2 = 0$, $c_3 = 0$

So $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}$ is independent.

Ex Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ linearly independent?

Ex Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \right\}$ linearly independent?

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ is <u>de</u>pendent

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$ indicates dependency.

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \right\}$ is <u>in</u>dependent

(a Vandermonde matrix)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Downarrow$ indicates independent.

<u>Ex</u> In $\mathbb{R}^3$, is $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ linearly independent?

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

no free columns

$\Longrightarrow$ the set is <u>in</u>dependent.

Could you have a set of <u>one</u> vector and the set is linearly <u>dependent</u>?

$\{\vec{v}_1\}$

Consider if $c_1\vec{v}_1 = \vec{0}$ has nontrivial sols.

If $\vec{v}_1 \neq \vec{0}$, c_1 would have to be 0.

then it's an independent set

If $\vec{v}_1 = \vec{0}$ then c_1 could be nontrivial.

So $\{\vec{0}\}$ is a <u>dependent</u> set.

If you have $\{\bar{v}_1, \bar{v}_2\}$, neither of which are $\vec{0}$, could this set be dependent?

Consider $c_1\bar{v}_1 + c_2\bar{v}_2 = \vec{0}$ with nontrivial c_i.

$$\Longrightarrow c_1\bar{v}_1 = -c_2\bar{v}_2 \quad (\text{assuming } c_1 \neq 0)$$

$$\Longrightarrow \bar{v}_1 = \left(\frac{-c_2}{c_1}\right)\bar{v}_2 \quad (\text{shows } \bar{v}_1 \ \&\ \bar{v}_2 \text{ are parallel}).$$

If there are more than two vectors, the set can be linearly dependent in more complicated ways. ~~[illegible]~~ One vector need <u>not</u> be a scalar mult of another vector.

1.8 Intro to Linear Transformations

We've seen $\underset{?}{x_1}\vec{a}_1 + \underset{?}{x_2}\vec{a}_2 + \cdots + \underset{?}{x_n}\vec{a}_n = \vec{b}$

we viewed x_i as variables to solve for:

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{matrix} | \\ \vec{b} \\ | \end{matrix}$$

still viewing all the x_i as things to solve for.

Flip this. What if we have A, we have $\vec{x}$ established.

Computing $A \cdot \vec{x}$, getting $\vec{b}$ can be interpreted as "A transformed $\vec{x}$ into $\vec{b}$."

Ex Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 8 \end{bmatrix}$.

Calculate $A\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

$$2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 12 \end{bmatrix}$$

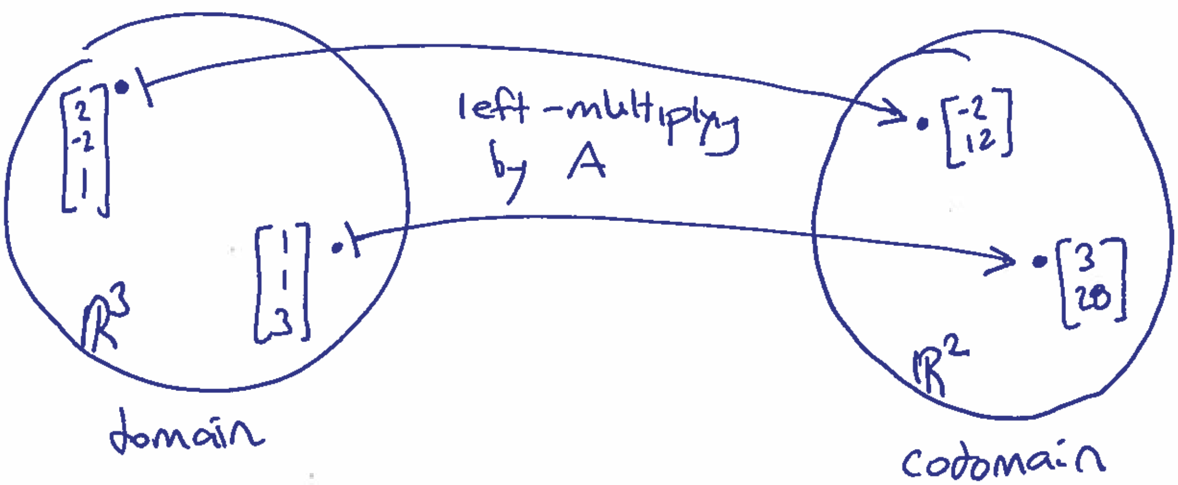
"A transformed $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} -2 \\ 12 \end{bmatrix}$."

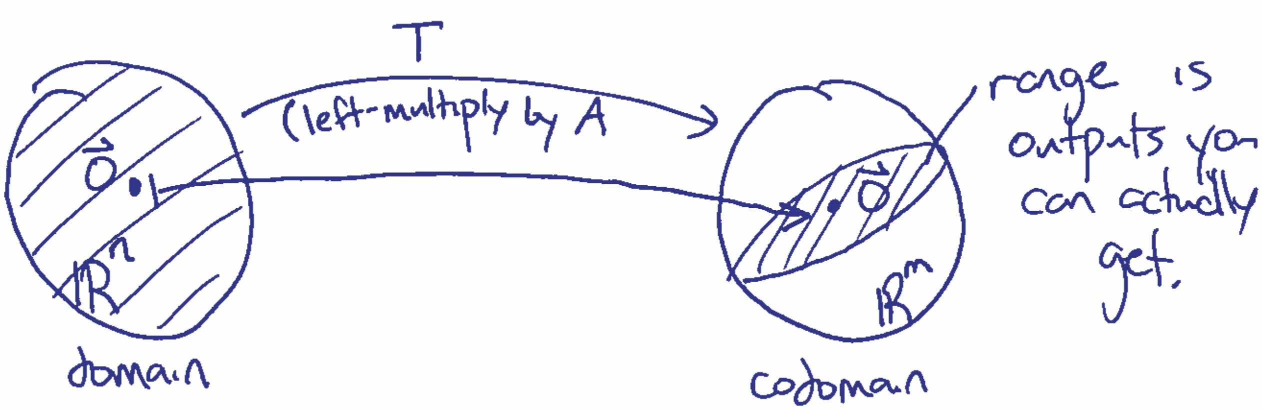
Calculate $A\begin{bmatrix}1\\1\\3\end{bmatrix}$.

$$\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}+3\begin{bmatrix}0\\8\end{bmatrix} = \begin{bmatrix}3\\28\end{bmatrix}$$

"A transformed $\begin{bmatrix}1\\1\\3\end{bmatrix}$ into $\begin{bmatrix}3\\28\end{bmatrix}$."

Left-multiplication by A is turning vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$. We call this process a <u>transformation</u>. (A function where the domain is $\mathbb{R}^n$, and the codomain is $\mathbb{R}^m$.)





Can discuss things like $A \cdot \vec{x}$

matrix

matrix vector multiplication

vector

Or we can discuss it like $T(\vec{x})$

transformation

input notation.

vector $\vec{x}$

In general a transformation is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

<u>Ex</u> $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ \sin(y) \end{bmatrix}$$

A <u>matrix transformation</u> is a specific kind of transformation where you have a matrix A, and $T(\vec{x}) = A \cdot \vec{x}$. Since T is defined by A we often write T_A.

Ex $\quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 1 \end{bmatrix}$. $\quad$ Find $T_A\left(\begin{bmatrix} 3 \\ -4 \end{bmatrix}\right)$.

3×2 matrix

T_A has $\mathbb{R}^2$ for its domain

T_A has $\mathbb{R}^3$ for its codomain

$$= A \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

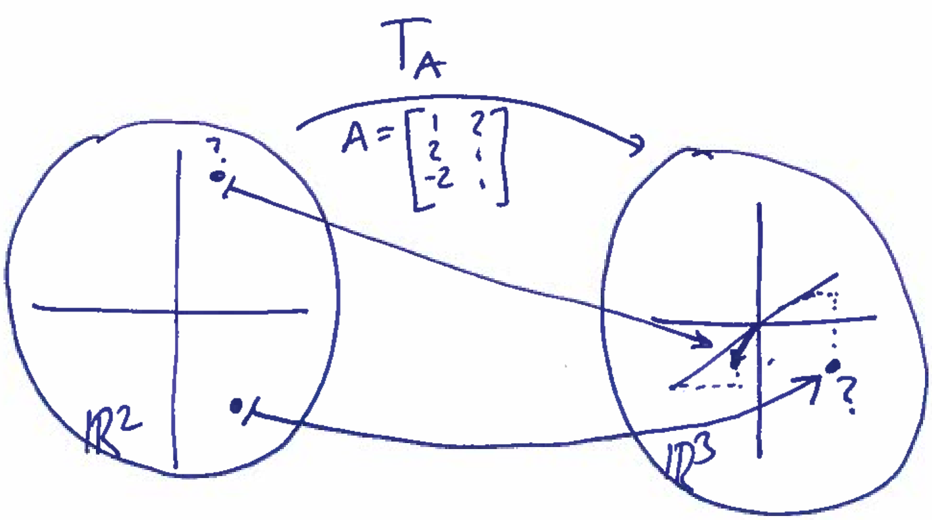
$$= 3\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + -4\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ -10 \end{bmatrix}$$

Ex $\quad$ Find a vector $\vec{x}$ such that $T_A(\vec{x}) = \begin{bmatrix} 19 \\ 14 \\ 2 \end{bmatrix}$.

$$\Longrightarrow \quad A \cdot \vec{x} = \begin{bmatrix} 19 \\ 14 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 19 \\ 2 & 1 & 14 \\ -2 & 1 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 3$
$x_2 = 8$



So $\vec{x}$ must be $\begin{bmatrix} 3 \\ 8 \end{bmatrix}$.

Is there more than one solution to $T_A(\vec{x}) = \vec{b}$ for a particular $\vec{b}$?

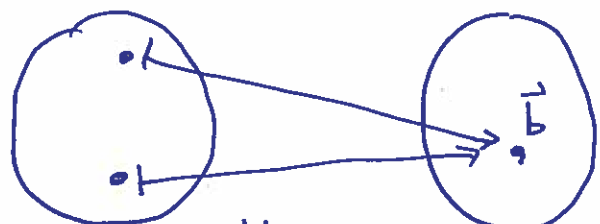
$A \cdot \vec{x} = \vec{b}$

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right] \xleftarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 2 & \uparrow \\ 2 & 1 & \vec{b} \\ -2 & 1 & \downarrow \end{array}\right]$$

May be 0
may be something else...

no free column...

the equation $T_A(\vec{x}) = \vec{b}$ has at most one solution in this example.



this isn't possible in our example.

Here, we found T_A is <u>one-to-one</u>.

Is T_A <u>onto</u>?

Every vector in the codomain is a possible output...

When we set up $T_A(\vec{x}) = \vec{b}$ there <u>will</u> be a solution for all $\vec{b}$.

~~there~~ the matrix A, once row-reduced has a row without a pivot.

For some $\vec{b}$, $A\vec{x} = \vec{b}$ has no solution. So T_A is not onto.

Ex Let $B = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 1 \end{bmatrix}$.

- Is T_B one-to-one?
- Is T_B onto?

We'd want $T_B(\vec{x}) = \vec{b}$ to have at most one solution for each $\vec{b}$.

No matter what $\vec{b}$ is, will $T_B(\vec{x}) = \vec{b}$ always have a solution?

$B \cdot \vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 1 & 2 & | & \sim \\ 0 & 1 & -7 & | & \sim \end{bmatrix} \xleftarrow{\text{start RREF}} \begin{bmatrix} 1 & 1 & 2 & | & \vec{b} \\ 4 & 5 & 1 & | & \end{bmatrix}$$

we have a free column. whenever there is a solution, there will be only many.

So T_B is not one-to-one.

(Shortcut, B's RREF didn't have a pivot in every column. You need a pivot in every column to be one-to-one.)

Since every row has a pivot, there's no pivot in aug column

$\Downarrow$

T_B IS onto.

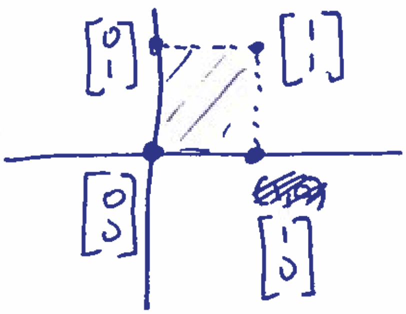
$\underline{\text{Ex}}$ $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ Consider T_A.

$$T_A : \underset{\text{domain}}{\mathbb{R}^2} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^2}$$

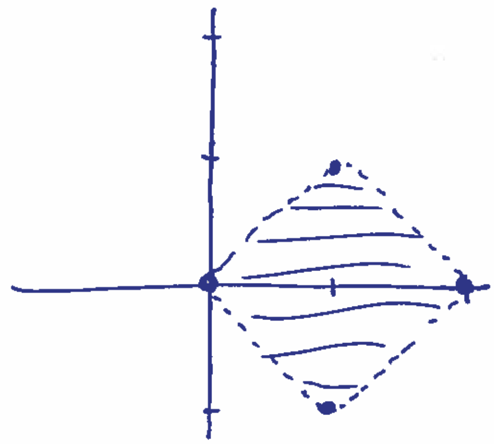
$\Big(A_{RREF} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

pivot in every col $\Rightarrow$ one-to-one

pivot in every row $\Rightarrow$ onto $\Big)$



domain



codomain

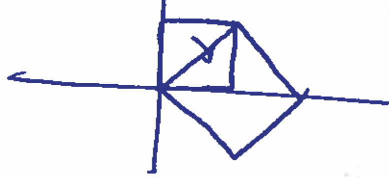
$$T_A\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

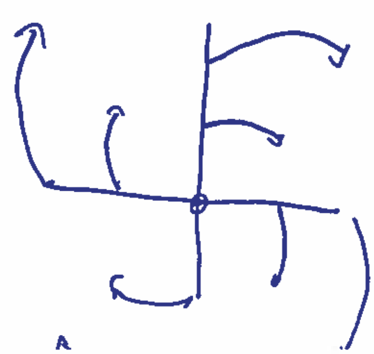
$$T_A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \qquad = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \qquad = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

the action T_A is doing.



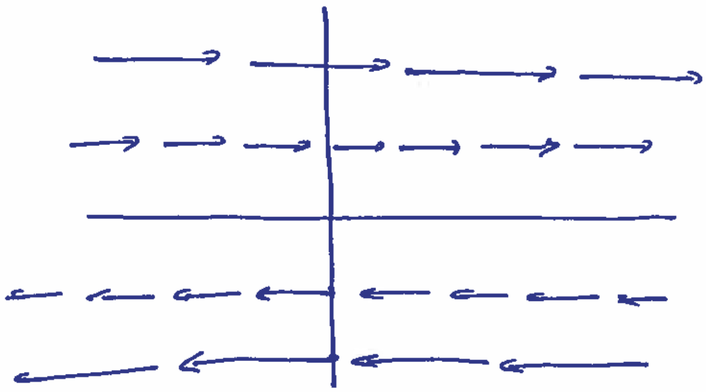


$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Ex $M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Similarly, explore T_M as an action.

We saw with GGB...



is the action that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ does to $\mathbb{R}^2$.

This is a shearing transformation.

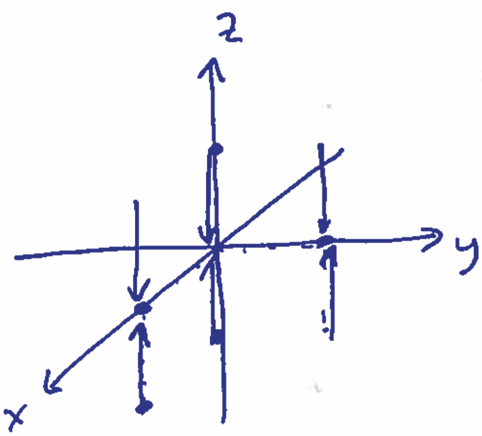
Ex $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Similarly explore T_A as an action

$$T_A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



every vector gets "projected" onto the xy-plane. (turn vector's z-coord to 0.)

A transformation in general just turns vectors in $\mathbb{R}^n$ into vectors in $\mathbb{R}^m$.

A <u>linear transformation</u> is a transformation T, where

for all $\vec{u}, \vec{v}, c$.

- $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
- $T(c\cdot\vec{u}) = c\cdot T(\vec{u})$

<u>Ex</u> $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} e^x \\ 2+y \end{bmatrix}$ Is T linear?

Well, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Does $T(\vec{u}+\vec{v}) \overset{?}{=} T(\vec{u}) + T(\vec{v})$?

$$T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) \overset{?}{=} T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

$$\begin{bmatrix} e^{u_1+v_1} \\ u_2+v_2+2 \end{bmatrix} = \begin{bmatrix} e^{u_1} \\ u_2+2 \end{bmatrix} + \begin{bmatrix} e^{u_2} \\ v_2+2 \end{bmatrix}$$

$e^{u_1+v_1} \overset{?}{=} e^{u_1} + e^{v_1}$ No

$u_2+v_2+2 \overset{?}{=} u_2+v_2+4$ No

So T is not <u>linear</u>.

<u>Fact</u> <u>Matrix transformations</u> <u>are</u> linear!

a transformation defined by $T(\vec{x}) = A\cdot\vec{x}$

Suppose $T(\vec{x}) = A\cdot\vec{x}$. Let $\vec{u}, \vec{v}$ be vectors in T's domain.

- $$T(\vec{u}+\vec{v}) = A\cdot(\vec{u}+\vec{v})$$
  $$= A\cdot\vec{u} + A\cdot\vec{v}$$
  $$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$
  (condition 1 is checked)

- $$T(c\cdot\vec{u}) = A(c\cdot\vec{u})$$
  $$= c\cdot A\cdot\vec{u}$$
  $$T(c\vec{u}) = c\cdot T(\vec{u})$$
  (condition 2 is checked)

<u>Ex</u> Suppose $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y-z \\ x \end{bmatrix}$. Is ~~this~~ T a linear transformation?

$$= \begin{bmatrix} x + 2y & \\ \quad y & -z \\ x & \end{bmatrix}$$

$$= x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Aha, T is a matrix transformation. So T is a linear transformation.

... Explore ... take a linear transformation T,

$$T(\vec{0}) = T(\vec{0}+\vec{0})$$

$$\Rightarrow T(\vec{0}) = T(\vec{0}) + T(\vec{0})$$

$$\Rightarrow \vec{0} = T(\vec{0})$$

Fact: just by virtue of T being linear, $T(\vec{0}) = \vec{0}$.

Ex Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2-y \\ 2+y \end{bmatrix}$ Is T a linear transformation?

well, $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

did not get $\vec{0}$. So T can't possibly be linear.