1 (a) $\det\left(\begin{bmatrix} 0-\lambda & -6 \\ -1 & -1-\lambda \end{bmatrix}\right) = 0$

$-\lambda(-1-\lambda) - 6 = 0$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda+3)(\lambda-2) = 0$

$\lambda = -3$ or 2

Solve $(A+3I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} 3 & -6 & 0 \\ -1 & 2 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}$ Take $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Solve $(A-2I)\vec{x} = \vec{0}$

$\left[\begin{array}{cc|c} -2 & -6 & 0 \\ -1 & -3 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{x} = \begin{bmatrix} -3x_2 \\ x_2 \end{bmatrix}$ Take $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

And $A = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/5 & 3/5 \\ -1/5 & 2/5 \end{bmatrix}$

(b) $\det\left(\begin{bmatrix} 1.4-\lambda & 0.4 & 0.9 \\ 0 & 2-\lambda & 0 \\ -1.2 & 0.8 & 3.6-\lambda \end{bmatrix}\right) = 0$

$(2-\lambda)\cdot\det\left(\begin{bmatrix} 1.4-\lambda & 0.9 \\ -1.2 & 3.6-\lambda \end{bmatrix}\right) = 0$

$(2-\lambda)\left[(1.4-\lambda)(3.6-\lambda) + .96\right] = 0$

$(2-\lambda)\left[\lambda^2 - 5\lambda + 5.04 + .96\right] = 0$

$(2-\lambda)\left[\lambda^2 - 5\lambda + 6\right]$

$(2-\lambda)(\lambda-2)(\lambda-3)$

$\lambda = 2$ or 3

Solve $(A-2I)\vec{x} = \vec{0}$

$\left[\begin{array}{ccc|c} -0.6 & 0.4 & 0.8 & 0 \\ 0 & 0 & 0 & 0 \\ -1.2 & 0.8 & 1.6 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -2/3 & -4/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\vec{x} = \begin{bmatrix} \frac{2}{3}x_2 + \frac{4}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix}$

Take $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$

Solve $(A-3I)\vec{x} = \vec{0}$

$\left[\begin{array}{ccc|c} -1.6 & 0.4 & 0.8 & 0 \\ 0 & -1 & 0 & 0 \\ -1.2 & 0.8 & 0.6 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\vec{x} = \begin{bmatrix} \frac{1}{2}x_3 \\ 0 \\ x_3 \end{bmatrix}$

Take $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$P = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix} \qquad \text{And} \qquad A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix}^{-1}$$

2. (a)

$$A^k = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}^k \begin{bmatrix} 1/5 & 3/5 \\ -1/5 & 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-3)^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1/5 & 3/5 \\ -1/5 & 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-3)^k & -3(2^k) \\ (-3)^k & 2^k \end{bmatrix} \begin{bmatrix} 1/5 & 3/5 \\ -1/5 & 2/5 \end{bmatrix} = \begin{bmatrix} \frac{2}{5}(-3)^k + \frac{3}{5}(2^k) & \frac{6}{5}(-3)^k - \frac{6}{5}(2^k) \\ \frac{1}{5}(-3)^k - \frac{1}{5}(2^k) & \frac{3}{5}(-3)^k + \frac{2}{5}(2^k) \end{bmatrix}$$

(b)

$$A^k = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2(2^k) & 4(2^k) & 3^k \\ 3(2^k) & 0 & 0 \\ 0 & 3(2^k) & 2(3^k) \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 \\ 2/5 & -4/15 & -1/5 \\ -3/5 & 2/5 & 4/5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{5}(2^k) - \frac{3}{5}(3^k) & \overbrace{\frac{2}{3}(2^k) + \frac{16}{15}(2^k)}^{\text{can be combined}} + \frac{2}{5}(3^k) & \frac{-4}{5}(2^k) + \frac{4}{5}(3^k) \\ 0 & 2^k & 0 \\ \frac{6}{5}(2^k) - \frac{6}{5}(3^k) & \frac{-4}{5}(2^k) + \frac{4}{5}(3^k) & \frac{-3}{5}(2^k) + \frac{8}{5}(3^k) \end{bmatrix}$$

find P^{-1} explicitly

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 6 & 12 & 3 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 12 & 3 & 3 & -2 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -5 & 3 & -2 & -4 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -5 & 3 & -2 & -4 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 15 & 10 & 0 & 0 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -5 & 3 & -2 & -4 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 15 & 0 & 6 & -4 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 15 & 0 & 6 & -4 & -3 \\ 0 & 0 & -5 & 3 & -2 & -4 \end{array}\right]$$

3. To find A's eigenvalues:

$$\det\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)^2 = 0$$

$\Longrightarrow \lambda = 1$ So it only has one.

Solve $(A - 1I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\to x_2 = 0$

$\to x_1$ free

$\vec{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ Eigenspace has basis $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

We are only able to find one linearly independent eigenvector. So we can't build a P to diagonalize A.

4. $$\det\left(\begin{bmatrix} 0.94-\lambda & 0.10 \\ 0.06 & 0.9-\lambda \end{bmatrix}\right) = 0$$

$$(0.94-\lambda)(0.9-\lambda) - 0.006 = 0$$

$$\lambda^2 - 1.84\lambda + 0.846 - 0.006 = 0$$

$$\lambda^2 - 1.84\lambda + 0.84 = 0$$

$$(\lambda - 1)(\lambda - 0.84) = 0$$

$$\lambda = 1 \text{ or } 0.84$$

Solve $(A - 1I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{cc|c} -0.06 & 0.10 & 0 \\ 0.06 & -0.10 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -5/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{x} = \begin{bmatrix} \frac{5}{3}x_2 \\ x_2 \end{bmatrix}$ basis $\left\{\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right\}$

Solve $(A - 0.84I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 0.1 & 0.1 & 0 \\ 0.06 & 0.06 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$ basis $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

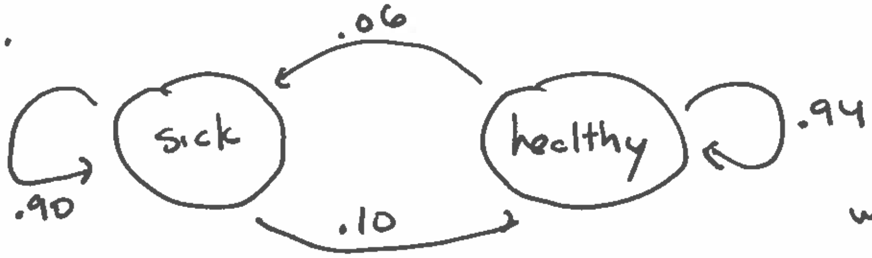
$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.84 \end{bmatrix} \cdot \frac{1}{8}\begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.84^k \end{bmatrix}\begin{bmatrix} 1/8 & 1/8 \\ -3/8 & 5/8 \end{bmatrix}$$

$$\lim_{k\to\infty}(A^k) = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/8 & 1/8 \\ -3/8 & 5/8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1/8 & 1/8 \\ -3/8 & 5/8 \end{bmatrix} = \begin{bmatrix} 5/8 & 5/8 \\ 3/8 & 3/8 \end{bmatrix}$$

5.



$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{\text{this week}} \\ & & \text{healthy} & \text{sick} \\ \hline \text{next week} & \begin{array}{c}\text{healthy}\\ \text{sick}\end{array} & \multicolumn{2}{c}{\begin{bmatrix} .94 & .10 \\ .06 & .90 \end{bmatrix}} \end{array}$$

(the matrix from #4)

One week later: $\begin{bmatrix} .94 & .10 \\ .06 & .90 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} .52 \\ .48 \end{bmatrix} \begin{array}{l} \to \text{healthy} \\ \to \text{sick} \end{array}$

Two weeks: $\begin{bmatrix} .94 & .10 \\ .06 & .90 \end{bmatrix}^2 \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} .94 & .10 \\ .06 & .90 \end{bmatrix} \begin{bmatrix} .52 \\ .48 \end{bmatrix} = \begin{bmatrix} .5368 \\ .4632 \end{bmatrix} \begin{array}{l} \to \text{healthy} \\ \to \text{sick} \end{array}$

"∞" weeks later: $\lim_{k \to \infty} \begin{bmatrix} .94 & .10 \\ .06 & .90 \end{bmatrix}^k \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} 5/8 & 5/8 \\ 3/8 & 3/8 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} 5/8 \\ 3/8 \end{bmatrix}$

$= \begin{bmatrix} .625 \\ .375 \end{bmatrix} \begin{array}{l} \to \text{healthy} \\ \to \text{sick} \end{array}$