Section 1.3

"The Interview" in 2014 ... in first 4 days of digital release, \$15 M in revenue from sales & rentals. There were 2M transactions. Each rental cost \$6, and to buy: \$15.

Question: how much of this was in rentals, & how much in sales?

Let r be the # rentals...

— s ——— # sales ..

(in millions)

count transactions ⟶ $r + s = 2$

count revenue ⟶ $6r + 15s = 15$

\$/trans. million trans \$M

A linear system $\begin{cases} r + s = 2 \\ 6r + 15s = 15 \end{cases}$

Someone had the idea to view by column instead of row

$$\begin{bmatrix} r \\ 6r \end{bmatrix} + \begin{bmatrix} s \\ 15s \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

a column vector

rule for adding two column vectors: add by corresponding entries:

$$\begin{bmatrix} r + s \\ 6r + 15s \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

$$r \cdot \begin{bmatrix} 1 \\ 6 \end{bmatrix} + s \begin{bmatrix} 1 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

rule for scalar—vector multiplication is to multiply scalar by each entry.

A linear system like

$$\begin{cases} r + s = 2 \\ 6r + 15s = 15 \end{cases} \longleftrightarrow r\begin{bmatrix} 1 \\ 6 \end{bmatrix} + s\begin{bmatrix} 1 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

A vector equation

unknown scalars

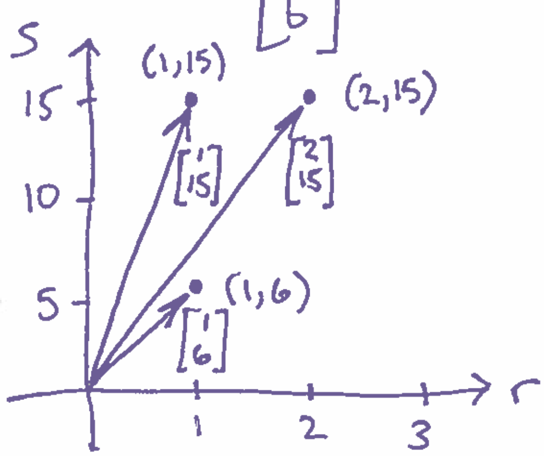
known vectors

Definition: $\mathbb{R}^2$ <u>is</u> the set of all column vectors with two entries.

That is $\mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

"in"

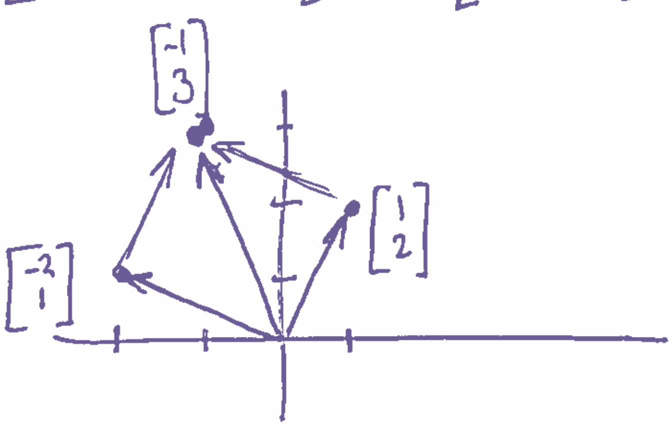
Easy to visualize $\mathbb{R}^2$ by identifying $\begin{bmatrix} a \\ b \end{bmatrix}$ with (a, b).



visualize as points <u>or</u> as arrows emanating from the origin

We can add column vectors...

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+(-2) \\ 2+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



Same ded for $\mathbb{R}^3$, $\mathbb{R}^n$, ...

$$\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

In $\mathbb{R}^n$, ... imagine vectors $\vec{v}_1, \vec{v}_2, \dots \vec{v}_p$

then make:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_p \vec{v}_p$$

scaler

vector

this is a <u>linear combinaton</u> of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ using <u>weights</u> $c_1, c_2, \dots, c_p$.

<u>Ex</u> The set of all lniear conbinactions of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is ... $\mathbb{R}^2$.

<u>Ex</u> The set of all linear combinactions of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is ... a straight lie through the origin within $\mathbb{R}^2$.

Find the linear combination of $\begin{bmatrix}1\\7\\3\end{bmatrix}$ and $\begin{bmatrix}-2\\0\\4\end{bmatrix}$ using weights 3 and -2.

$$\underbrace{3\begin{bmatrix}1\\7\\3\end{bmatrix} + (-2)\begin{bmatrix}-2\\0\\4\end{bmatrix}}_{\text{one particular linear combo of } \begin{bmatrix}1\\7\\3\end{bmatrix}, \begin{bmatrix}-2\\0\\4\end{bmatrix}.} = \begin{bmatrix}7\\21\\1\end{bmatrix}$$

Can we visualize all linear combos of $\begin{bmatrix}1\\7\\3\end{bmatrix}, \begin{bmatrix}-2\\0\\4\end{bmatrix}$ as a collection.

We did with GGB...

we saw a plane intersecting the origin.

Remember

$$\begin{cases} r + s = 2 \\ 6r + 15s = 15 \end{cases} \xrightarrow[\text{to}]{\text{converted}} r\begin{bmatrix}1\\6\end{bmatrix} + s\begin{bmatrix}1\\15\end{bmatrix} = \begin{bmatrix}2\\15\end{bmatrix}$$

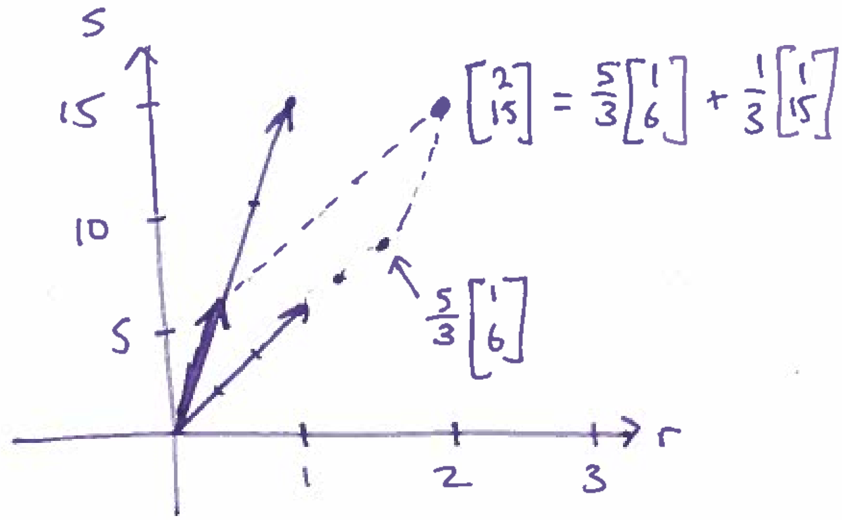
Find values for r, s that make two true equations

Find values for r and s to use as weights on $\begin{bmatrix}1\\6\end{bmatrix}$ and $\begin{bmatrix}1\\15\end{bmatrix}$ to land at $\begin{bmatrix}2\\15\end{bmatrix}$.

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 6 & 15 & 15 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{cc} r & s \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 5/3 \\ 0 & 1 & 1/3 \end{array}\right]$$

$$r = 5/3 \approx 1.666$$
$$s = 1/3 \approx 0.333$$

in context,
1.666 million rentals
0.333 million sales



Notation: The <u>span</u> of $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p\}$ is the collection of all linear combinations of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p$. $\text{Span}\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p\}$.

$$\text{Span}\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p\} = \left\{ c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_p\bar{v}_p \;\middle|\; c_1, c_2, \ldots, c_p \in \mathbb{R} \right\}$$

<u>Ex</u> $\text{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$, the origin in $\mathbb{R}^3$.

<u>Ex</u> $\text{Span}\left\{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right\}$ is a line in $\mathbb{R}^3$ passing through the origin.

Ex $\text{Span}\left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\} = \mathbb{R}^2$

Ex $\text{Span}\left\{\begin{bmatrix}1\\7\\3\end{bmatrix}, \begin{bmatrix}-2\\0\\4\end{bmatrix}\right\}$ is a plane in $\mathbb{R}^3$ passing through origin.

Ex $\text{Span}\left\{\begin{bmatrix}1\\7\\3\end{bmatrix}, \begin{bmatrix}-2\\-14\\-6\end{bmatrix}\right\}$ is a line in $\mathbb{R}^3$
(one vector was a multiple of the other)

Ex $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is $\mathbb{R}^3$.

Ex $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is the xy-plane in $\mathbb{R}^3$.

System

$$\begin{cases} r + s = 2 \\ 6r + 15s = 15 \end{cases}$$

Vector Equation

$$r\begin{bmatrix}1\\6\end{bmatrix} + s\begin{bmatrix}1\\15\end{bmatrix} = \begin{bmatrix}2\\15\end{bmatrix}$$

Is $\begin{bmatrix}2\\15\end{bmatrix}$ in the Span of $\begin{bmatrix}1\\6\end{bmatrix}, \begin{bmatrix}1\\15\end{bmatrix}$?
And if so, with what weights?

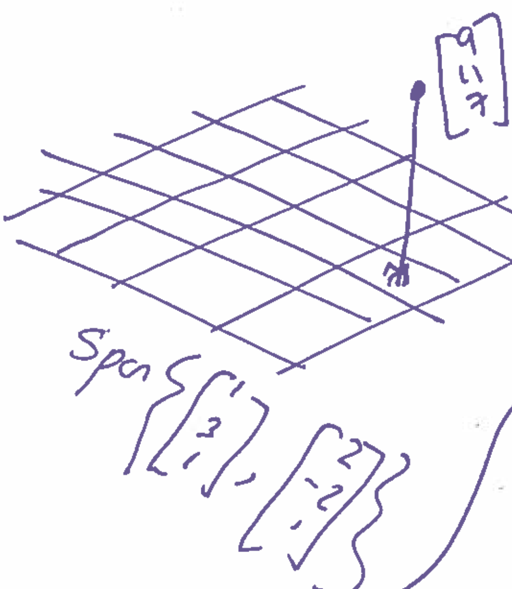
3 takes on same question.

Ex
$$\begin{cases} x+2y=9 \\ 3x-2y=11 \\ x+y=7 \end{cases} \quad \underset{\text{as}}{\overset{\text{same}}{\longleftrightarrow}} \quad x\begin{bmatrix}1\\3\\1\end{bmatrix} + y\begin{bmatrix}2\\-2\\1\end{bmatrix} = \begin{bmatrix}9\\11\\7\end{bmatrix}$$

Is $\begin{bmatrix}9\\11\\7\end{bmatrix}$ in $\text{Span}\left\{\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix}\right\}$?

To answer:
$$\left[\begin{array}{cc|c} 1 & 2 & 9 \\ 3 & -2 & 11 \\ 1 & 1 & 7 \end{array}\right] \xrightarrow[\text{steps}]{\text{some}} \left[\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{array}\right]$$

- inconsistent system
- the vector eq has <u>no</u> solution.
- $\begin{bmatrix}9\\11\\7\end{bmatrix}$ is <u>not</u> in $\text{Span}\left\{\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix}\right\}$



Ex In $\mathbb{R}^4$, is $\begin{bmatrix}13\\8\\2\\12\end{bmatrix}$ in
$$\text{Span}\left\{\begin{bmatrix}2\\1\\-2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\4\\-1\end{bmatrix}\right\} ?$$

(If so, with what weights...)

$$\left[\begin{array}{cc|c} 2 & 1 & 13 \\ 1 & 1 & 8 \\ -2 & 4 & 2 \\ 3 & -1 & 12 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 2 & 1 & 13 \\ -2 & 4 & 2 \\ 3 & -1 & 12 \end{array}\right]$$

$$\xrightarrow[\substack{2R_1+R_3 \to R_3 \\ -3R_1+R_4 \to R_4}]{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & -1 & -3 \\ 0 & 6 & 18 \\ 0 & -4 & -12 \end{array}\right] \xrightarrow[-4R_2+R_4]{6R_2+R_3} \left[\begin{array}{cc|c} \boxed{1} & 1 & 8 \\ 0 & \boxed{-1} & -3 \\ 0 & 0 & 0 \checkmark \\ 0 & 0 & 0 \checkmark \end{array}\right]$$

there <u>is</u> a solution ... $\begin{bmatrix} 13 \\ 8 \\ 2 \\ 12 \end{bmatrix}$ <u>is</u> in Span$\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right\}$.

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

first weight is 5

2nd weight is 3

$$5\begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 13 \\ 8 \\ 2 \\ 12 \end{bmatrix} \quad \checkmark$$

1.4

The Matrix Equation $A\vec{x}=\vec{b}$

Our convention: a matrix

$$2 \text{ rows} \downarrow \begin{bmatrix} 1 & 3 & 8 \\ 2 & 0 & 1 \end{bmatrix}$$

$\xrightarrow{\text{3 cols}}$

this is ~~2x2~~. 2×3.

Consider: $\begin{cases} 3x + 2y - z = 8 \\ x + y + 2z = 5 \end{cases}$ (system of lin. eq.)

$$x\begin{bmatrix}3\\1\end{bmatrix} + y\begin{bmatrix}2\\1\end{bmatrix} + z\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}8\\5\end{bmatrix}$$ (vector equation)

Someone got lazy & clever

put weights here

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix}8\\5\end{bmatrix}$$ (a matrix equation of form $A\vec{x}=\vec{b}$)

Define Matrix-Vector multiplication this way:

$$\underset{\text{(m×n) matrix}}{\begin{bmatrix} | & | & & | \\ \bar{a}_1 & \bar{a}_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}} \underset{\text{(n×1) matrix}}{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}} = \underbrace{c_1\bar{a}_1 + c_2\bar{a}_2 + \cdots + c_n\bar{a}_n}_{\text{m×1 matrix}}$$

must match

Ex

$$\underbrace{\begin{bmatrix} 3 & 2 & 1 \\ 8 & 0 & -2 \end{bmatrix}}_{2\times 3}\underbrace{\begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}}_{3\times 1} = 5\begin{bmatrix} 3 \\ 8 \end{bmatrix} + -2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ 40 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ 32 \end{bmatrix}$$

2×1 result

trying to make a lin. combo of the matrix's columns using the vector's entries as weights.

Ex

$$\begin{bmatrix} 7 & 1 \\ 1 & 7 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (1)\begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -6 \\ 1 \end{bmatrix}$$

Ex

$$\underbrace{\begin{bmatrix} 3 & 1 & 8 \\ 4 & 2 & 9 \end{bmatrix}}_{(2\times 3)}\underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{(2\times 1)}$$ is undefined.

don't match!

System, solve for variable

$$\begin{cases} r + s = 2 \\ 6r + 15s = 15 \end{cases}$$

Vector equation, solve for variables

$$r\begin{bmatrix} 1 \\ 6 \end{bmatrix} + s\begin{bmatrix} 1 \\ 15 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

Is $\begin{bmatrix} 2 \\ 15 \end{bmatrix}$ in the span of $\begin{bmatrix} 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 15 \end{bmatrix}$?

If so, with what weights?

Solve for vector $\vec{x}$ in the matrix eq.

$$\begin{bmatrix} 1 & 1 \\ 6 & 15 \end{bmatrix}\vec{x} = \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} r \\ s \end{bmatrix}$$

$\underline{\text{Ex}}$ Does $\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ have a solution for $\vec{x}$?

to ~~do~~ find out... $\left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ -4 & 2 & -6 & 2 \\ -3 & -2 & -7 & 3 \end{array}\right] \longrightarrow$

$\xrightarrow[\text{work}]{\text{some}} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & 14 & 10 & 6 \\ 0 & 0 & 0 & 3 \end{array}\right] \Longrightarrow$ No solution.

There is no $\vec{x}$ that makes $\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ true.

You try!

Does $\begin{bmatrix} 4 & 2 \\ 1 & 7 \\ -2 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}$ have a solution?

$\left[\begin{array}{cc|c} 4 & 2 & 6 \\ 1 & 7 & 5 \\ -2 & 1 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 12 \end{array}\right] \Longrightarrow$ No solution!

Ex Do $\begin{bmatrix}5\\1\\-4\end{bmatrix}, \begin{bmatrix}-4\\2\\3\end{bmatrix}, \begin{bmatrix}6\\4\\-5\end{bmatrix}$ span $\mathbb{R}^3$?

$$\left(\text{Is } \operatorname{Span}\left\{\begin{bmatrix}5\\1\\-4\end{bmatrix}, \begin{bmatrix}-4\\2\\3\end{bmatrix}, \begin{bmatrix}6\\4\\-5\end{bmatrix}\right\} = \mathbb{R}^3 ? \right)$$

If yes... then no matter what $\vec{b}$ is...

$\begin{bmatrix}5 & -4 & 6\\1 & 2 & 4\\-4 & 3 & 5\end{bmatrix}\vec{x} = \vec{b}$ should have a solution.

$$\sim \left[\begin{array}{ccc|c}5 & -4 & 6 & b_1\\1 & 2 & 4 & b_2\\-4 & 3 & 5 & b_3\end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c}1 & 2 & 4 & b_2\\5 & -4 & 6 & b_1\\-4 & 3 & 5 & b_3\end{array}\right]$$

$$\xrightarrow[4R_1+R_3]{-5R_1+R_2} \left[\begin{array}{ccc|c}1 & 2 & 4 & b_2\\0 & -14 & -14 & -5b_2+b_1\\0 & 11 & 21 & 4b_2+b_3\end{array}\right] \xrightarrow[\text{steps}]{\text{skipping}} \left[\begin{array}{ccc|c}1 & 2 & 4 & b_2\\0 & 1 & * & ☆\\0 & 0 & 1 & ☆\end{array}\right]$$

there is a solution.

Yes the span is $\mathbb{R}^3$.

Theorem 4 from 1.4

Let A be an $m \times n$ matrix.
The following 4 things are equivalent
(either all true, or all false, depending on A)

a) $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ in $\mathbb{R}^m$.

b) Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combo of A's columns.

c) The columns of A span $\mathbb{R}^m$.

d) The matrix A has a pivot position in every row.

How do we use this?

Do $\begin{bmatrix}1\\8\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-3\end{bmatrix}, \begin{bmatrix}2\\4\\1\end{bmatrix}$ span $\mathbb{R}^3$? (Consider $A = \begin{bmatrix}1 & 2 & 2\\8 & 1 & 4\\1 & -3 & 1\end{bmatrix}$)

Theorem 4 pointing to item c....
Look into d)... $\begin{bmatrix}1 & 2 & 2\\8 & 1 & 4\\1 & -3 & 1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

there <u>is</u> a pivot in every row... they <u>do</u> span $\mathbb{R}^3$.

<u>Ex</u> Do $\begin{bmatrix}1\\3\\4\end{bmatrix}, \begin{bmatrix}-3\\1\\4\end{bmatrix}, \begin{bmatrix}-5\\5\\12\end{bmatrix}, \begin{bmatrix}3\\4\\4\end{bmatrix}$

span $\mathbb{R}^3$? Let $A = \begin{bmatrix}1 & -3 & -5 & 3\\ 3 & 1 & 5 & 4\\ 4 & 4 & 12 & 4\end{bmatrix}$.

use Thm 4. (Does it have a pivot in every row?)

$$\xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 1 & 3/2\\ 0 & \boxed{1} & 2 & -1/2\\ 0 & 0 & 0 & 0\end{bmatrix} \leftarrow \text{no pivot.}$$

by Thm 4,
these 4 vectors do <u>not</u> span $\mathbb{R}^3$.

<u>Ex</u> Do $\begin{bmatrix}1\\8\\7\\1\\2\end{bmatrix}, \begin{bmatrix}3\\1\\0\\1\\4\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\\2\end{bmatrix}$ span $\mathbb{R}^5$?

$A = \begin{bmatrix}1 & 3 & -1\\ 8 & 1 & 1\\ 7 & 0 & 0\\ 1 & 1 & 1\\ 2 & 4 & 2\end{bmatrix}$ will have <u>at most</u> 3 pivots (one per column). There <u>will</u> be rows without pivots.
By thm 4, these vectors don't span $\mathbb{R}^5$.