Section 5.3 Diagonalization

<u>Ex</u> $A = \begin{bmatrix} 7 & -15 \\ 2 & -4 \end{bmatrix}$.

Find A^{10}.

Directly..

$\underbrace{A \cdot A} \cdot A \cdots A$

$= \begin{bmatrix} 7 & -15 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 7 & -15 \\ 2 & -4 \end{bmatrix} \cdot A \cdots A$

$= \begin{bmatrix} 7\cdot7 + (-15)\cdot2 & 7(-15)+(-15)(-4) \\ 2\cdot7 + (-4)(2) & 2(-15)+(-4)(-4) \end{bmatrix} \cdot A \cdots A$

8 multiplications...
4 additions..

To get to A^{10} would require a lot!

Today we learn that:

inverses

$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$A = P \cdot D \cdot P^{-1}$

With this established

$A^{10} = (PDP^{-1})^{10}$

$= (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$

$A^{10} = P \cdot D^{10} \cdot P^{-1}$

$= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{10}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

with a diagonal matrix...

$= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2^{10} & 0 \\ 0 & 1^{10} \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1024 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$= \cdots$

$= \begin{bmatrix} 6139 & -9191 \\ 2046 & -3076 \end{bmatrix}$

two exponentiations
16 multiplications
8 additions.

Starting over... $A = \begin{bmatrix} 7 & -15 \\ 2 & -4 \end{bmatrix}$.

Let's find A's eigenvalues.

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 7-\lambda & -15 \\ 2 & -4-\lambda \end{bmatrix}\right) = 0$

$(7-\lambda)(-4-\lambda) + 30 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda-2)(\lambda-1) = 0$

$\lambda = 2$ or $\lambda = 1$

$\lambda = 2$ or $\mu = 1$

Now eignvectors...

With $\lambda = 2$...

Want solutions to

$A\vec{x} = 2\vec{x}$

$(A - 2I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} 5 & -15 & 0 \\ 2 & -6 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 free... take $x_2 = 1$

$x_1 = 3x_2$ would be 3. $\Longrightarrow$ $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an e'vector for $\lambda = 2$.

With ~~λ~~ $\mu = 1$.

$A\vec{x} = \vec{x}$

~~Ax~~ $(A - I)\vec{x} = \vec{0}$

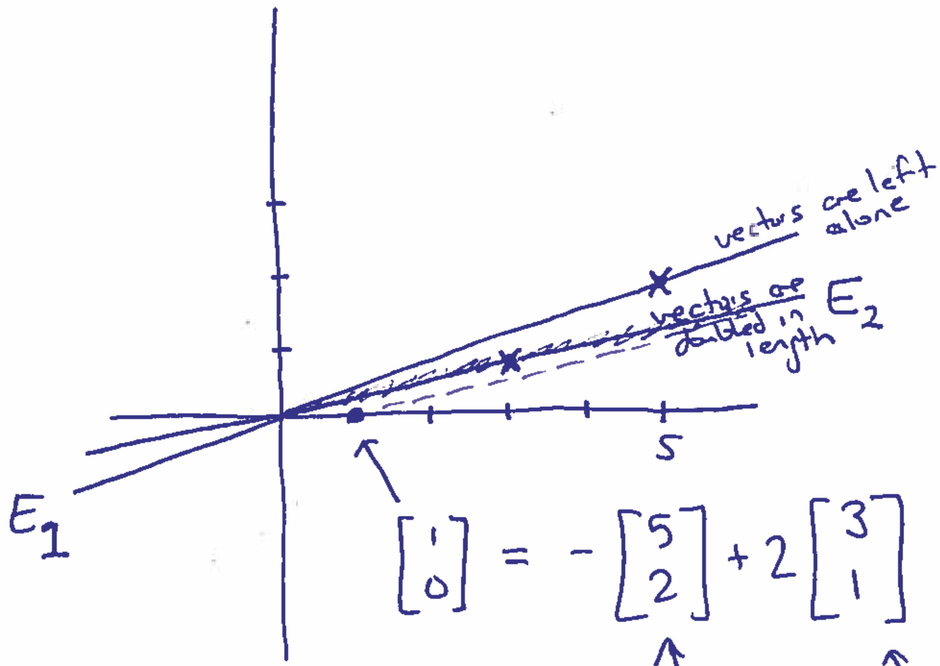
$\left[\begin{array}{cc|c} 6 & -15 & 0 \\ 2 & -5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 free ... take $x_2 = 2$

$x_1 = \frac{5}{2}x_2$ x_1 would be 5

$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is an e'vector for $\mu = 1$

Our matrix A leaves vectors alone when they are in E_1



$$\begin{bmatrix}1\\0\end{bmatrix} = -\begin{bmatrix}5\\2\end{bmatrix} + 2\begin{bmatrix}3\\1\end{bmatrix}$$

in E_1 (A leaves it alone)

in E_2 (A doubles its length)

What is $A\begin{bmatrix}1\\0\end{bmatrix}$?

So $A\begin{bmatrix}1\\0\end{bmatrix} = -\begin{bmatrix}5\\2\end{bmatrix} + 2\cdot 2\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}7\\2\end{bmatrix}$

Make $\mathcal{B} = \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix}\right\}$. If $\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}2\\-1\end{bmatrix}$

For this matrix A, this basis of eigenvectors is in some more natural than the standard basis. With respect to basis $\mathcal{B}$, A stretches by 2 in 1st coordinate, leaves alone 2nd coordinate.

Let's examine A's action in $\mathcal{B}$-coordinate vectors...

$$\cancel{[A\cdot\vec{x}]_\beta}$$

$\begin{bmatrix}5\\2\end{bmatrix}$ $\mu=1$

$\begin{bmatrix}3\\1\end{bmatrix}$ $\lambda=2$

$$[\vec{x}]_\beta = \begin{bmatrix}x_{\beta1}\\x_{\beta2}\end{bmatrix}$$

$$\left(\vec{x} = x_{\beta1}\cdot\vec{b}_1 + x_{\beta2}\cdot\vec{b}_2\right)$$

$$\Downarrow$$

$$A\vec{x} = \underbrace{x_{\beta1}\cdot 2}\vec{b}_1 + \underbrace{x_{\beta2}\cdot}\vec{b}_2$$

$$[A\vec{x}]_\beta = \begin{bmatrix}2x_{\beta1}\\x_{\beta2}\end{bmatrix} \Longleftarrow$$

$$[A\vec{x}]_\beta = \begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}x_{\beta1}\\x_{\beta2}\end{bmatrix} \Longrightarrow [A\vec{x}]_\beta = \begin{bmatrix}2&0\\0&1\end{bmatrix}[\vec{x}]_\beta$$

$$\underset{\beta\leftarrow std}{P}\cdot A\vec{x} = \begin{bmatrix}2&0\\0&1\end{bmatrix}\underset{\beta\leftarrow std}{P}\cdot\vec{x}$$

$$\Longrightarrow A\vec{x} = \underset{std\leftarrow\beta}{P}\begin{bmatrix}2&0\\0&1\end{bmatrix}\underset{\beta\leftarrow std}{P}\vec{x}$$

since ... for all $\vec{x}$ ... $\Longrightarrow$

$$A = \underset{std\leftarrow\beta}{P}\begin{bmatrix}2&0\\0&1\end{bmatrix}\underset{\beta\leftarrow std}{P}$$

diagonal built out of e'values

built as

$$\begin{bmatrix}\vec{b}_1 & \vec{b}_2\end{bmatrix}$$

eigenvectors for A

inverse of front...

Ex $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$. Diagonalize A.

(write A as $P \cdot D \cdot P^{-1}$)

find these! and this!

First find e'vales.

$$\det(A - \lambda I) = 0$$

$$(1-\lambda)(5-\lambda) - 12 = 0$$

$$\lambda^2 - 6\lambda - 7 = 0$$

$$(\lambda+1)(\lambda-7) = 0$$

$$\lambda = -1 \text{ or } \lambda = 7$$

Nice to have a convention and arrange in increasing order: $-1, 7$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}.$$

Next e'vectors...

$\lambda = -1$

$$A\vec{x} = -\vec{x}$$

$$(A+I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 is free take to be 2.

$$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$\lambda = 7$

$$A\vec{x} = 7\vec{x}$$

$$(A-7I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} -6 & 3 & 0 \\ 4 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 free take to be 2

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

⟨thinking $\left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a good basis, consisting of e'vectors⟩

$$P = \underset{std \leftarrow \beta}{P} = \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-8} \begin{bmatrix} 2 & -1 \\ -2 & -3 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}} = \underset{P}{\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}} \underset{D}{\begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}} \cdot \underset{P^{-1}}{\frac{-1}{8} \begin{bmatrix} 2 & -1 \\ -2 & -3 \end{bmatrix}}$$

You try: Diagonalize $A = \begin{bmatrix} 31 & -10 \\ 100 & -34 \end{bmatrix}$

Find e'values: $(31-\lambda)(-34-\lambda) + 1000 = 0$

$\lambda^2 + 3\lambda - 54 = 0$

$(\lambda+9)(\lambda-6) = 0$

$\lambda = -9 \quad , \quad \lambda = 6$

find e'vectors

$A\vec{x} = -9\vec{x}$

$(A+9I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} 40 & -10 & 0 \\ 100 & -25 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

take $x_2 = 4$ $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$A\vec{x} = 6\vec{x}$

$(A-6I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} 25 & -10 & 0 \\ 100 & -40 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$

take $x_2 = 5$ $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

So $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -9 & 0 \\ 0 & 6 \end{bmatrix} \cdot \frac{-1}{3} \begin{bmatrix} 5 & -2 \\ -4 & 1 \end{bmatrix}$

Curve balls...

Ex Diagonalize $\begin{bmatrix} 12 & -8 \\ -7 & 3 \end{bmatrix}$.

e'values...

$$(12-\lambda)(3-\lambda)-(-8)(-7)=0$$
$$\lambda^2-15\lambda-20=0$$
$$\lambda=\frac{15\pm\sqrt{225-4(1)(-20)}}{2}$$
$$\lambda=\frac{15\pm\sqrt{305}}{2}$$

do not ever round these...

$$\lambda=\frac{15-\sqrt{305}}{2} \qquad \mu=\frac{15+\sqrt{305}}{2}$$

conjugate roots

e'vectors...

$$A\vec{x}=\lambda\vec{x}$$
$$(A-\lambda I)\vec{x}=\vec{0}$$
$$\left[\begin{array}{cc|c} 12-\frac{15-\sqrt{305}}{2} & -8 & 0 \\ -7 & 3-\frac{15-\sqrt{305}}{2} & 0 \end{array}\right]$$

RREF

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{7}+\frac{15-\sqrt{305}}{14} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$=\left[\begin{array}{cc|c} 1 & \frac{9-\sqrt{305}}{14} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

e'vector $=\begin{bmatrix} -9+\sqrt{305} \\ 14 \end{bmatrix}$

other e'vector will be

$$\begin{bmatrix} -9-\sqrt{305} \\ 14 \end{bmatrix}$$

$$\text{So} \quad \begin{bmatrix} 12 & -8 \\ -7 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} -9+\sqrt{305} & -9-\sqrt{305} \\ 14 & 14 \end{bmatrix}}_{P} \begin{bmatrix} \frac{15-\sqrt{305}}{2} & 0 \\ 0 & \frac{15+\sqrt{305}}{2} \end{bmatrix} \cdot P^{-1}$$

Ex Diagonalize $\begin{bmatrix} 12 & 8 \\ -7 & 11 \end{bmatrix}$.

Find e'values ..

$$(12-\lambda)(11-\lambda)+56 = 0$$
$$\lambda^2 - 23\lambda + 188 = 0$$
$$\lambda = \frac{23 \pm \sqrt{23^2 - 4(1)(188)}}{2}$$
$$= \frac{23 \pm \sqrt{529 - 752}}{2}$$
$$= \frac{23 \pm \sqrt{\text{negative!}}}{2}$$

the e'values are not real.

A has no real e'values...

$\Longrightarrow$ It can't be written $P \cdot D \cdot P^{-1}$ with these being real.

Ex Diagonalize $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

It is diagonal...
$A = P \cdot A \cdot P^{-1}$
will be true...

find e'values.

$(2-\lambda)(2-\lambda) - 0 = 0$

$(\lambda-2)^2 = 0$

$\lambda = 2$ or... $\lambda = 2$

repeated eigenvalue.

$A\vec{x} = 2\vec{x}$

$(A-2I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 free

x_1 free too.

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is fine...

Ex Diagonalize $\begin{bmatrix} 5 & -9 \\ 1 & -1 \end{bmatrix}$.

e'values: $(5-\lambda)(-1-\lambda) + 9 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda-2)(\lambda-2) = 0$

$(\lambda-2)^2 = 0 \longrightarrow \lambda = 2$ is the only e'value...

in both situations the eigenvalue 2 has algebraic multiplicity 2

set up $A\vec{x} = 2\vec{x}$

$(A-2I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} 3 & -9 & 0 \\ 1 & -3 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

e'vector

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

In this example, E_2 has dimension 1 because $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ is a basis for E_2. So ~~the~~ $\lambda = 2$ has geometric multiplicity 1.

Geometric multiplicity of an e'value λ is the dimension of that e'value's e'space.

geometric multiplicity $\leq$ algebraic multiplicity

need equality to be able to diagonalize A.

<u>Ex</u> Diagonalize $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Find e'values: $\det(A-\lambda I)=0$

$$\det\begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(-5-\lambda)(1-\lambda) + 3(-3)(3) + 3(-3)(3)$$
$$-(1-\lambda)(-3)(3) - 3(-3)(1-\lambda) - 3(-5-\lambda)(3) = 0$$

....

$$-\lambda^3 - 3\lambda^2 + 4 = 0$$
$$\lambda^3 + 3\lambda^2 - 4 = 0$$

with a cubic.. try divisors (pos & neg) of ~~last~~ constant.

Like 1....

$$1^3 + 3\cdot 1^2 - 4 \stackrel{\checkmark}{=} 0$$

$$(\lambda - 1)\left(\text{need to engineer here...} \right) = 0$$

$$(\lambda - 1)(\lambda^2 + c\lambda + 4) = 0$$

$$\lambda^3 + c\lambda^2 + 4\lambda - \lambda^2 - c\lambda - 4 = 0$$

$$\lambda^3 + (c-1)\lambda^2 + (4-c)\lambda - 4 = 0$$

compare and $c = \ldots$

$c = 4$

$$(\lambda-1)(\lambda^2 + 4\lambda + 4) = 0$$

$$(\lambda-1)(\lambda+2)^2 = 0$$

$\lambda = 1$ with multiplicity 1

$\lambda = -2$ with <u>algebraic</u> multiplicity 2.

$$A\vec{x} = \vec{x}$$

$$(A - I)\vec{x} = 0$$

$$\left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free... take to be 1

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$A\vec{x} = -2\vec{x}$$

$$(A + 2I)\vec{x} = \vec{0}$$

$$\cancel{\begin{bmatrix} -1 & 3 & 3 \\ -3 & -7 & -3 \\ 3 & 3 & -1 \end{bmatrix}} \quad \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \vec{x}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free

x_2 free

$x_1 = -x_2 - x_3$

General solution is $\vec{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

take these e'vectors.

So geometric multiplicity of -2 is 2.

So $A = \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}^{P} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \cdot P^{-1}$

or, following convention...

$A = \underbrace{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}}_{P} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P^{-1}$

Diagonalize

$\begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}.$

Note 5, 0, 2 are the e'values... 0, 2, 5

$\lambda = 0$

$A\vec{x} = 0\vec{x}$

$A\vec{x} = 0$

$$\left[\begin{array}{ccc|c} 5 & -8 & 1 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \downarrow \left[\begin{array}{ccc|c} 1 & -8/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \downarrow \begin{bmatrix} 8 \\ 5 \\ 0 \end{bmatrix}$$

$\lambda = 2$

$A\vec{x} = 2\vec{x}$

$\cancel{A\vec{x}}\ (A - 2I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 3 & -8 & 1 & 0 \\ 0 & -2 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \downarrow \left[\begin{array}{ccc|c} 1 & 0 & -9 & 0 \\ 0 & 1 & -7/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \downarrow \begin{bmatrix} 18 \\ 7 \\ 2 \end{bmatrix}$$

$\lambda = 5$

$A\vec{x} = 5\vec{x}$

$(A - 5I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 0 & -8 & 1 & 0 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \downarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \downarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So... $A = \overbrace{\begin{bmatrix} 8 & 18 & 1 \\ 5 & 7 & 0 \\ 0 & 2 & 0 \end{bmatrix}}^{P} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \cdot P^{-1}$

Ex Diagonalize $\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Find e'values...

$$\det \begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} = 0$$

⋮ leads to...

$$\lambda^3 + 3\lambda^2 - 4 = 0$$
$$(\lambda-1)(\lambda+2)^2 = 0$$

← Same as an earlier example.

$\lambda = 1$

$A\bar{x} = \bar{x}$

$(A - I)\bar{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = -2$

$A\bar{x} = -2\vec{x}$

$(A + 2I)\bar{x} = 0$

$$\left[\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

geometric multiplicity is 1

not going to be able to diagonalize.

With 2x2 matrices, with real entries,

like $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, maybe there is a non-real eigenvalue, $\lambda_1 = s + it$

there will be nonreal eigenvectors...

$$\begin{bmatrix} p+qi \\ u+vi \end{bmatrix}$$

Observe $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underbrace{\begin{bmatrix} p+qi \\ u+vi \end{bmatrix}}_{\text{2x1 complex vector}} = \begin{bmatrix} ap + aqi + bu + bvi \\ cp + cqi + du + dvi \end{bmatrix}$

$$= \begin{bmatrix} ap + bu + i(aq+bv) \\ cp + du + i(cq+dv) \end{bmatrix}$$

And: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underbrace{\begin{bmatrix} p & q \\ u & v \end{bmatrix}}_{\text{2x2 real matrix}} = \begin{bmatrix} ap+bu & aq+bv \\ cp+du & cq+dv \end{bmatrix}$

It is valid to treat $A \cdot \begin{bmatrix} \text{comp} \\ \text{vector} \end{bmatrix}$ as $A \begin{bmatrix} \text{2x2} \\ \text{matrix} \end{bmatrix}$

Ex $A = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$

It's e'vales..

$(3-\lambda)(1-\lambda)+10=0$

$\lambda^2-4\lambda+13=0$

$\lambda = \frac{4 \pm \sqrt{16-4(1)(13)}}{2}$

$\lambda = 2 \pm \sqrt{4-13}$

$\lambda = 2 \pm \sqrt{-9}$

$\lambda = 2 \pm 3i$

$\lambda_1 = 2-3i$, $\lambda_2 = 2+3i$

(if e'vales are nonreal, λ_1 and λ_2 are conjugates)

Look for e'vectors

$A\vec{x} = (2-3i)\vec{x}$

$(A-(2-3i)I)\vec{x} = 0$

$$\left[\begin{array}{cc|c} 3-(2-3i) & -2 & 0 \\ 5 & 1-(2-3i) & 0 \end{array}\right]$$

RREF

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{5}+\frac{3}{5}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 free ... take 5

$x_1 = (\frac{1}{5}-\frac{3}{5}i)x_2$ ... so $1-3i$

$\begin{bmatrix} 1-3i \\ 5 \end{bmatrix}$

Take $\vec{x} = \begin{bmatrix} 1-3i \\ 5 \end{bmatrix}$ ... an e'vector for λ_1 $(2-3i)$

Then $A \cdot \vec{x} = \lambda \cdot \vec{x}$

$$\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1-3i \\ 5 \end{bmatrix} = (2-3i) \begin{bmatrix} 1-3i \\ 5 \end{bmatrix}$$

$$= (2-3i)\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} + i \begin{bmatrix} -3 \\ 0 \end{bmatrix} \right)$$

$$= 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3\begin{bmatrix} -3 \\ 0 \end{bmatrix} + i\left(-3\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + i \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + i \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right)$$

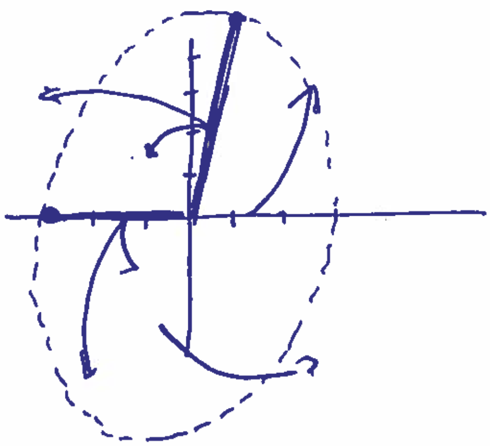
$$\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1-3i \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2-3i \\ 3+2i \end{bmatrix}$$

$$\approx \quad \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

$$\Longrightarrow \quad \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix}^{-1}$$

like diagonalizing ... $\longrightarrow$ special form $\begin{bmatrix} T & -S \\ S & T \end{bmatrix}$

$$\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} = \underset{std \leftarrow \beta}{P} \cdot \begin{bmatrix} T & -S \\ S & T \end{bmatrix} \cdot P^{-1}$$



consider $\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix} \right\}$

Examine $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ --- determinant is 13

$$= \underbrace{\sqrt{13}}_{\approx 3.6} \underbrace{\begin{bmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}}_{\text{determinant } 1}$$

Fact..

$$\frac{2}{\sqrt{13}} = \cos\theta$$

$$\frac{3}{\sqrt{13}} = \sin\theta$$

for the right θ...

$$\theta = \cos^{-1}(2/\sqrt{13}) \approx 0.98 \text{ radians} \approx 56^\circ$$

$$= \sqrt{13} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

So... $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ rotates by $\approx 56^\circ$, then stretches by ≈ 3.6.

Summary: with a non-real λ for a 2×2 matrix. basically A is rotating and stretching wrt some basis of $\mathbb{R}^2$.