

MTH 261

LINEAR ALGEBRA

SPRING 2017

Basis and Dimension

Find partners, and follow the instructions. You will not turn this in, but you must be working diligently to get attendance credit.

For each matrix A below, respond to each of the following:

- Find a basis for $\text{Nul } A$.
- What is the dimension of $\text{Nul } A$ (the nullity of A)?
- Find a basis for $\text{Col } A$.
- What is the dimension of $\text{Col } A$ (the rank of A)?
- What is $(\text{rank } A) + (\text{nullity } A)$? (Do you notice any pattern once you have completed this for all four?)

1. $\begin{bmatrix} 1 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix}$

RREF $\rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Col A has basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}$ and $\text{rank}(A) = 2$

$\vec{x} \in \text{Nul } A \Rightarrow \vec{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$ So $\text{Nul } A$ has basis $\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$

and nullity $(A) = 1$

2. $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & -1 & 3 \\ 0 & 3 & 2 \end{bmatrix}$

RREF $\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Col A has basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 2 \end{bmatrix} \right\}$

and $\text{rank}(A) = 3$.

$\Rightarrow \text{Nul}(A) = \{ \vec{0} \}$ --- the $\vec{0}$ space has no basis.

nullity $(A) = 0 \Leftarrow$ the $\vec{0}$ space has dimension 0 by convention.

3. $\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 2 \\ 1 & 0 & 3 & 3 & 6 \end{bmatrix}$

PREP $\Rightarrow$ (at most 3 pivots $\Rightarrow$ at most 3 pivot cols $\Rightarrow$ rank A is at most 3)

Col A has a basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

rank A = 3

$\Rightarrow$ nullity A = 2

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 3 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$\rightarrow \vec{x} \in \text{Nul } A$

$x_1 = -3x_4 - 3x_5$

$x_2 = x_4 + x_5$

$x_3 = -x_5$

x_4 free

x_5 free

$$\vec{x} = \begin{bmatrix} -3x_4 - 3x_5 \\ x_4 + x_5 \\ -x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_4 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

4. $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

just for fun, in $\mathbb{F}_2$. (binary). $= \{0, 1\}$.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Nul A has a basis $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

don't feel a need to learn this!

Col A has a basis (in $\mathbb{F}_2$): $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$
rank A (as a matrix over $\mathbb{F}_2$) is 3

nullity A is 3
Nul A has a basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$