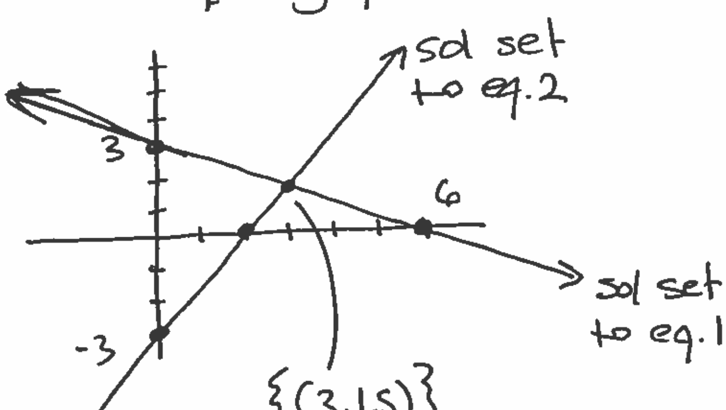
Section 1.5 (Geometric Shapes of Solution Sets to systems of linear equations)

Ex
$$\begin{cases} x + 2y = 6 \\ 3x - 2y = 6 \end{cases}$$

convert to

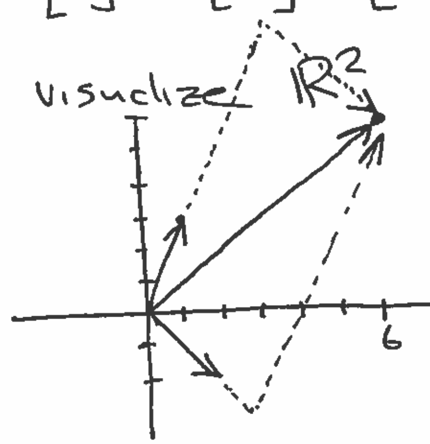
$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}6\\6\end{bmatrix}$$

Cartesian xy-plane



$\{(3,1.5)\}$ is the sol set to the system.

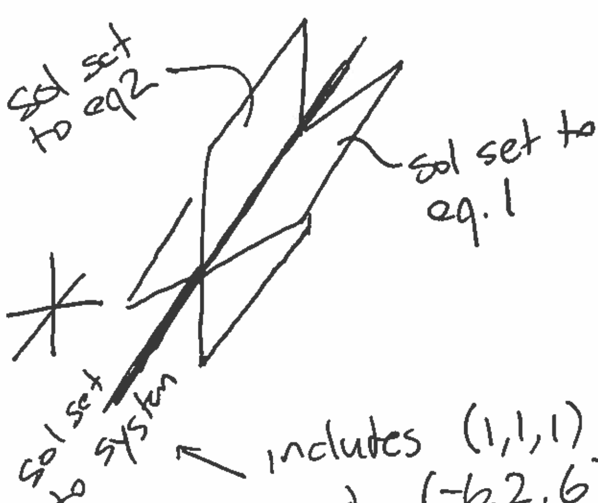
visualize $\mathbb{R}^2$



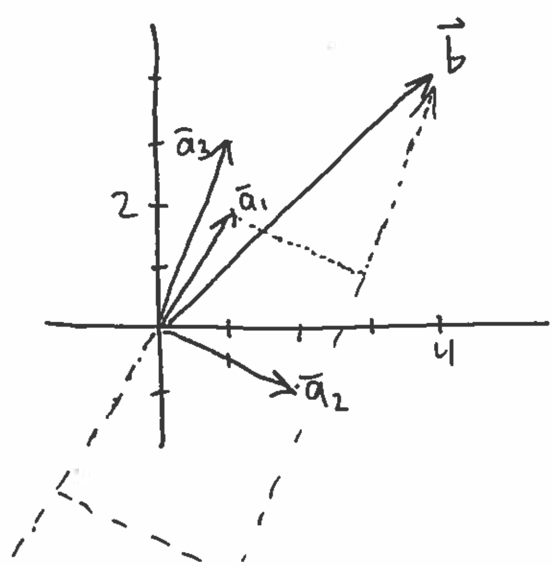
Ex
$$\begin{cases} x + 2y + z = 4 \\ 2x - y + 3z = 4 \end{cases}$$

convert

$$x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}2\\-1\end{bmatrix} + z\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix}$$



includes $(1,1,1)$ and $(-6,2,6)$



A system of linear equations:

$$\begin{aligned} \_\_\_\_\_\_ &= \_\_ \\ \_\_\_\_\_\_ &= \_\_ \\ &\vdots \\ \_\_\_\_\_\_ &= \_\_ \end{aligned} \quad \xrightarrow{\text{equivalent to}} \quad A\vec{x} = \vec{b}$$

So I can speak of "a system of linear equations where $\vec{b} = \vec{0}$."

when a linear system has $\vec{b} = \vec{0}$, we call it a homogeneous system.

Fact A homogeneous system is automatically consistent. Because $\vec{x} = \vec{0}$ will be a solution to $A\vec{x} = \vec{0}$.

A more interesting question: What does it take for $\underset{\text{homog.}}{A\vec{x} = \vec{0}}$ to have nontrivial solutions? (solutions other than $\vec{0}$)

$$[A \mid \vec{0}] \xrightarrow{\text{RREF}} [A_{\text{RREF}} \mid \vec{0}]$$

So we'll have nontrivial solutions exactly when A has a free column.

In other words, $A\vec{x} = \vec{0}$ has a unique solution precisely when A has a pivot in every column.

~~Ex~~

Consider $\begin{bmatrix} 3 & 4 & 1 \\ 1 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

~~Is this~~ Does this have a solution? Yes!

Is it unique? Need to find A's pivot pos.

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

free column $\Longrightarrow$ $\begin{bmatrix} 3 & 4 & 1 \\ 1 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix} \vec{x} = \vec{0}$ has nontrivial sols.

Follow up: can we describe them.

$$\left[\begin{array}{ccc|c} 3 & 4 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_3 = 0$

$x_2 + x_3 = 0$

⟨ We were trying to solve for $\bar{x}$. where $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ⟩

$$\begin{cases} x_1 = x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases}$$

So... $\vec{x} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix}$ is a vector presentation of sol. set.

That is... $\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

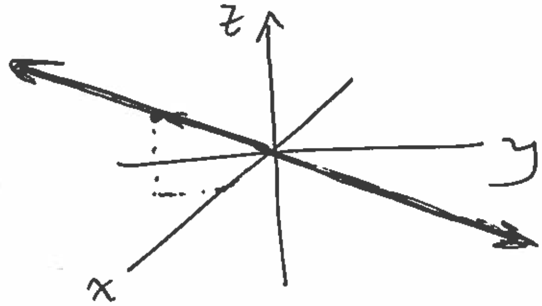
So solutions are all some scalar multiple of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

So... $\left\{ r \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$ is the solution set.

... So the solution set to

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix} \vec{x} = \vec{0} \quad \text{is} \quad \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

... so the solution set looks like

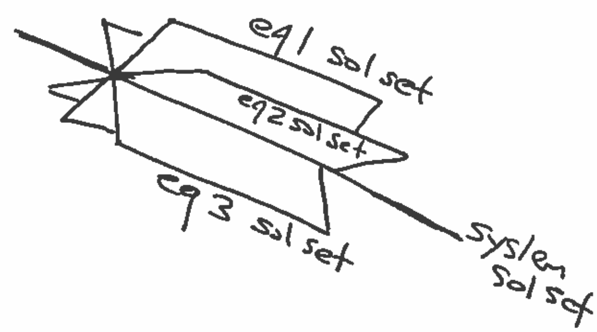


picture of sol. set in $\mathbb{R}^3$

it's a line that passes through $\vec{0}$.

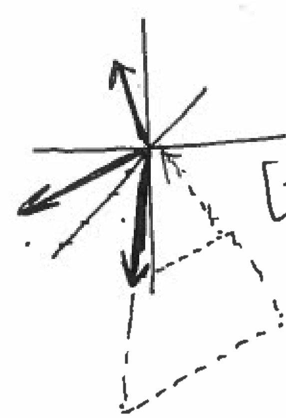
recall the linear system:

$$\begin{cases} 3x_1 + 4x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \\ -2x_1 + x_2 + 3x_3 = 0 \end{cases}$$



From vector perspective ...

$$x_1 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ are sols.

Ex Consider systems with only one equation...

$$\begin{cases} 4x_1 + 5x_2 - 3x_3 = 0 \end{cases}$$

<u>is</u> homogeneous.

($\overset{\Longrightarrow}{\text{is}}$ consistent)

Does it have non-trivial solutions?

$$A = \begin{bmatrix} 4 & 5 & -3 \end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & 1.25 & -0.75 \end{bmatrix}$$

has free columns

$\Longrightarrow$ there <u>are</u> nontrivial solutions.

$$\Longrightarrow \begin{cases} x_1 = -1.25x_2 + 0.75x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

(parametrized solution set)

As a vector presentation, $\vec{x} = \begin{bmatrix} -1.25x_2 + 0.75x_3 \\ x_2 \\ x_3 \end{bmatrix}$

So... $\vec{x} = \begin{bmatrix} -1.25x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.75x_3 \\ 0 \\ x_3 \end{bmatrix}$

So... $\vec{x} = x_2 \begin{bmatrix} -1.25 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.75 \\ 0 \\ 1 \end{bmatrix}$

We realize solutions $\vec{x}$ are the linear combinations of $\begin{bmatrix} -1.25 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.75 \\ 0 \\ 1 \end{bmatrix}$.

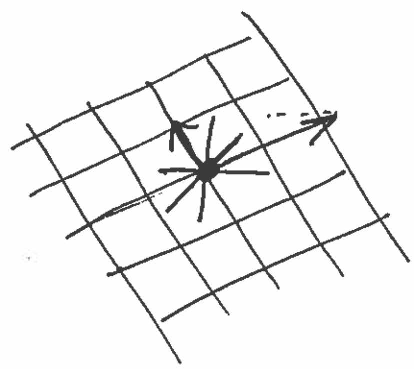
A <u>vector parametrization</u>:

$$\vec{x} = r\begin{bmatrix} -1.25 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0.75 \\ 0 \\ 1 \end{bmatrix}$$

.. So the solution set is $\text{Span}\left\{\begin{bmatrix} -1.25 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.75 \\ 0 \\ 1 \end{bmatrix}\right\}$.

We started with $4x_1 + 5x_2 - 3x_3 = 0$.

Visually, this sol. set is a plane in $\mathbb{R}^3$ passing through the origin.



For example, $(3,0,4)$, or $(1,1,3)$, or ...

visualizing the solutionset to a system of eq. and describing it with a vector parametrization.

Vector Eq: $x_1[4] + x_2[5] + x_3[-3] = [0]$

lots of ways to stretch and add these to get 0.

Ex
$$\begin{cases} 3x + 5y - 4z = 0 \\ -3x - 2y + 4z = 0 \\ 6x + y - 8z = 0 \end{cases}$$

Give a vector parametrization for the sol. set. / Express the sol. set as a span.

Note homog.

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x - \tfrac{4}{3}z = 0$$
$$y = 0$$

parametric solution.
$$\begin{cases} x = \frac{4}{3}z \\ y = 0 \\ z \text{ is free} \end{cases}$$

vector solution
$$\begin{bmatrix} 4/3\, z \\ 0 \\ z \end{bmatrix} = z\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

The solution set is
$$\left\{ r\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} : r \in \mathbb{R} \right\} = \text{Span}\left\{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Ex
$$\begin{cases} x_1 + x_2 - 3x_3 + x_4 = 0 \\ 2x_1 + 3x_2 + x_3 - x_4 = 0 \end{cases}$$

homog ✓

Give a vector param. of the solution set...

$$A = \begin{bmatrix} 1 & 1 & -3 & 1 \\ 2 & 3 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{RREF} \underbrace{\begin{bmatrix} 1 & 0 & -10 & 4 \\ 0 & 1 & 7 & -3 \end{bmatrix}}_{\text{coeff matrix}}$$

$$\begin{cases} x_1 = 10x_3 - 4x_4 \\ x_2 = -7x_3 + 3x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{cases}$$

$$\bar{x} = \begin{bmatrix} 10x_3 - 4x_4 \\ -7x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\bar{x} = x_3 \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

the vector param is $\left\{ r \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix} : r, s \in \mathbb{R} \right\}$

Sol set is $\operatorname{Span}\left\{ \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

the solution set is a plane in $\mathbb{R}^4$

Describe solution sets to these systems...

vector param,
as a span,
geometric...

$$\begin{cases} 2x_1 - 3x_2 - 4x_3 = 0 \\ -2x_1 + 3x_2 + 4x_3 = 0 \\ 5x_2 + 5x_3 = 0 \end{cases} \quad \& \quad \begin{cases} x_1 - 4x_2 - 2x_3 + 3x_5 = 0 \\ x_3 - x_5 = 0 \\ x_4 - 4x_5 = 0 \end{cases}$$

$\Downarrow$

Sol. set is $\left\{ r \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} : r \in \mathbb{R} \right\}$

(maybe you got $\left\{ r \begin{bmatrix} 1/2 \\ -1 \\ 1 \end{bmatrix} : r \in \mathbb{R} \right\}$)

~~Set~~ Sol set is $\text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\}$

(maybe you got $\text{Span}\left\{ \begin{bmatrix} 1/2 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\Downarrow$

RREF of A is

$$\begin{bmatrix} 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

x_5 free

$x_4 = 4x_5$

$x_3 = x_5$

x_2 is free

$x_1 = 4x_2 - x_5$

$$\bar{x} = \begin{bmatrix} 4x_2 - x_5 \\ x_2 \\ x_5 \\ 4x_5 \\ x_5 \end{bmatrix}$$

$$\bar{x} = r \cdot \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 4 \\ 1 \end{bmatrix}$$

Non homogeneous

$$\begin{cases} 3x + 5y - 4z = 7 \\ -3x - 2y + 4z = -1 \\ 6x + y - 8z = -4 \end{cases}$$

Geometrically describe the solution set.

(vector param... but not exactly a span---)

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & 8 & -4 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \text{consistent} \checkmark$$

z is free

$y = 2$

$x = \frac{4}{3}z - 1$

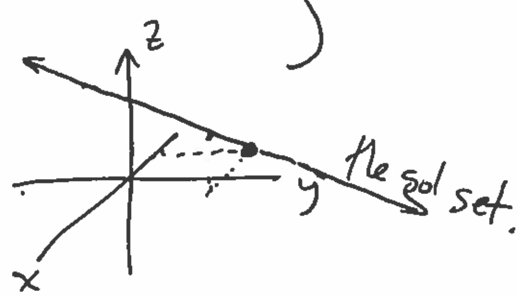
mix of const. terms & param. terms.

$$\bar{x} = \begin{bmatrix} \frac{4}{3}z - 1 \\ 2 \\ z \end{bmatrix}$$

$$\bar{x} = z\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

The sol. set is $\left\{ r \cdot \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} : r \in \mathbb{R} \right\}$

Visualize: Go here then expand by stretches of



1.6 Applications

Ex Integrate $\int \frac{x+10}{x^2(x+1)}\,dx$

all linear factors $x, x, (x+1)$

Fact: $\frac{x+10}{x^2(x+1)} = \frac{A?}{x} + \frac{B?}{x^2} + \frac{C?}{x+1}$

partial fraction decomposition

give a comm. denom.

Lin. Alg to find A, B, C.

$$\frac{x+10}{x^2(x+1)} = \frac{A\cdot x(x+1) + B(x+1) + C\cdot x^2}{x^2(x+1)}$$

$$x + 10 = Ax^2 + Ax + Bx + B + Cx^2$$

Constants:	$10 =$	B
linear terms	$1 =$	$A + B$
quad terms	$0 =$	$A \quad + C$

$$\left\{\begin{array}{lll} & B & = 10 \\ A + & B & = 1 \\ A & \quad + C & = 0 \end{array}\right.$$

RREF, solve, go back & integrate.

$A = -9,\ B = 10,\ C = 9$

$$\int \left(\frac{-9}{x} + \frac{10}{x^2} + \frac{9}{x+1} \right) dx$$

$$= -9\ln|x| - \frac{10}{x} + 9\ln|x+1| + K$$

Ex $\displaystyle\int \frac{1}{x(x^2+10)}\,dx$

quadratic term denom.

$$\frac{1}{x(x^2+10)} = \frac{A}{x} + \frac{Bx+C}{x^2+10}$$

$$\Longrightarrow \frac{1}{x(x^2+10)} = \frac{A(x^2+10)+(Bx+C)x}{x(x^2+10)}$$

$$\Longrightarrow 1 = Ax^2 + 10A + Bx^2 + Cx$$

const. / lin. / quad

$$\begin{cases} 1 = 10A \\ 0 = C \\ 0 = A + B \end{cases} \qquad \left[\begin{array}{ccc|c} 10 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

Solution: $A = \frac{1}{10}$, $B = \frac{-1}{10}$, $C=0$

$$\int \left(\frac{1/10}{x} + \frac{-1/10\, x}{\underbrace{x^2+10}_{u=}} \right) dx$$

$$= \frac{1}{10}\ln|x| - \frac{1}{20}\ln|x^2+10| + K$$

Ex Suppose $f(x) = \overset{?}{a}e^{x} + \overset{?}{b}e^{2x} + \overset{?}{c}e^{3x}$

$$f(0) = 2$$
$$f'(0) = 0$$
$$f(1) = 5$$

You can find a, b, c.

$$f'(x) = ae^{x} + 2be^{2x} + 3ce^{3x}$$

$$
\begin{aligned}
a + b + c &= 2 \\
a + 2b + 3c &= 0 \\
e \cdot a + e^2 \cdot b + e^3 \cdot c &= 5
\end{aligned}
$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 0 \\ e & e^2 & e^3 & 5 \end{array}\right] \xrightarrow{\text{RREF}}$$ to solve for a, b, c.

$$
\begin{cases}
3a = 30 \\
a + 8b = 18 \\
4b - 2c = 2 \\
a + 3b + c = h
\end{cases}
$$

$$\left[\begin{array}{ccc|c} \overset{1}{\cancel{3}} & 0 & 0 & \overset{10}{\cancel{30}} \\ 1 & 8 & 0 & 18 \\ 0 & 4 & -2 & 2 \\ 1 & 3 & 1 & h \end{array}\right]$$

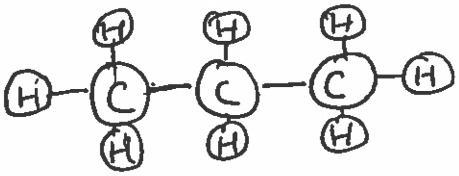
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & \cancel{8}\,1 & 0 & \cancel{8}\,1 \\ 0 & 4 & -2 & 2 \\ 0 & 3 & 1 & -10+h \end{array}\right]$$

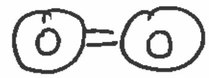
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \cancel{-2}\,1 & \cancel{-2}\,1 \\ 0 & 0 & 1 & -13+h \end{array}\right]
\qquad
\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -14+h \end{array}\right]$$

$h = 14$

Ex Burning Propane

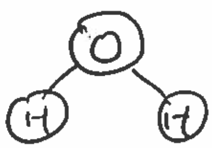
C_3H_8 mixes with O_2





makes $\longrightarrow$

H_2O and CO_2



$$\underset{?}{n_1}\, C_3H_8 + \underset{?}{n_2}\, O_2 = \underset{?}{n_3}\, H_2O + \underset{?}{n_4}\, CO_2$$

LinAlg

A system

Hydrogen: $8n_1 = 2n_3$

Carbon: $3n_1 = n_4$

Oxygen: $2n_2 = n_3 + 2n_4$

$$\begin{cases} 8n_1 - 2n_3 = 0 \\ 3n_1 - n_4 = 0 \\ 2n_2 - n_3 - 2n_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 8 & 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -5/3 & 0 \\ 0 & 0 & 1 & -4/3 & 0 \end{array}\right]$$

n_4 free

$n_3 = \frac{4}{3} n_4$

$n_2 = \frac{5}{3} n_4$

$n_1 = \frac{1}{3} n_4$

choose

$n_4 = 3$

$n_3 = 4$

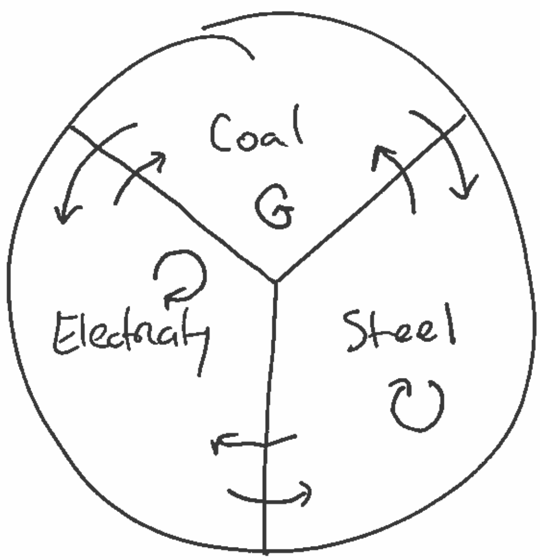
$n_2 = 5$

$n_1 = 1$

Report $C_3H_8 + 5O_2 \longrightarrow 4H_2O + 3CO_2$

An "economy" has "sectors"

Ex

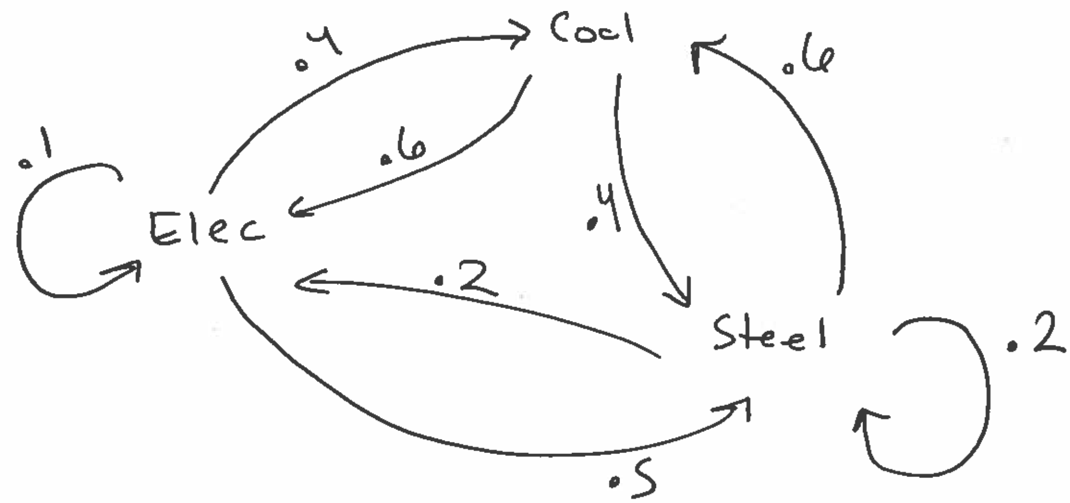


The whole economy is valued at some dollar amount.

On any given day P_C, P_E, P_S are the valuations of the three sectors.

Suppose each week, the electric sector

- passes 40% of value to coal
- passes 50% of value to steel
- holds on to 10% of value.



Over time, no matter what the three values start at ... they stabilize.
Find ~~that~~ those steady state values for P_C, P_E, P_S.

valuation at beginning of week

$$
\begin{array}{l}
C \\ E \\ S
\end{array}
\left\{
\begin{array}{lcl}
p_C & = & 0.4p_E + 0.6p_S \\
p_E & = & 0.6p_C + 0.1p_E + 0.2p_S \\
p_S & = & 0.4p_C + 0.5p_E + 0.2p_S
\end{array}
\right.
$$

hom.

$$\begin{cases} -p_C + 0.4p_E + 0.6p_S = 0 \\ 0.6p_C - 0.9p_E + 0.2p_S = 0 \\ 0.4p_C + 0.5p_E - 0.8p_S = 0 \end{cases}$$

Suppose this economy's total value is 1 (billion \$).

$$p_C + p_E + p_S = 1$$

$$\left[\begin{array}{ccc|c} -1 & 0.4 & 0.6 & 0 \\ 0.6 & -0.9 & 0.2 & 0 \\ 0.4 & 0.5 & -0.8 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & .3369 \\ 0 & 1 & 0 & .3043 \\ 0 & 0 & 1 & .3586 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Coal sector is worth 0.3369 \$B,

elect. 0.3043 \$B

steel 0.3586 \$B