Solve for x & y : $\begin{cases} 2x + y = 3 \\ 3x - 2y = 8 \end{cases}$

isolate ... y

$y = -2x + 3$

sub into other...

$3x - 2(-2x+3) = 8$

only <u>one</u> variable!

$3x + 4x - 6 = 8$

$7x = 14$

$x = 2$

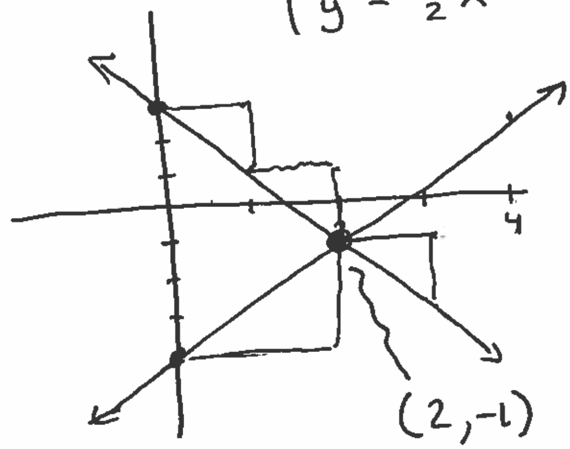
$y = -2(2) + 3$

$y = -1$

Solution: $x = 2$
$y = -1$

graphically

$\begin{cases} y = -2x + 3 \\ y = \frac{3}{2}x - 4 \end{cases}$



elimination

$\begin{cases} 2x + \boxed{y} = 3 \\ 3x \boxed{-2y} = 8 \end{cases}$

x2

$\begin{cases} 4x + 2y = 6 \\ 3x - 2y = 8 \end{cases}$

Add

$7x = 14$

$x = 2$

LCM of 2,3 is 6

$\begin{cases} \boxed{2x} + y = 3 \\ \boxed{3x} - 2y = 8 \end{cases}$

x3

x2

$\begin{cases} 6x + 3y = 9 \\ 6x - 4y = 16 \end{cases}$

Subtract

$7y = -7$

$y = -1$

Solve for x, y, z

$$\begin{cases} \boxed{3x} + y + 4z = 3 \\ -5x - 2y - 10z = 1 \\ 2x - 2y + 3z = -4 \end{cases}$$

LCM is 30

This is laborious, tedious, prone to human error... if you use substitution

$$\begin{array}{l} \times 10 \\ \times 6 \\ \times 15 \end{array} \begin{cases} 30x + 10y + 40z = 30 \\ -30x - 12y - 60z = 6 \\ 30x - 30y + 45z = -60 \end{cases} \xrightarrow{\text{leave alone, add, sub.}} \begin{cases} 30x + 10y + 40z = 30 \\ -2y - 20z = 36 \\ 40y - 5z = 90 \end{cases}$$

$$\begin{array}{l} \\ \div 2 \\ \div 5 \end{array} \begin{cases} 30x + 10y + 40z = 30 \\ -y - 10z = 18 \\ 8y - z = 18 \end{cases} \xrightarrow{\times 8} \begin{cases} 30x + 10y + 40z = 30 \\ -8y - 80z = 144 \\ 8y - z = 18 \end{cases} \text{ add}$$

LCM: 8

$$\begin{cases} 30x + 10y + 40z = 30 \\ -8y - 80z = 144 \\ -81z = 162 \end{cases} \qquad \begin{cases} 30x + 10y + 40z = 30 \\ -8y - 80z = 144 \\ z = -2 \end{cases}$$

$80R_3 + R_2 \longrightarrow R_2$

$-40R_3 + R_1 \longrightarrow R_1$

$$\begin{cases} 30x + 10y = 110 \\ -8y = -16 \\ z = -2 \end{cases} \qquad \begin{cases} 30x + 10y = 110 \\ y = 2 \\ z = -2 \end{cases}$$

$-10R_2 + R_1 \longrightarrow R_1$

$$\begin{cases} 30x = 90 \\ y = 2 \\ z = -2 \end{cases}$$

$x = 3$
$y = 2$
$z = -2$

Check!

Solve
$$\begin{cases} \cancel{3x + y + 4z} \\ 2x - 7y + 2z = 7 \\ 2x + 2y - 7z = -2 \\ 3x - 5y - 4z = 2 \end{cases}$$

still tedious, prone to human errors...

takes up space..

Look for anything to streamline this...

$$\begin{array}{ccc|c} x & y & z & \\ \downarrow & \downarrow & \downarrow & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & -7 & 2 & 7 \\ 2 & 2 & -7 & -2 \\ 3 & -5 & -4 & 2 \end{array}\right] \xrightarrow{-R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 2 & -7 & 2 & 7 \\ 0 & 9 & -9 & -9 \\ 3 & -5 & -4 & 2 \end{array}\right]$$

~~An~~ Augmented Matrix for the linear system.

Coefficient Matrix for the linear system.

still stands for
$$9y - 9z = -9$$

Smaller Example

Solve

$$\begin{cases} 2x + y = 3 \\ 3x - 2y = 8 \end{cases}$$

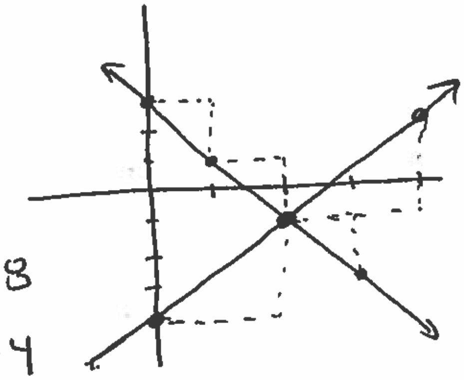
by using an augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ \boxed{3} & -2 & 8 \end{array}\right]$$

target.
make it 0

$2x + y = 3$
$y = -2x + 3$

$3x - 2y = 8$
$y = \frac{3}{2}x - 4$



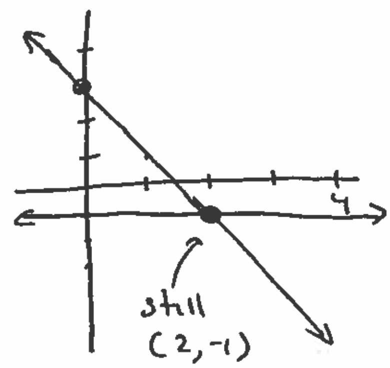
$3R_1 \to R_1$
$2R_2 \to R_2$
$$\left[\begin{array}{cc|c} 6 & 3 & 9 \\ 6 & -4 & 16 \end{array}\right]$$

$6x + 3y = 9$
$6x - 4y = 16$

same plot

$-R_1 + R_2 \to R_2$
$$\left[\begin{array}{cc|c} 6 & 3 & 9 \\ 0 & -7 & 7 \end{array}\right]$$

$6x + 3y = 9$
$-7y = 7$
$(y = -1)$

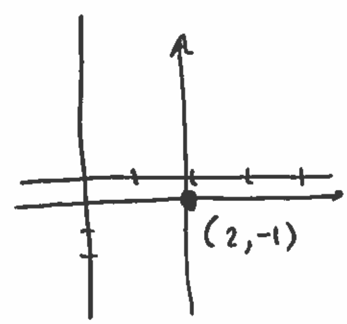


target

$-\frac{1}{7}R_2 \to R_2$
$$\left[\begin{array}{cc|c} 6 & \boxed{3} & 9 \\ 0 & 1 & -1 \end{array}\right]$$

$-3R_2 + R_1 \to R_1$
$$\left[\begin{array}{cc|c} 6 & 0 & 12 \\ 0 & 1 & -1 \end{array}\right]$$

$6x = 12$
$(x = 2)$
$y = -1$



$\frac{1}{6}R_1 \to R_1$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$x = 2$
$y = -1$

Solve
$$\left\{\begin{array}{rrrl} 2x & -7y^{(4)} & +2z^{(3)} & = 7 \\ 2x & +2y & -7z & = -2 \\ 3x_{(1)} & -5y_{(2)} & -4z & = 2 \end{array}\right.$$

using an augmented matrix.

More of the same?

$$\begin{cases} x + 2y + z = 3 \\ 2x - y + z = 4 \\ -4x + 7y - z = 2 \end{cases}$$

Solve...

coefficient +1

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 2 & -1 & 1 & 4 \\ -4 & 7 & -1 & 2 \end{array}\right] \xrightarrow[4R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & \boxed{-5} & -1 & -2 \\ 0 & 15 & 3 & 14 \end{array}\right]$$

$$3R_2+R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & -1 & -2 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

$0x + 0y + 0z = 8$

There is no solution!

The system is <u>inconsistent</u>.

$$\begin{cases} x + y + z = 8 \\ 2x + y = 4 \\ y + 2z = 12 \end{cases}$$

Solve...

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 8 \\ 2 & 1 & 0 & 4 \\ 0 & 1 & 2 & 12 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & \boxed{-1} & -2 & -12 \\ 0 & 1 & 2 & 12 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 8 \\ 0 & \boxed{-1} & -2 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -4 \\ 0 & -1 & -2 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2 \to R_2}$$

$0x+0y+0z = 0$

free column

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & -4 \\ 0 & \boxed{1} & 2 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot columns

$$\begin{aligned} x \quad - z &= -4 \\ y + 2z &= 12 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x &= z - 4 \\ y &= -2z + 12 \end{aligned}$$

free to let z take <u>any</u> value... then x, y follow

there are ∞-ly many solutions...

$$\begin{cases} x = z - 4 \\ y = -2z + 12 \end{cases}$$

1.1 Summary

To solve a system of lin. eq.

1) Arrange in form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

2) Make augmented matrix

3) Use row reductions...

- Scaling (Like $3R_2 \to R_2$)
- Interchange (Like $R_1 \leftrightarrow R_2$)
- Replacement (Like $7R_2 + \underbrace{R_3}_{\text{match}} \to R_3$)

A system may have one, no, or $\underbrace{\infty\text{-ly many sols}}$.

may need to parametrize solution set

Like $\begin{cases} x = z - 4 \\ y = -2z + 12 \\ z \text{ is free} \end{cases}$

Section 1.2

We're seen a matrix like this:

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

Some matrices are in <u>(row) echelon form.</u>

More examples

$$\begin{bmatrix} \boxed{2} & 1 & 3 \\ 0 & \boxed{1} & 3 \end{bmatrix} \quad \begin{bmatrix} \boxed{8} & 1 \\ 0 & \boxed{3} \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & 3 & 5 & 9 \\ 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Read row by row...
Identify first nonzero entry
Moving down a row needs to move the leading entry to right.

<u>Not</u> like $\begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 3 \\ \boxed{3} & 1 & 5 \end{bmatrix}$

In ~~some~~ row echelon form these leading nonzero entries: <u>pivot positions</u>

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix} \quad \begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\blacksquare$ nonzero for sure

$*$ any number possible

A matrix is in <u>reduced (row) echelon form</u>

if
- it's in row echelon form
- the pivot positions have 1 in them.
- positions above each pivot position are 0.

<u>Ex</u>

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

This matrix <u>maybe</u> came from a system of equations as the augmented matrix.

$$\Longrightarrow \quad \begin{aligned} x &= 2 \\ y &= 3 \\ z &= 4 \end{aligned}$$

Ex

free column

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & \boxed{1} & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>is</u> in RREF

$$\begin{cases} x = -2y + 3 \\ y \text{ is free} \\ z = 8 \end{cases}$$

<u>Ex</u>

$$\begin{bmatrix} \boxed{1} & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

<u>is</u> in RREF

tells us there is no solution.

Ex $\begin{bmatrix} 2 & 6 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ -4 & -12 & 0 & 2 \end{bmatrix}$

Find the RREF of this matrix

(Apply valid row operations to obtain an RREF matrix)

$$\begin{bmatrix} 2 & 6 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ -4 & -12 & 0 & 2 \end{bmatrix} \xrightarrow[-2R_2 \to R_2]{2R_1+R_3 \to R_3} \begin{bmatrix} 2 & 6 & 1 & 0 \\ -2 & -6 & -2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 2 & 6 & 1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 2 & 6 & 1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[-R_3 \to R_3]{-R_2 \to R_2} \begin{bmatrix} 2 & 6 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_2 \to R_2} \begin{bmatrix} 2 & 6 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-R_2+R_1 \to R_1 \begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 1 & 0 \\ 1 & 3 & 1 & 1 \\ -4 & -12 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 1 & 0 \\ -4 & -12 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow[4R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 4 & 6 \end{bmatrix}$$

$$\xrightarrow{4R_2+R_3 \to R_3} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

both in row echelon form

$$\xrightarrow[-R_2 \to R_2]{-\frac{1}{2}R_3 \to R_3} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[-2R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$-R_2+R_1 \to R_1$

RREF for a matrix is **unique** no matter what path you take!

$$\begin{bmatrix} \boxed{2} & 6 & 1 & 0 \\ 1 & 3 & \boxed{1} & 1 \\ -4 & -12 & 0 & \boxed{2} \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

these are pivot positions.

– because RREF is unique... these may be called its pivot positions.

Ex
$$\begin{cases} \quad\quad\; -3x_2 - 6x_3 + 4x_4 = 9 \\ -x_1 - 2x_2 - x_3 + 3x_4 = 1 \\ -2x_1 - 3x_2 \quad\quad\;\; + 3x_4 = -1 \\ x_1 + 4x_2 + 5x_3 - 9x_4 = -7 \end{cases}$$

Are there ∞, 0, or 1 solutions to this system...?

$$\left[\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right] \xrightarrow[\text{work}]{\text{lots of}} \left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Wasn't asked to solve... I've got REF ↗ ...
I don't need RREF...

It's consistent ✓
(no $\left[\begin{array}{ccc|c} 0 & 0 \cdots & 0 & 1 \end{array}\right]$)

There's a free column
$\Longrightarrow$ ∞-ly many sols.

Summary of 1.2

Theorem

You have a linear system...

You build its augmented matrix...

You reduce to REF...

Then the system is consistent (at least one sol.) if there are no rows

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & \text{pivot} \end{array}\right]$$

And then there are ∞-ly many sols if there is a free column.

(There is only one solution if there are no free columns)

A column with no pivot pos. in the coefficient part of the aug. matrix.