Ch 6: Inner Products & Orthogonality

6.1 Inner Products

An "inner product" on a vector space is a way to "multiply" two vectors and the result is a number in $\mathbb{R}$.

The dot product is one example of an inner product

For $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

or...

$$\vec{u} \cdot \vec{v} = [u_1 \; u_2 \cdots u_n] \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

"dot"

regular matrix mult.

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

Ex $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix} = 1 \cdot 3 + 2(-4) = -5$

Facts about dot product

* $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
* $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c \cdot (\vec{u} \cdot \vec{v})$
* $\vec{u} \cdot \vec{u} \geq 0$ (and $\vec{u} \cdot \vec{u} = 0$ only when $\vec{u} = \vec{0}$)

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

Well, an <u>inner product</u> on a vector space V is denoted $\langle \vec{u}, \vec{v} \rangle$ (looks like an ordered pair, but really represents a single number.)

where

* $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
* $\langle c\vec{u}, \vec{v} \rangle = \langle \vec{u}, c\vec{v} \rangle = c \cdot \langle \vec{u}, \vec{v} \rangle$
* $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$.

<u>Ex</u> Define $\langle \vec{u}, \vec{v} \rangle$ on $\mathbb{R}^2$ by $\underbrace{\vec{u}^T}_{1\times 2} \cdot \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}}_{2\times 2} \underbrace{\vec{v}}_{2\times 1}$

Is this an inner product?

<u>Ex</u> $\left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ -14 \end{bmatrix} = -33$

$\langle \vec{u}, \vec{v}+\vec{w} \rangle = \vec{u}^T A (\vec{v}+\vec{w})$

$= \vec{u}^T A \vec{v} + \vec{u}^T A \cdot \vec{w}$

$= \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ ✓

$\langle c\vec{u}, \vec{v} \rangle = (c\vec{u})^T \cdot A \cdot \vec{v} = \vec{u}^T A (c\vec{v}) = c \cdot (\vec{u}^T A \vec{v})$ ✓

$= \langle \vec{u}, c\vec{v} \rangle = c \cdot \langle \vec{u}, \vec{v} \rangle$

$\langle \vec{u}, \vec{u} \rangle = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} u_1 + 2u_2 \\ 2u_1 + 5u_2 \end{bmatrix} = u_1^2 + 4u_1u_2 + 5u_2^2$

$$\langle \vec{u}, \vec{u} \rangle = \cdots = u_1^2 + 4u_1u_2 + 5u_2^2$$
$$= (u_1^2 + 4u_1u_2 + 4u_2^2) + u_2^2$$
$$= (u_1 + 2u_2)^2 + u_2^2 \geq 0 \checkmark$$

And $\langle \vec{u}, \vec{u} \rangle = 0 \implies u_2 = 0 \implies \vec{u}_1 = 0$.

So $\vec{u}$ would have to be $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

<u>Fact</u> If $\langle *, * \rangle$ is an inner product on $\mathbb{R}^n$, Build $A = \begin{bmatrix} \langle \bar{e}_1, \bar{e}_1 \rangle & \langle \bar{e}_1, e_2 \rangle & \cdots \\ \langle \bar{e}_2, \bar{e}_1 \rangle & \langle \bar{e}_2, \bar{e}_2 \rangle & \\ \vdots & \vdots & \ddots \end{bmatrix}$

Then $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \cdot A \cdot \vec{v}$.

(Every inner product on $\mathbb{R}^n$ is basically $\vec{u}^T \cdot A \cdot \vec{v}$.)

<u>Fact</u> If you have a symmetric A, and all of A's eigenvalues are positive, then $\vec{u}^T A \cdot \vec{v}$ <u>will</u> be an inner product.

Ex Let $V = C^{\infty}_{[0,1]}(\mathbb{R}) = \left\{ \begin{array}{l} \text{all functions from } [0,1] \text{ to } \mathbb{R} \\ \text{that are differentiable any number of times} \end{array} \right\}$

this is a vector space

has the same features as $\mathbb{R}^n$

vectors can still be added, scaled, ...

Look for an inner product on V.

Need a way to combine two functions and get a number...

$$\text{Define } \langle f, g \rangle = \int_0^1 f(x) \cdot g(x) \, dx$$

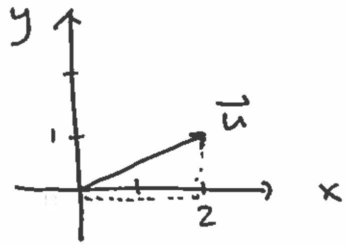
Does this meet inner product criteria?

$$\begin{aligned} \langle f, g+h \rangle &= \int_0^1 f(x)\big(g(x) + h(x)\big) \, dx \\ &= \int_0^1 \big[f(x) \cdot g(x) + f(x) \cdot h(x) \big] \, dx \\ &= \int_0^1 f(x) g(x) \, dx + \int_0^1 f(x) h(x) \, dx \\ &= \langle f, g \rangle + \langle f, h \rangle \quad \checkmark \end{aligned}$$

Other two rules $\checkmark$

Length and Norm

Length of a vector in $\mathbb{R}^n$...



How "long" is $\vec{u}$?

$$\|\vec{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\|\vec{u}\| = \sqrt{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$= \sqrt{\vec{u}^T \cdot \vec{u}}$$

Definition of Length

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

Ex $\left\| \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right\| = \sqrt{\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix}} = \sqrt{9 + 16} = \sqrt{25} = 5$

Define the norm of $\vec{u}$: $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

Ex In $\mathbb{R}^3$, define $\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + 2 u_2 v_2 + 3 u_3 v_3$

Find $\left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\|$ using this inner product. $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

$$= \sqrt{1 \cdot 1 + 2 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot 1} = \sqrt{6}$$

It turns out...

$$\|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos(\underset{\substack{\uparrow \\ \text{angle} \\ \text{between } u \,\&\, v.}}{\theta}) = \vec{u}\cdot\vec{v} = \text{what we've defined it as}$$

$$\Longrightarrow \cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

Ex In $\mathbb{R}^3$ with the dot product, find angle between $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\0\end{bmatrix}$.

$$\cos(\theta) = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\\0\end{bmatrix}\right\|\cdot\left\|\begin{bmatrix}1\\1\\0\end{bmatrix}\right\|}$$

$$\Longrightarrow \cos(\theta) = \frac{1}{1\cdot\sqrt{2}} = \frac{1}{\sqrt{2}} \Longrightarrow \theta = \frac{\pi}{4}.$$

But In $\mathbb{R}^3$ with $\langle\vec{u},\vec{v}\rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3 = \vec{u}^T\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\vec{v}$

Find the "angle" between $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\0\end{bmatrix}$.

$$\cos(\theta) = \frac{\left\langle\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\rangle}{\left\|\begin{bmatrix}1\\0\\0\end{bmatrix}\right\|\cdot\left\|\begin{bmatrix}1\\1\\0\end{bmatrix}\right\|} = \frac{1\cdot1 + 2\cdot0\cdot1 + 3\cdot0\cdot0}{\sqrt{\left\langle\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix}\right\rangle}\cdot\sqrt{\left\langle\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\rangle}}$$

norm

$$= \frac{1}{\sqrt{1\cdot1+2\cdot0\cdot0+3\cdot0\,0}\;\sqrt{1\cdot1+2\cdot1\cdot1+3\cdot0\cdot0}} = \frac{1}{\sqrt{3}}$$

$$\cos(\theta) = \frac{1}{\sqrt{3}} \Longleftrightarrow \theta = \arccos\left(\frac{1}{\sqrt{3}}\right)$$

Fact

$$\|c \cdot \vec{u}\| = \sqrt{\langle c\vec{u}, c\vec{u}\rangle}$$

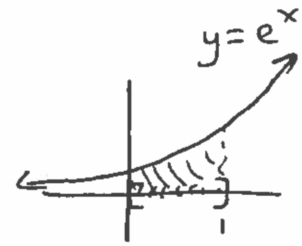
length or norm

$$= \sqrt{c^2 \cdot \langle \vec{u}, \vec{u}\rangle}$$

$$= |c| \sqrt{\langle \vec{u}, \vec{u}\rangle}$$

$$= |c| \cdot \|\vec{u}\|$$

Ex Recall $V = C^{\infty}_{[0,1]}(\mathbb{R})$ where



$$\langle f, g\rangle = \int_0^1 f(x) \cdot g(x)\, dx.$$

Take $f(x) = e^x$. Find $\|e^x\|$.

$$\|e^x\| = \sqrt{\langle e^x, e^x\rangle}$$

$$= \sqrt{\int_0^1 e^x \cdot e^x \cdot dx} = \sqrt{\int_0^1 e^{2x}\, dx}$$

$$= \sqrt{\left[\tfrac{1}{2} \cdot e^{2x}\right]_0^1} = \sqrt{\tfrac{1}{2} e^2 - \tfrac{1}{2}}$$

So $\|e^x\| = \sqrt{\frac{1}{2} e^2 - \frac{1}{2}}$

Ex Let $V = C^{\infty}_{[0,1]}$. Propose

$$\langle f, g \rangle = \int_0^1 f'(x) \cdot g(x)\, dx$$

differentiated....

Is this an inner product?

Consider $\langle f, f \rangle = \int_0^1 f'(x) \cdot f(x)\, dx$

(which we need to be ≥ 0...)

For example e^{-x}.

Then $f'(x) = -e^{-x}$.

So we don't have an inner product.

$$\text{Then } \int_0^1 -e^{-x} \cdot e^{-x}\, dx = \int_0^1 -e^{-2x}\, dx$$

$$= -\left[\tfrac{1}{-2} e^{-2x}\right]_0^1 = \tfrac{1}{2} e^{-2} - \tfrac{1}{2} e^{0}$$

$$= \tfrac{1}{2} e^{-2} - 1$$

is negative!

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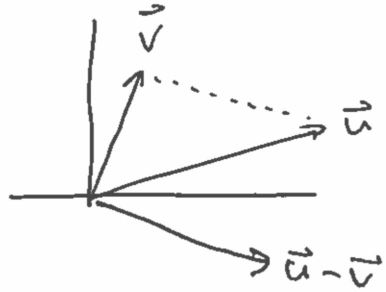
distance between $\vec{u}$ & $\vec{v}$.



$$\text{distance}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

norm

Ex In $\mathbb{R}^3$ with dot product,

$$\operatorname{dist}\left(\begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}-2\\-2\\1\end{bmatrix}\right) = \left\| \begin{bmatrix}2\\1\\3\end{bmatrix} - \begin{bmatrix}-2\\-2\\1\end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix}4\\3\\2\end{bmatrix} \right\| = \sqrt{\begin{bmatrix}4\\3\\2\end{bmatrix} \cdot \begin{bmatrix}4\\3\\2\end{bmatrix}}$$

$$= \sqrt{16+9+4} = \sqrt{29}$$

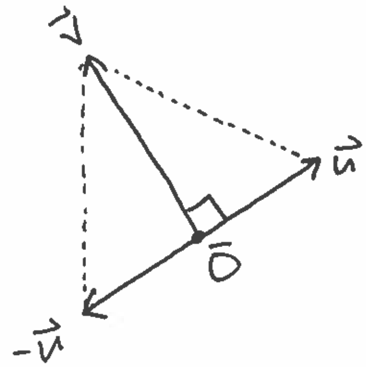
Orthogonality:

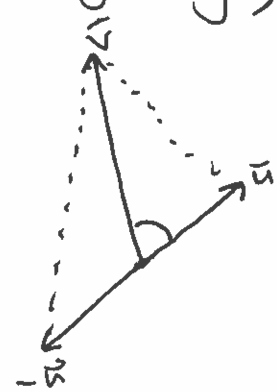
Ortho — straight

gon — angle, corner

Take $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ with the dot product...

If $\vec{u}, \vec{v}$ have a right angle,





$\vec{u}, \vec{v}$ have a right angle ($\vec{u}$ is orthogonal to $\vec{v}$) $\iff \operatorname{dist}(\vec{u}, \vec{v}) = \operatorname{dist}(-\vec{u}, \vec{v})$

So $\vec{u}, \vec{v}$ orthogonal $\implies \|\vec{u} - \vec{v}\| = \|-\vec{u} - \vec{v}\|$

$$\implies \sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})} = \sqrt{(-\vec{u}-\vec{v})\cdot(-\vec{u}-\vec{v})}$$

$$\implies (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$

$$\Rightarrow (\vec{u}-\vec{v})\cdot\vec{u} - (\vec{u}-\vec{v})\cdot\vec{v} = (\vec{u}+\vec{v})\cdot\vec{u} + (\vec{u}+\vec{v})\cdot\vec{v}$$

$$\Rightarrow \underset{\sim}{\vec{u}\cdot\vec{u}} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \underline{\underline{\vec{v}\cdot\vec{v}}} = \underset{\sim}{\vec{u}\cdot\vec{u}} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \underline{\underline{\vec{v}\cdot\vec{v}}}$$

$$-2\vec{u}\cdot\vec{v} = 2\vec{u}\cdot\vec{v}$$

$$\Rightarrow -4\vec{u}\cdot\vec{v} = 0$$

$\vec{u}, \vec{v}$ are orthogonal $\quad \Rightarrow \vec{u}\cdot\vec{v} = 0$

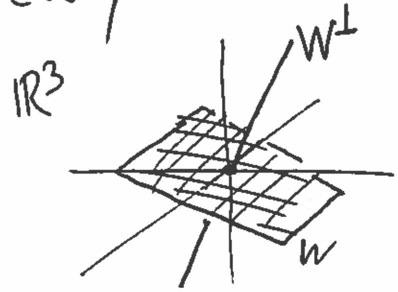
(goes both ways)

<u>Ex</u> Are $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ orthogonal?

Well $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = -6 + 4 + 2 = 0$ ✓ Yes!

Let W be a subspace of $\mathbb{R}^n$.

Consider all the vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W... we call this $W^\perp$



"W perp", "The orthogonal complement of W,"

Fact For any **set** W in $\mathbb{R}^n$,

$W^\perp$ (collection of all vectors $\perp$ to all the vectors in W)

is a subspace of $\mathbb{R}^n$.

To be a subspace ...

when $\vec{v} \in W^\perp$, $c \cdot \vec{v}$ needs to be in it too.

$\Downarrow$

$\vec{v} \cdot \vec{w} = 0$ for all $\vec{w}$ in W. $\implies$ $c \cdot \vec{v} \cdot \vec{w} = 0$ for all $\vec{w}$ in W

$\Downarrow$

So $c\vec{v}$ is in $W^\perp$ too.

when $\vec{u}, \vec{v}$ are $W^\perp$,

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$
$$= 0 + 0 = 0$$

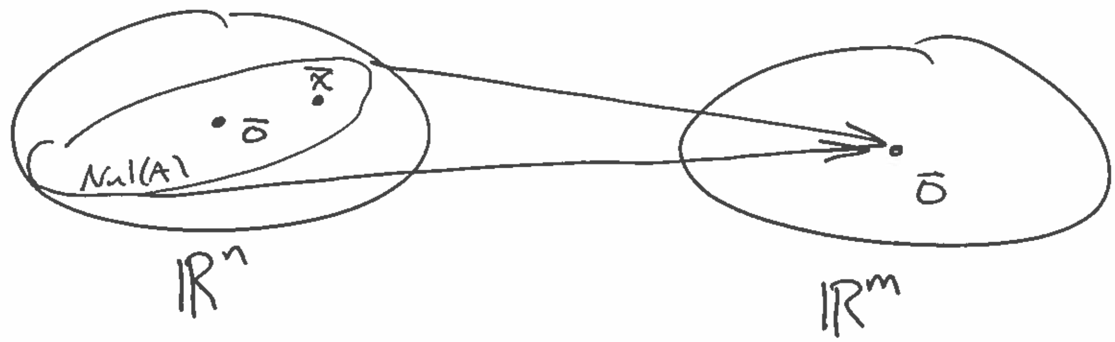
$\Rightarrow$ $\vec{u} + \vec{v}$ is in $W^\perp$ too.

Ex Let $W =$ yz-plane in $\mathbb{R}^3$.

Then $W^\perp$ is the x-axis.

Suppose A is a $m \times n$ matrix...

And $\vec{x}$ is in $\text{Nul}(A)$. $\left(A \cdot \vec{x} = \vec{0}\right)$



$$A \cdot \vec{x} = \vec{0}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

A's first row $\cdot \, \vec{x} = \vec{0}$

A's first row & $\vec{x}$ are $\perp$

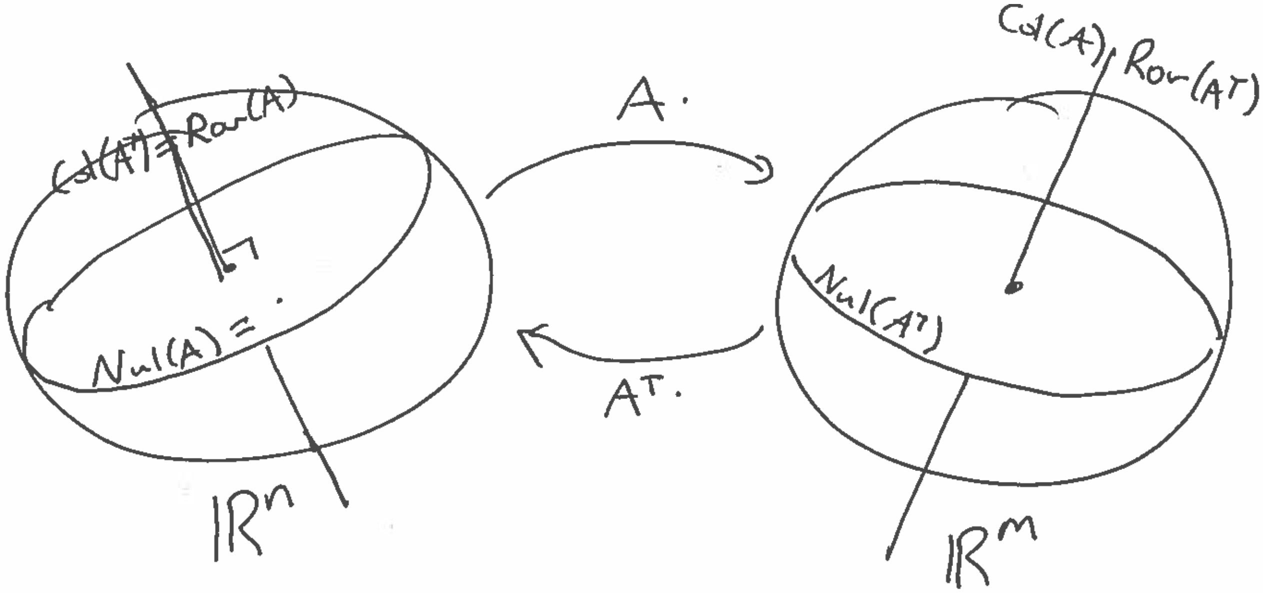
$\vec{x}$ is $\perp$ to all of A's rows...

$\vec{x}$ is $\perp$ to the span of A's rows...

$\vec{x}$ is $\perp$ $\text{Row}(A) = \text{Col}(A^T)$

$$\Longrightarrow \text{Nul}(A) = \text{Row}(A)^{\perp} = \text{Col}(A^T)^{\perp}$$

Consequence: $\text{Row}(A^T) = \text{Col}(A) = \text{Nul}(A^T)^{\perp}$



Ex Let $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$. Find $W^\perp$.

(Find a basis for $W^\perp$.)

$\vec{x} \in W^\perp \implies \vec{w} \cdot \vec{x} = 0$ for all $\vec{w}$ in W.

$$c \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \vec{x} = 0$$

$\implies c\left(2x_1 + x_2 - x_3\right) = 0$ for all c

$\implies 2x_1 + x_2 - x_3 = 0$

Same task as finding a basis for sol. set to

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \end{array}\right]$$

x_3 free

x_2 free

$x_1 = -\frac{1}{2}x_2 + \frac{1}{2}x_3$

$$\begin{bmatrix} -\frac{1}{2}x_2 + \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$

So a basis for $W^\perp$ is $\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Ex Let $V = \mathbb{R}^4$. Consider $\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Show this is an orthogonal set.

Compute all dot products:

$$\langle 0,2,1,-1\rangle \cdot \langle 1,3,-6,0\rangle = 0+6-6+0 = 0$$
$$\langle 0,2,1,-1\rangle \cdot \langle -9,1,-1,1\rangle = 0+2-1-1 = 0$$
$$\langle 1,3,-6,0\rangle \cdot \langle -9,1,-1,1\rangle = -9+3+6+0 = 0$$

An orthonormal set is an orthogonal set, where each vector has norm 1 (magnitude)

Ex Rescale the above to get an orthonormal set.

$$\left\| \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\| = \sqrt{\begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}} = \sqrt{0+4+1+1} = \sqrt{6} \rightsquigarrow \begin{bmatrix} 0 \\ 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$ is one unit long.

$$\ldots \rightarrow \left\{ \begin{bmatrix} 0 \\ 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{46} \\ 3/\sqrt{46} \\ -6/\sqrt{46} \\ 0 \end{bmatrix}, \begin{bmatrix} -9/\sqrt{84} \\ 1/\sqrt{84} \\ -1/\sqrt{84} \\ 1/\sqrt{84} \end{bmatrix} \right\}$$

Fact An orthogonal set is automatically linearly independent.

~~Pf~~ Suppose $c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_k\bar{v}_k = \vec{0}$

$$\bar{v}_7 \cdot (c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_k\bar{v}_k) = \bar{v}_7 \cdot \vec{0}$$

$$0 + 0 + \cdots c_7 \cdot \bar{v}_7 \cdot \bar{v}_7 + 0 \cdots + 0 = 0$$

$$c_7 \cdot \underbrace{\bar{v}_7 \cdot \bar{v}_7}_{\text{def not } 0 \text{ if } \bar{v}_7 \neq \vec{0}} = 0$$

$$\Longrightarrow c_7 = \frac{0}{\bar{v}_7 \cdot \bar{v}_7} = 0$$

So all c_i would have to be 0.

So $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ is linearly indep.

Fact When $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ is orthogonal, Automatic that $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ is a basis for $\mathrm{Span}\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$

Better: Suppose H has a basis $\mathcal{B} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$, then when $\mathcal{B}$ is also an <u>orthonormal</u> set, calculating with $\mathcal{B}$ is easy!

~~Ex~~ $\cancel{\mathcal{B} = \left\{ \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix} \right\}}$ ~~is an ortho~~

Ex $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} \right\}$.

- orthogonal ✓
- orthonormal ?

no....

$\sqrt{20} = 2\sqrt{5}$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix} \right\}$$

is an orthormal set.

Let $H = \mathrm{Span}(\mathcal{B})$. Take $\vec{x} = \begin{bmatrix} 9 \\ -2 \\ 3 \end{bmatrix}$.

We promise $\vec{x}$ is in H....

Calculate $[\vec{x}]_{\mathcal{B}}$. Try to find $c_1 \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} + c_2 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1/\sqrt{14} & 2/\sqrt{5} & 9 \\ 2/\sqrt{14} & -1/\sqrt{5} & -2 \\ 3/\sqrt{14} & 0 & 3 \end{array}\right] \xrightarrow{\text{RREF}}$$

To find c_1, c_2...

Instead, to find c_1, c_2

$$c_1 \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} + c_2 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ 3 \end{bmatrix}$$

Because of having an o'normal basis

Dot with this...

$$c_1 \cdot 1 + c_2 \cdot 0 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \cdot \begin{bmatrix} 9 \\ -2 \\ 3 \end{bmatrix}$$

$$c_1 = \frac{14}{\sqrt{14}} = \sqrt{14}$$

Dot with $\bar{v}_2$

$$c_1 \cdot 0 + c_2 \cdot 1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ -2 \\ 3 \end{bmatrix} = \frac{20}{\sqrt{5}}$$

$$c_2 = \frac{20}{\sqrt{5}}$$

Punchline: $$[\vec{x}]_\beta = \begin{bmatrix} \sqrt{14} \\ 20/\sqrt{5} \end{bmatrix}.$$

Orthonormal bases are valuable!

Ex $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$ This A is "symmetric" $(A^T = A)$

Diagonalize A...

$$\det\left(\begin{bmatrix} 3-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix}\right) = 0$$

$$(3-\lambda)^2 - 16 = 0$$

$$\lambda^2 - 6\lambda - 7 = 0$$

$$(\lambda-7)(\lambda+1) = 0$$

$\lambda = 7$

$\lambda = -1$

$$D = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$$

$\lambda = 7$

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

take $x_2 = 1$

$x_1 = x_2 = 1$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -1$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_2 = 1$

$x_1 = -x_2 = -1$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Observe $\perp$

with a symmetric matrix,
it's always orthogally diagonlizable.

($\mathbb{R}^n$ has a basis of orthogonal eigenvectors.)

we've found $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^2$...

make it orthonormal!

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}.$$

$$\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\text{an orthogonal matrix}} \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{-1}$$

an orthogonal matrix.
(a matrix with orthonormal matrix)

when P is orthogonal,
$$P^{-1} = P^T$$