Date ___________

MTH 261
Linear Algebra
Spring 2017
Column Space and Null Space

Find partners, and follow the instructions. You will not turn this in, but you must be working diligently to get attendance credit.

1. Find the null space and column space for $A = \begin{bmatrix} 1 & -2 & 2 & 2 \\ 0 & 3 & 1 & -1 \end{bmatrix}$. Specifically, I mean write Nul A and Col A as the span of some vector(s).

$$\text{Col}A = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}-2\\3\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}2\\-1\end{bmatrix}\right\}$$

Nul A = ... ?

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 2 & 0 \\ 0 & 3 & 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 8/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \end{array}\right]$$

x_4 free

x_3 free

$x_2 = -\frac{1}{3}x_3 + \frac{1}{3}x_4$

$x_1 = -\frac{8}{3}x_3 - \frac{4}{3}x_4$

$$\vec{x} = \begin{bmatrix} -\frac{8}{3}x_3 - \frac{4}{3}x_4 \\ -\frac{1}{3}x_3 + \frac{1}{3}x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\Longleftarrow$$

$$\text{Nul}A = \text{Span}\left\{\begin{bmatrix}-8/3\\-1/3\\1\\0\end{bmatrix},\begin{bmatrix}-4/3\\1/3\\0\\1\end{bmatrix}\right\}$$

2. In the last problem you probably wrote Col A as the span of four vectors. Show that both $\vec{e}_1$ and $\vec{e}_2$ are in this span. (Note that $\vec{e}_i$ is one thing in the context of Col A and something different in the context of Nul A.) Does this mean Col $A = \text{Span}\{\vec{e}_1, \vec{e}_2\}$?

is this true in this example?

$\vec{e}_1 \in \text{Col}A$? trivially ✓

$\vec{e}_2 \in \text{Col}A$? $-2\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}2\\1\end{bmatrix} = \vec{e}_2$ ✓

Yes! $\text{Span}\{\vec{e}_1,\vec{e}_2\} \subset \text{Span}\{\vec{a}_1,\vec{a}_2,\vec{a}_3,\vec{a}_4\}$ (because $\vec{e}_1, \vec{e}_2$ are linear combos of A's columns)

And $\text{Span}\{\vec{a}_1,\dots\vec{a}_4\} \subset \text{Span}\{\vec{e}_1,\vec{e}_2\}$ because each $\vec{a}_i$ is a linear combo of $\vec{e}_1, \vec{e}_2$.

$\Downarrow$

Here $\text{Col}A = \text{Span}\{\vec{e}_1,\vec{e}_2\}$

$\text{Col}A = \mathbb{R}^2$

3. Find the null space and column space for $A = \begin{bmatrix} 1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$. Specifically, I mean write Nul A and Col A as the span of some vector(s).

$$\text{Col}A = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}-2\\0\\0\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\2\end{bmatrix},\begin{bmatrix}0\\2\\0\end{bmatrix}\right\}$$

$$= \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \mathbb{R}^3$$

$\frac{1}{2}\vec{a}_5$ $\quad$ $\frac{1}{2}\left(\vec{a}_4 + \frac{1}{2}\vec{a}_5\right)$

RREF

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Gen sol is

$$\begin{bmatrix} 2x_2 + 6x_5 \\ x_2 \\ -2x_5 \\ 0 \\ x_5 \end{bmatrix} \Longrightarrow \text{Nul}A = \text{Span}\left\{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}6\\0\\-2\\0\\1\end{bmatrix}\right\}$$

$$\text{RREF} \rightarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Row reduce $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & -4 \end{bmatrix}$ into a reduced echelon matrix B. Do A and B have they have the same null space? Do they have the same column space?

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} \right\}$$

includes vectors with nonzero entries in 3rd coordinate

$$\text{Col RREF} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} \right\}$$

only has vectors with 0 in 3rd coordinate

In general Col A $\neq$ Col RREF

Sols to $A\vec{x} = \vec{0}$

Same as Sols to $\underbrace{E_p \cdot E_{p-1} \cdots E_1 A}\vec{x} = \vec{0}$

Same as sols to $(\text{RREF})\vec{x} = \vec{0}$

$$\text{Nul } A \overset{\checkmark}{=} \text{Nul RREF}$$

5. Let $S = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 2 \right\}$. Is S a subspace of $\mathbb{R}^3$? Why or why not?

$\vec{0}$ is not in S!

$a_1 + b_1 + c_1 = 2 \quad (a_1, b_1, c_1) \in S$

$a_2 + b_2 + c_2 = 2 \quad (a_2, b_2, c_2) \in S$

$(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = 4$

$(a_1 + a_2, b_1 + b_2, c_1 + c_2) \notin S$

6. Let $A = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix}$. Find a set of vectors that spans Nul A. Show that $\vec{a}_3$ and $\vec{a}_5$ are in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$. Explain why T_A is neither one-to-one nor onto. ('Onto' means that it's image is the entire codomain.)

$$\text{RREF} \begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{bmatrix} -\frac{1}{3}x_3 - \frac{10}{3}x_5 \\ -\frac{1}{3}x_3 + \frac{26}{3}x_5 \\ x_3 \\ 4x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{3} \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 26 \\ 0 \\ 12 \\ 3 \end{bmatrix} \right\}$$

$\Downarrow$

Nul A is not one-to-one because many vectors map to $\vec{0}$.

$-\vec{a}_1 - \vec{a}_2 + 3\vec{a}_3 = \vec{0} \Rightarrow \vec{a}_3 = \frac{1}{3}\vec{a}_1 + \frac{1}{3}\vec{a}_2$

$-10\vec{a}_1 + 26\vec{a}_2 + 12\vec{a}_4 + 3\vec{a}_5 = \vec{0}$

$\Rightarrow \vec{a}_5 = \frac{10}{3}\vec{a}_1 - \frac{26}{3}\vec{a}_2 - 4\vec{a}_4$

So $\vec{a}_3$ & $\vec{a}_5$ are in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$

T_A is not onto because A has 4 rows but only 3 pivot positions