6.2, 6.3 cont'd

It's nice when a basis is orthonormal.

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\} \quad (\text{in } \mathbb{R}^n)$$

pairwise orthogonal (perpendicular)

and unit length.

Then when looking for $[\vec{y}]_{\mathcal{B}}$ ...

$$= \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

$$\vec{y} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_k \vec{b}_k$$

$\checkmark \quad ? \checkmark \quad ? \checkmark \quad ? \checkmark$

(could use row reduction to solve for c_i's)

$\Downarrow$ Dot product with $\vec{b}_i$.

$$\vec{b}_i \cdot \vec{y} = c_i \underbrace{\vec{b}_i \cdot \vec{b}_i}_{1} \implies c_i = \vec{b}_i \cdot \vec{y}$$

$\longrightarrow$ So we get $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 \checkmark \\ c_2 \checkmark \\ \vdots \\ c_k \checkmark \end{bmatrix}$

Ex A basis for $\mathbb{R}^3$ ...

$$\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\-2\\0\end{bmatrix}, \begin{bmatrix}-3\\-6\\5\end{bmatrix} \right\}$$

dot to 0

dot to 0

dot to 0

$\Longrightarrow$ orthogonal basis . .

Let $\beta = \left\{ \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}, \frac{1}{\sqrt{20}}\begin{bmatrix}4\\-2\\0\end{bmatrix}, \frac{1}{\sqrt{70}}\begin{bmatrix}-3\\-6\\5\end{bmatrix} \right\}$.

β is an orthonormal basis.

Let $\vec{y} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Find $[\vec{y}]_\beta$.

$\vec{y} \cdot \vec{b}_1 = \frac{6}{\sqrt{14}}$

$\vec{y} \cdot \vec{b}_2 = \frac{2}{\sqrt{20}}$

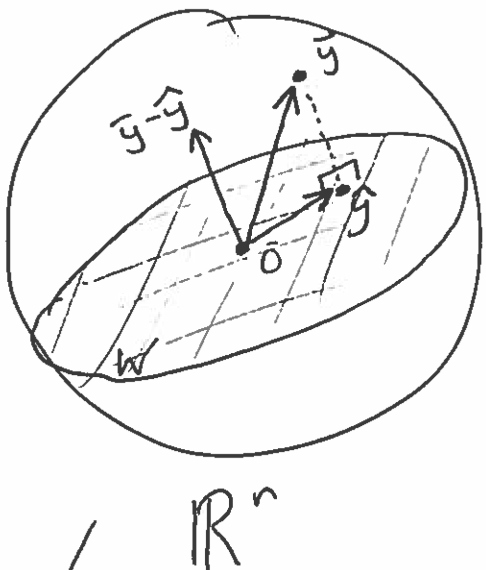
$\vec{y} \cdot \vec{b}_3 = \frac{-4}{\sqrt{70}}$

So $[\vec{y}]_\beta = \begin{bmatrix} 6/\sqrt{14} \\ 2/\sqrt{20} \\ -4/\sqrt{70} \end{bmatrix}$.

Check:

$$\frac{6}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} + \frac{2}{\sqrt{20}}\frac{1}{\sqrt{20}}\begin{bmatrix}4\\-2\\0\end{bmatrix} + \frac{-4}{\sqrt{70}}\frac{1}{\sqrt{70}}\begin{bmatrix}-3\\-6\\5\end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Given a subspace W of $\mathbb{R}^n$, and W has an <u>orthonormal</u> basis $\beta = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_k\}$



Suppose $\vec{y}$ is not in W...
✓

There is always a vector $\hat{y}$ inside W that is as close as possible to $\bar{y}$.
?

Can we find $\hat{y}$?

Well, $\bar{y} - \hat{y}$ is $\perp$ to W.

So $\bar{y} - \hat{y}$ is $\perp$ to each $\vec{b}_i$.

So $\vec{b}_i \cdot (\vec{y} - \hat{y}) = 0$

$\Longrightarrow \vec{b}_i \cdot \vec{y} = \vec{b}_i \cdot \hat{y}$

✓ ✓ ✓ ?

If I wanted $[\hat{y}]_\beta$... I'd calculate $\vec{b}_i \cdot \hat{y}$...

✓ ?

So we can find $[\hat{y}]_\beta$ by calculating $\vec{b}_i \cdot \vec{y}$....

Ex $\mathbb{R}^3$ ... W, with basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} \right\}$.

orthogonal ✓

Take $\vec{y} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$. Find $\hat{y}$ in W, closest possible vector to $\vec{y}$.

Take $\beta = \left\{ \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{26}}\begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} \right\}$.

Find $[\hat{y}]_\beta$ by ...

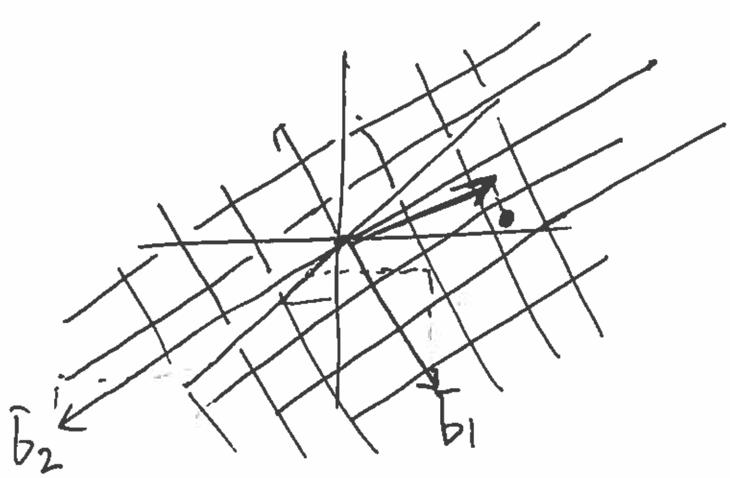
$$\vec{b}_1 \cdot \vec{y} = \frac{4}{3}$$

$$\vec{b}_2 \cdot \vec{y} = \frac{-3}{\sqrt{26}}$$

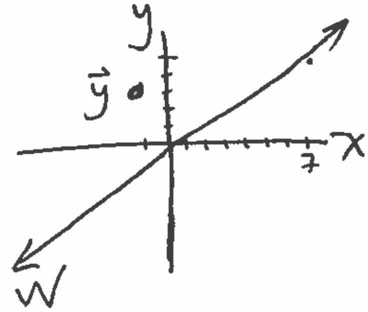
$$[\hat{y}]_\beta = \begin{bmatrix} 4/3 \\ -3/\sqrt{26} \end{bmatrix}$$

So $\hat{y} = \frac{4}{3} \cdot \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} - \frac{3}{\sqrt{26}}\frac{1}{\sqrt{26}}\begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 4/9 \\ 8/9 \\ -8/9 \end{bmatrix} - \begin{bmatrix} 12/26 \\ -9/26 \\ -3/26 \end{bmatrix} = \begin{bmatrix} \text{result} \\ \text{result} \\ \text{result} \end{bmatrix}$$



In $\mathbb{R}^2$, Let $W = \text{Span}\left\{ \begin{bmatrix} 7 \\ 5 \end{bmatrix} \right\}$.



Let $\vec{y} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Find closest point on the line to $\vec{y}$.

$$\beta = \left\{ \frac{1}{\sqrt{74}} \begin{bmatrix} 7 \\ 5 \end{bmatrix} \right\}$$

So... $\vec{b}_1 \cdot \vec{y} = \dfrac{8}{\sqrt{74}}$. So $[\hat{y}]_\beta = \begin{bmatrix} 8/\sqrt{74} \end{bmatrix}$

So $\hat{y} = \dfrac{8}{\sqrt{74}} \cdot \dfrac{1}{\sqrt{74}} \cdot \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \cancel{\begin{bmatrix} 56/\sqrt{74} \\ 5/\sqrt{74} \end{bmatrix}} \begin{bmatrix} 56/74 \\ 40/74 \end{bmatrix}$

An orthonormal set: all vectors one unit long, mutually orthogonal.

When a matrix's columns are an orthonormal set, like $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$, it's called an orthogonal matrix

<u>Fact</u>: If U is an orthogonal matrix,

$$U^T \cdot U = \begin{bmatrix} — \bar{u}_1 — \\ — \bar{u}_2 — \\ \vdots \\ — \bar{u}_n — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & 1 \end{bmatrix} = I_{n \times n}$$

If, additionally, U is <u>square</u>, then $U^{-1} = U^T$.

<u>Ex</u> Find $\begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}^{-1}$.

orthonormal !!

$$= A^T = \begin{bmatrix} 3/\sqrt{11} & 1/\sqrt{11} & 1/\sqrt{11} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{66} & -4/\sqrt{66} & 7/\sqrt{66} \end{bmatrix}$$

<u>Facts</u> $\quad \| U \cdot \vec{x} \| = \| \vec{x} \|$

well, $\| U\vec{x} \| = \sqrt{(U\vec{x}) \cdot (U\vec{x})}$

$$= \sqrt{(U\vec{x})^T (U\vec{x})}$$

$$= \sqrt{\vec{x}^T \cdot \underbrace{U^T U}\, \vec{x}}$$

$$= \sqrt{\vec{x}^T \vec{x}}$$

$$= \sqrt{\vec{x} \cdot \vec{x}}$$

$$\| U\vec{x} \| = \| \vec{x} \|$$

This means orthogonal matrices make "rigid transformations".

Also $\quad (U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$

$$(U\vec{x})^T (U\vec{y})$$

$$\vec{x}^T U^T U \vec{y}$$

$$\vec{x}^T y$$

Fact If U is a square orthogonal matrix... what kind of eigenvalues are possible?

Well, if $\vec{x}$ is an eigenvector for λ...

$$U\vec{x} = \lambda\vec{x}.$$

$$\Longrightarrow \quad \|U\vec{x}\| = \|\lambda\vec{x}\|$$

$$\vec{x} \neq \vec{0}$$

$$\Longrightarrow \quad \|\vec{x}\| = |\lambda| \cdot \|\vec{x}\|$$

$$\|\vec{x}\| \neq 0.$$

$$\Longrightarrow \quad 1 = |\lambda|$$

$$\Longrightarrow \quad \lambda = 1 \text{ or } \lambda = -1.$$

Also $\det(U)$ must be.... 1 or -1.

6.4 Gram-Schmidt orthogonalization process

W is a subspace of $\mathbb{R}^n$... (like Col(A), or Nul(A) or Row(A) or ....)

And you have a basis for W...
but it's not orthonormal....

Plan: swap out given basis with an orthonormal basis.

Ex $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ Find a basis for Col(A).

$$\xrightarrow[\text{RREF}]{} \begin{bmatrix} \boxed{1} & 0 & 0 & +1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis for Col(A)

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

definitely not orthogonal, nor normal...

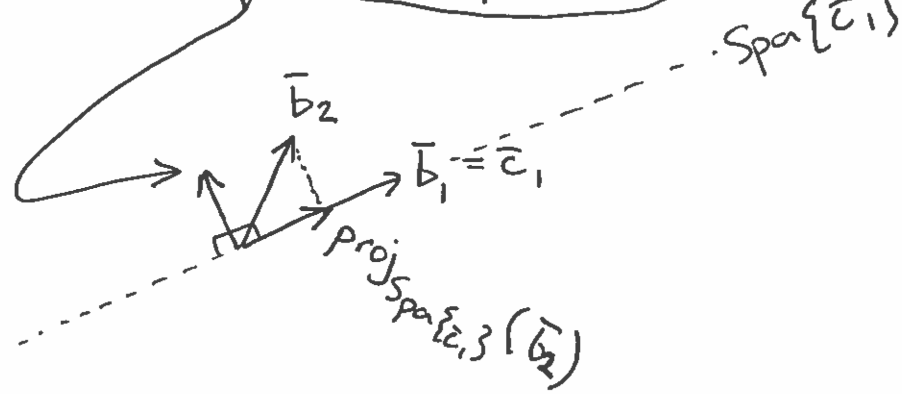
Goal: create $C = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ that is orthonormal.

ignore at first...

Step 1 : Let $\vec{c}_1 = \vec{b}_1$ $\quad C = \left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, ?, ? \right\}$

Now need a $\bar{c}_2$, $\perp$ to $\bar{c}_1$, but also such that $\mathrm{Span}\{\vec{b}_1, \vec{b}_2\} = \mathrm{Span}\{\bar{c}_1, \bar{c}_2\}$ ✓

Consider: $\bar{c}_2 = \vec{b}_2 - \mathrm{proj}_{\mathrm{Span}\{\bar{c}_1\}}(\bar{b}_2)$



To calculate $\mathrm{proj}_{\mathrm{Span}\{\bar{c}_1\}}(\bar{b}_2)$ ... It's $\hat{b}_2$, the closest to $\vec{b}_2$ that is inside $\mathrm{Span}\{\bar{c}_1\}$

Has an orthonormal basis $\left\{ \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}$

$$\bar{b}_2 \cdot \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{3}{2}$$

is the coef of

$$\frac{3}{2} \cdot \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}. \implies \vec{c}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-3/4\\1/4\\1/4\\1/4\end{bmatrix}$$

Now... $C = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}}_{\text{has span equal to } \operatorname{span}\{\vec{b}_1, \vec{b}_2\}.}, ? \right\}$

How'd we get $\vec{c}_2$?

$$\vec{c}_2 = \vec{b}_2 - \operatorname{proj}_{\operatorname{Span}\{\vec{c}_1\}}(\vec{b}_2)$$

$$\vec{c}_2 = \vec{b}_2 - \left(\vec{b}_2 \cdot \frac{\vec{c}_1}{\|\vec{c}_1\|}\right) \cdot \frac{\vec{c}_1}{\|\vec{c}_1\|}$$

$$\vec{c}_2 = \vec{b}_2 - \frac{\vec{b}_2 \cdot \vec{c}_1}{\vec{c}_1 \cdot \vec{c}_1} \cdot \vec{c}_1$$

Lastly for $\vec{c}_3$...

$$\vec{c}_3 = \vec{b}_3 - \operatorname{proj}_{\operatorname{Span}\{\vec{c}_1, \vec{c}_2\}}(\vec{b}_3)$$

$$\vec{c}_3 = \vec{b}_3 - \left(\left(\vec{b}_3 \cdot \frac{\vec{c}_1}{\|\vec{c}_1\|}\right) \frac{\vec{c}_1}{\|\vec{c}_1\|} + \left(\vec{b}_3 \cdot \frac{\vec{c}_2}{\|\vec{c}_2\|}\right) \frac{\vec{c}_2}{\|\vec{c}_2\|} \right)$$

$$\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$\vec{c}_3 = \vec{b}_3 - \left(\frac{\vec{b}_3 \cdot \vec{c}_1}{\vec{c}_1 \cdot \vec{c}_1} \vec{c}_1 + \frac{\vec{b}_3 \cdot \vec{c}_2}{\vec{c}_2 \cdot \vec{c}_2} \cdot \vec{c}_2 \right)$$

$$\vec{c}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1/2}{12/16} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \right) = \cdots = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

So we have $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}, \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \right\}$

$\qquad\qquad\qquad\quad \bar{c}_1 \qquad\quad \bar{c}_2 \qquad\quad \bar{c}_3$

is a basis for Col(A) just like

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Last step: normalize: $\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12/16}}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}, \frac{1}{\sqrt{6/9}}\begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \right\}$

is an orthonormal basis for Col(A).

$\underline{\text{Ex}}$ In $\mathbb{R}^4$, consider Nul(A) where
lives in domain

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}.$$

Find an orthonormal basis for Nul(A).

RREF

First, a plain basis:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -5 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & -1/5 & 2/5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 3/5 & -1/5 \\ 0 & 1 & -1/5 & 2/5 \end{bmatrix}$$

x_4 free... ~~take to be~~

x_3 free

$$x_2 = \frac{1}{5}x_3 - \frac{2}{5}x_4$$

$$x_1 = -\frac{3}{5}x_3 + \frac{1}{5}x_4$$

$$\vec{x} = \begin{bmatrix} -\frac{3}{5}x_3 + \frac{1}{5}x_4 \\ \frac{1}{5}x_3 - \frac{2}{5}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3/5 \\ 1/5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1/5 \\ -2/5 \\ 0 \\ 1 \end{bmatrix}$$

A basis for Nul(A) is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix} \right\}$.

Start building C ...

$$\bar{c}_1 = \vec{b}_1 = \begin{bmatrix}-3\\1\\5\\0\end{bmatrix}$$

$$\bar{c}_2 = \bar{b}_2 - \text{proj}_{\text{spn(previous } \bar{c}\text{'s)}}(\bar{b}_2)$$

$$= \vec{b}_2 - \frac{\bar{b}_2\cdot\bar{c}_1}{\bar{c}_1\cdot\bar{c}_1}\cdot\bar{c}_1 = \begin{bmatrix}1\\-2\\0\\5\end{bmatrix} - \frac{\begin{bmatrix}-3\\1\\5\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\0\\5\end{bmatrix}}{\begin{bmatrix}-3\\1\\5\\0\end{bmatrix}\cdot\begin{bmatrix}-3\\1\\5\\0\end{bmatrix}}\begin{bmatrix}-3\\1\\5\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\-2\\0\\5\end{bmatrix} \cancel{-\frac{-8}{35}} + \frac{1}{7}\begin{bmatrix}-3\\1\\5\\0\end{bmatrix}$$

$$= \begin{bmatrix}4/7\\-13/7\\5/7\\5\end{bmatrix} \quad \text{So} \quad C = \left\{\begin{bmatrix}-3\\1\\5\\0\end{bmatrix}, \begin{bmatrix}4/7\\-13/7\\5/7\\5\end{bmatrix}\right\}$$

normalize

welcome to replace with $\begin{bmatrix}4\\-13\\5\\35\end{bmatrix}$

$$\left\{\frac{1}{\sqrt{35}}\begin{bmatrix}-3\\1\\5\\0\end{bmatrix}, \frac{1}{\sqrt{1435}}\begin{bmatrix}4\\-13\\5\\35\end{bmatrix}\right\}$$