5.2 Characteristic Polynomial of a matrix

Ex What are the eigenvalues of $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$?

Let λ be an eigenvalue.

There is a nonzero vector $\vec{x}$ where $A\vec{x} = \lambda\vec{x}$.

$$\Longrightarrow A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\Longrightarrow (A - \lambda I)\vec{x} = \vec{0}$$

λ is an e'value $\Longrightarrow$ this equation has nontrivial solutions

$\Longrightarrow$ $A - \lambda I$ is not an invertible matrix

$\Longrightarrow$ So $\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (2-\lambda)(-6-\lambda) - 9 = 0$$

$$\Rightarrow \lambda^2 + 4\lambda - 21 = 0$$

small degree polynomial equation

$$(\lambda + 7)(\lambda - 3) = 0$$

$$\lambda = -7, \quad \lambda = 3$$

So... A's eigenvalues are -7 and 3.

Let's take $p(X) = X^2 + 4X - 21$

$$\Longrightarrow \quad p(A) = A^2 + 4A - 21I$$

$$= \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} + 4\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} 21 & 0 \\ 0 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -12 \\ -12 & 45 \end{bmatrix} + \begin{bmatrix} 8 & 12 \\ 12 & -24 \end{bmatrix} - \begin{bmatrix} 21 & 0 \\ 0 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

variable for a scalar

This polynomial $\det(A - XI)$ is the characteristic polynomial of A. Its roots tell us the eigenvalues for A.

<u>Ex</u> Let $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ Find A's eigenvalues.

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 5 & 0 \\ 2 & 4-\lambda & -1 \\ 0 & -2 & -\lambda \end{bmatrix} = 0$$

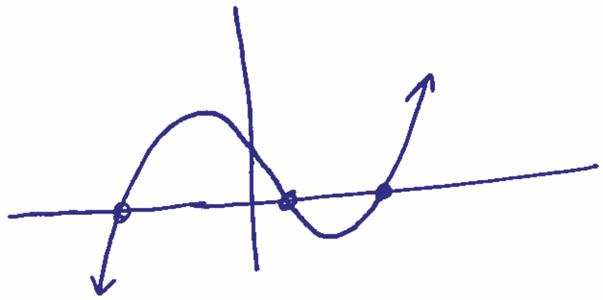
$$\underbrace{(1-\lambda)(4-\lambda)(-\lambda)}_{(\lambda^2 - 5\lambda + 4)(-\lambda)} + 5(-1)(0) + 0(2)(-2) - \underbrace{(1-\lambda)\overbrace{(-1)(-2)}^{2}}_{-2+2\lambda} - \underbrace{5(2)(-\lambda)}_{+10\lambda} - 0(4-\lambda)(0) = 0$$

$$-\lambda^3 + 5\lambda^2 - 4\lambda - 2 + 2\lambda + 10\lambda = 0$$

$$-\lambda^3 + 5\lambda^2 + 8\lambda - 2 = 0$$

$$\lambda^3 - 5\lambda^2 - 8\lambda + 2 = 0$$

solve this? in general we won't be able to solve exactly -- rely on



There are 3 distinct eigenvalues...

Note: $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

$(-1)^{n-1} \cdot tr(A)$

$$\chi(t) = -t^3 + 5t^2 + 8t - 2$$

$(-1)^n$ this number is $\det(A)$

trace of A...

$tr(A)$ = sum all diagonal entries

Ex Find the eigenvalues ~~characteristic polynomial~~ of $\begin{bmatrix} 1 & 6 \\ 5 & -7 \end{bmatrix}$.

Fine...

$$\det\begin{bmatrix} 1-\lambda & 6 \\ 5 & -7-\lambda \end{bmatrix} = 0$$

$\vdots$

OR

$$\lambda^2 + 6\lambda - 37 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{36 - 4(-37)}}{2}$$

$$= -3 \pm \sqrt{9 + 37}$$

$$= -3 \pm \sqrt{46}$$

Ex Find the e'vals of $\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

$$\det(\underbrace{A - \lambda I}_{\text{a triangular matrix.}}) = 0$$

with a triangular matrix, diagonal entries are the e'values.

$$\Longrightarrow (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$$

$$\Longrightarrow \lambda = 5 \text{ or } \lambda = 3 \text{ or } \lambda = 1.$$

5 is a doubled root of the characteristic polynomial.
The e'value 5 has <u>algebraic multiplicity</u> 2.

Ex Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. What are A's eigenvalues?

this is rotation by θ...
predict no real eigenvalues!

$$\det \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} = 0$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

see that unless $\theta = 0, \pi, 2\pi$, etc...
this equation has no real solutions...

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \underbrace{\cos\theta \pm \sqrt{\underbrace{\cos^2\theta - 1}_{\text{probably negative.}}}}_{\text{eigenvalues are non-real complex numbers.}}$$

<u>Definition</u> Two matrices A, B are <u>similar</u> if $A = P \cdot B \cdot P^{-1}$ for some invertible matrix P.

(We write $A \sim B$).

<u>Observation:</u> If A, B are <u>similar</u>, then

$$\det(A) = \det(PBP^{-1})$$
$$\det(A) = \det(P)\cdot\det(B)\cdot\det(P^{-1})$$
$$\det(A) = \underbrace{\det(P)\cdot\det(P^{-1})}_{=1}\cdot\det(B)$$
$$\det(A) = \det(B).$$

Similarly, if A, B are similar, they have the same... <u>eigenvalues</u>.

$$\det(A - \lambda I) = \det(PBP^{-1} - P(\lambda I)P^{-1})$$
$$= \det(P(B - \lambda I)P^{-1})$$
$$= \det(P)\cdot\det(B-\lambda I)\cdot\det(P^{-1}) = \det(B-\lambda I)$$

We had $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ ... we found $\chi(t) = t^2 + 4t - 21$

... so -7 and 3 are eigenvalues...

For $\lambda = 3$, what are the e'vectors?

Find vectors $\vec{x}$ where $A\vec{x} = 3\vec{x}$

$$\Longrightarrow (A\vec{x} - 3\vec{x}) = \vec{0}$$

$$\Longrightarrow (A - 3I)\vec{x} = \vec{0}$$

$$\Longrightarrow \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & 9 & 0 \end{array}\right]$$

RREF

$$\left[\begin{array}{cc|c} \boxed{1} & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 is free

$x_1 = 3x_2$

$$\Longrightarrow \vec{x} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix}.$$

So $\vec{x}$ is in $\mathrm{Span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$.

For $\lambda = -7$...

$$A\vec{x} = -7\vec{x}$$

$$(A + 7I)\vec{x} = 0$$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = -\frac{1}{3}x_2$

x_2 free

$$\vec{x} = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \end{bmatrix}$$

$\vec{x}$ in $\mathrm{span}\left\{\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}\right\}$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$