4.7 Change of basis matrix.

Given a subspace H of $\mathbb{R}^n$, where you have two bases for H: $\mathcal{B}$ and $\mathcal{C}$

$$\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_p\} \qquad \mathcal{C} = \{\vec{c}_1, \dots \vec{c}_p\}$$

When $\vec{x}$ is in H... $\vec{x}$, an n-entried vector

$[\vec{x}]_{\mathcal{B}}$ has p entries...

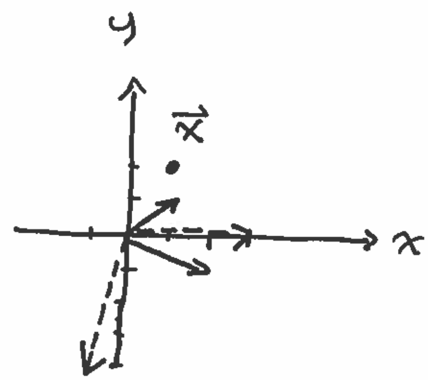
$[\vec{x}]_{\mathcal{C}}$ has p entries...

$$[\vec{x}]_{\mathcal{B}} \xrightarrow[\text{a } p \times p \text{ matrix (square)}]{\text{some matrix}} [\vec{x}]_{\mathcal{C}}$$

Ex $H = \mathbb{R}^2$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \end{bmatrix} \right\}$$



Take $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $[\vec{x}]_{\mathcal{B}}$.

well $\underset{std \leftarrow \mathcal{B}}{\begin{bmatrix} P \end{bmatrix}} [\vec{x}]_{\mathcal{B}} = \vec{x}$

$\Longrightarrow \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & -1 & 2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 5/3 \\ 0 & 1 & -1/3 \end{array}\right].$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix}$$

Find $[\vec{x}]_{\mathcal{C}}$.

$$\underset{std \leftarrow \mathcal{C}}{\begin{bmatrix} & P & \end{bmatrix}} [\vec{x}]_{\mathcal{C}} = \vec{x}$$

$$\begin{bmatrix} 3 & -1 \\ 0 & -4 \end{bmatrix} [\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 0 & -4 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1/6 \\ 0 & 1 & -1/2 \end{array}\right]$$

So $[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 1/6 \\ -1/2 \end{bmatrix}$

So this section is about finding a matrix that:

$$\begin{tikzcd} {[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix}} \arrow[r, bend left, "P."] & {\begin{bmatrix} 1/6 \\ -1/2 \end{bmatrix} = [\vec{x}]_{\mathcal{C}}} \arrow[l, bend left, "\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}"] \end{tikzcd}$$

We're looking for $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ where $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$

How to calculate this?

Think through each $\vec{c}_i \in \mathcal{C}$.

and calculate $[\vec{c}_i]_{\mathcal{B}}$.

$[\vec{c}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$[\vec{c}_2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Build $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} [\vec{c}_1]_{\mathcal{B}} & [\vec{c}_2]_{\mathcal{B}} & \cdots & [\vec{c}_p]_{\mathcal{B}} \end{bmatrix}$.

will turn $[\vec{c}_i]_{\mathcal{C}}$ into $[\vec{c}_i]_{\mathcal{B}}$

To calculate $[\bar{c}_i]_\beta$?

want $\vec{c}_i = \underset{?}{x_1}\vec{b}_1 + \underset{?}{x_2}\vec{b}_2 + \cdots + \underset{?}{x_p}\vec{b}_p$

we know ✓ ✓ ✓ ✓

In our example, $\vec{c}_1$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & -1 & 0 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -3 & -3 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\begin{matrix} x_1 = 1 \\ x_2 = 1 \end{matrix} \Longrightarrow [\vec{c}_1]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now for $\bar{c}_2$

$$\begin{bmatrix} -1 \\ -4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 1 & -1 & -4 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -3 & -3 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \quad \text{So } [\bar{c}_2]_\beta = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

$$\text{So } \underset{\beta \leftarrow C}{P} = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}.$$

$$\cancel{\text{Check: } \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 5/3 + 3/3 \\ \; \end{bmatrix}}$$

Check $\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/6 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/6 + 3/2 \\ 1/6 - 1/2 \end{bmatrix}$

$= \begin{bmatrix} 10/6 \\ -2/6 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \end{bmatrix}$ ✓

Ex H is a plane in $\mathbb{R}^3$.

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} \right\}$.

are both bases of H.

Find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$. taking each $\vec{c}_i$ and finding its $\mathcal{B}$-coordinate vector.

$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 8 \\ 1 & 0 & 1 & 3 \end{array}\right]$

ending basis vectors

starting basis vectors

RREF

$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 1 \\ 0 & -3 & -3 & 6 \\ 0 & -1 & -1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

this is... $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$

Use $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$. Would feed it a $\mathcal{C}$-coordinate vec.

$[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. $\vec{x} = \begin{bmatrix} 11 \\ 43 \\ 18 \end{bmatrix} \leftarrow 3\vec{c}_1 + 5\vec{c}_2$

Then $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 18 \\ -7 \end{bmatrix}$ ✓

$18\vec{b}_1 - 7\vec{b}_2$

$\underset{\beta \leftarrow C}{P}$ is ~~"the change of matrix from C to β".~~

"the change of basis matrix from C to β."

How to find $\underset{\beta \leftarrow C}{P}$:

$$\left[\begin{array}{c|c} \beta & C \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{p} & \overbrace{\underset{\beta \leftarrow C}{P}}^{p} \\ 0 & 0 \end{array}\right]$$

Fact: $\underset{C \leftarrow \beta}{P}$ either swap β & C here

or... $\underset{C \leftarrow \beta}{P} = \left(\underset{\beta \leftarrow C}{P}\right)^{-1}$

$\boxed{5.1}$ Eigenvectors and eigenvalues for a particular matrix A...

German: "special"

Ex
$$\begin{bmatrix} 35 & -42 \\ 20 & -23 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 35 \\ 25 \end{bmatrix}$$

!!!

this is

$$5 \cdot \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

A matrix A turned a vector into a parallel version of itself...

__Definitions__ Given an $n \times n$ matrix A, if you have a __nonzero__ vector $\vec{x}$ where

$$A \cdot \vec{x} = \lambda \cdot \vec{x}$$

(A turns $\vec{x}$ into a __parallel__ version of itself.)

λ: a scalar

we call $\vec{x}$ an __eigenvector__ of A with __eigenvalue__ λ.

__Ex__ $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an ~~eigenvect~~ eigenvector?

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

nope.

Is $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ an eigenvector for A?

Well, $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \cdots = \underbrace{\begin{bmatrix} -24 \\ 20 \end{bmatrix}}_{-4 \cdot \begin{bmatrix} 6 \\ -5 \end{bmatrix}}$

So yeah! $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector with eigenvalue -4.

Is -4 an eigenvalue? Yes

Is 5 an eigenvalue?

$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \cdot \vec{x} = 5 \cdot \vec{x}$ would have to have a nontrivial solution.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \cdot \vec{x} = 5 \cdot I \vec{x} \longrightarrow \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \vec{x} \Longrightarrow \begin{bmatrix} -4 & 6 \\ 5 & -3 \end{bmatrix} \vec{x} = \vec{0}$$

Since $\begin{bmatrix} -4 & 6 \\ 5 & -3 \end{bmatrix} \xrightarrow{\text{RREF's}}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, no nontrivial

Solutions... So 5 is not an eigenvalue

Is 7 an eigenvalue for $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \vec{x} = 7\vec{x}$ (Are there nontrivial solutions?)

$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \vec{x}$

$\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \vec{x} = \vec{0}$ $\xrightarrow{\text{RREF}}$ $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x_1 = x_2$

x_2 free

So.... $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

has nontrivial solutions, so 7 is an eigenvalue. And $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector...

Ex Is $\underbrace{\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}}_{\vec{v}}$ an eigenvector for $\underbrace{\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}}_{A}$?

Calculate: $A \cdot \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

this is $0 \cdot \vec{v}$

Yes. $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is an eigenvector, with eigenvalue 0.

Ex Is 3 an eigenvalue for $\overbrace{\begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ 3 & 3 & 3 \end{bmatrix}}^{A}$?

(Is there a nontrivial solution to $A\vec{x} = \underbrace{3\vec{x}}_{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\vec{x}}$?)

$\Longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix} \vec{x} = \vec{0}$ $\xrightarrow{\text{start RREF}}$ $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{bmatrix}$

we will have a free column.

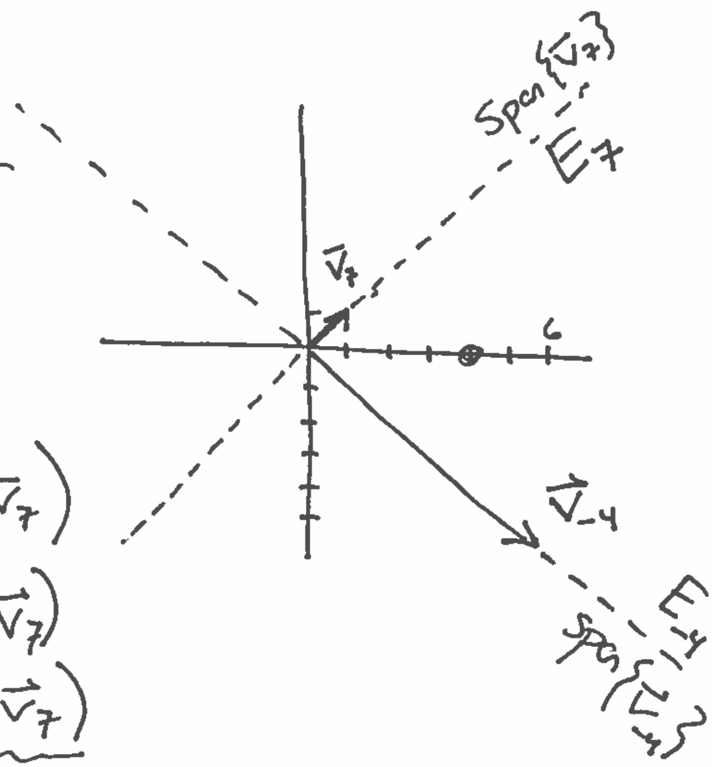
There will be nontrivial solution so 3 is an eigenvalue.

Ex $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. We found that -4 and 7 are eigenvalues for this matrix.

(it turns out there are no more...)

For -4, an eigenvector was $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$

For 7 an eigenvector was $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Observe: $A(\underbrace{c\cdot\vec{v}_7}) = c\cdot(A\vec{v}_7)$

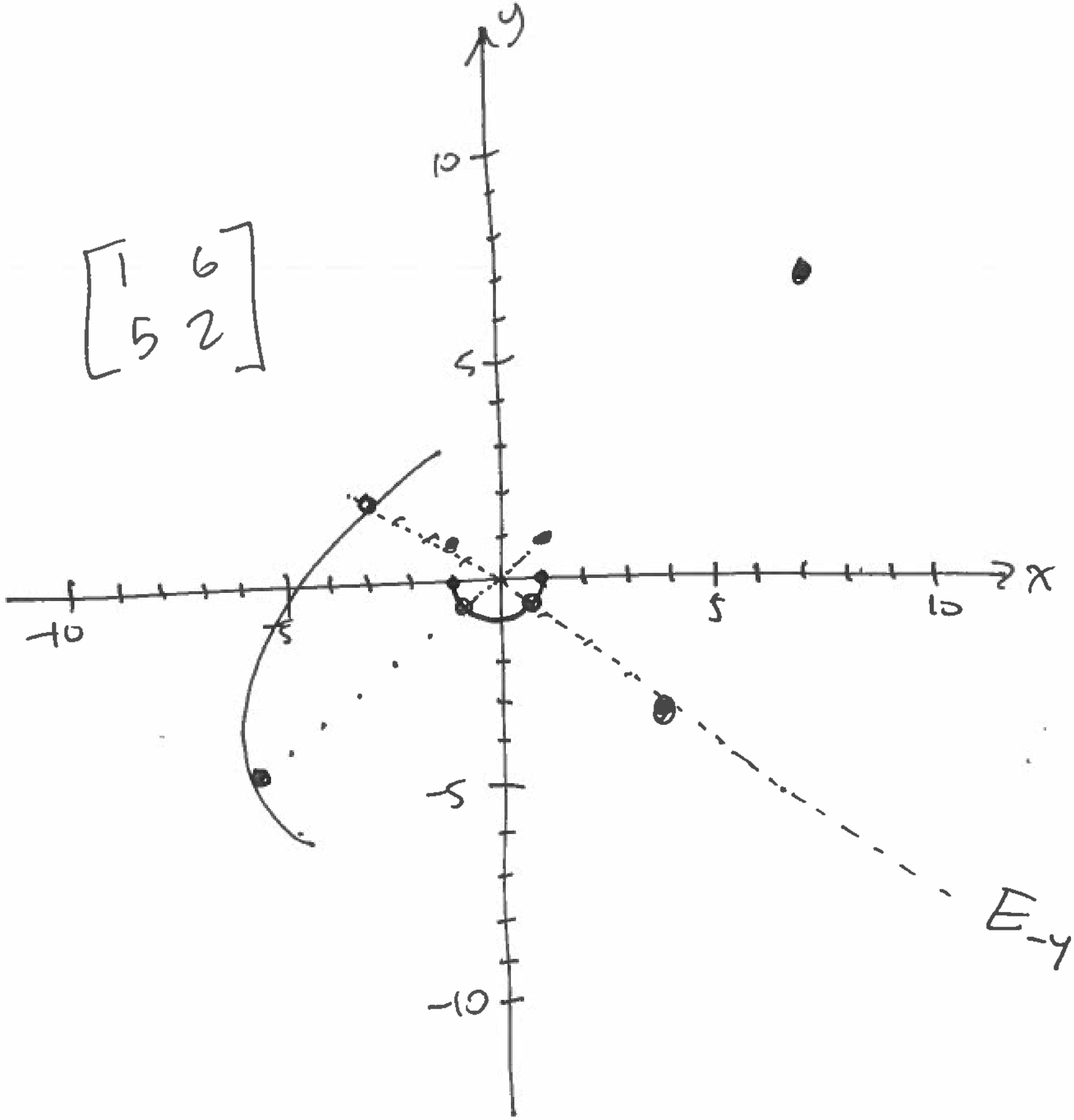
$= c\cdot(7\cdot\vec{v}_7)$

$= 7\,(\underbrace{c\vec{v}_7})$

Tells us scalar multiples of $\vec{v}_7$ are <u>also</u> eigenvectors of eigenvalue 7.

Define: An <u>eigenspace</u> of eigenvalue λ is the ~~[illegible]~~ Span of all eigenvectors of eigenvalue λ.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$



5.2 Characteristic Polynomial of a matrix.

<u>Ex</u> $\underbrace{\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}}_{A}$ .... Find its eigenvalues.

<u>If</u> λ is an eigenvalue, $A\vec{x} = \lambda\vec{x}$ has a nontrivial solution.

So $A\vec{x} - \lambda\vec{x} = \vec{0}$ has a nontrivial solution.

$A\vec{x} - \underbrace{\lambda \cdot I}\cdot\vec{x} = \vec{0}$

$\underbrace{(A - \lambda\cdot I)}\cdot\vec{x} = \vec{0}$. again, where $\vec{x}$ is nonzero...

what can we say about $A-\lambda I$ based on it annihilating some nonzero $\vec{x}$?

$A - \lambda I$ is not invertible

$\Longrightarrow \det(A - \lambda I) = 0$.

So... $\det\left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\left(\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}\right) = 0$

$(2-\lambda)(-6-\lambda) - 9 = 0$

$\lambda^2 + 4\lambda - 21 = 0$

$(\lambda+7)(\lambda-3) = 0$

$\Downarrow$

$\lambda = -7$ or $\lambda = 3$.

Ex $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

Find A's eigenvalues....

$A\vec{x} = \lambda \cdot \vec{x}$

$A\vec{x} - \lambda\vec{x} = \vec{0}$

$(A - \lambda I)\vec{x} = \vec{0}$

nontrivial

$\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} 1-\lambda & 5 & 0 \\ 2 & 4-\lambda & -1 \\ 0 & -2 & -\lambda \end{bmatrix}\right) = 0$

$(-2)(-1)\cdot\det\begin{bmatrix} 1-\lambda & 0 \\ 2 & -1 \end{bmatrix} + (-\lambda)\cdot\det\begin{bmatrix} 1-\lambda & 5 \\ 2 & 4-\lambda \end{bmatrix} = 0$

$2(-1+\lambda - 0) - \lambda\left(\underline{(1-\lambda)(4-\lambda)} - 10\right) = 0$

$2(\lambda - 1) - \lambda(\lambda^2 - 5\lambda \underbrace{+4-10}_{-6}) = 0$

$2\lambda - 2 - \lambda^3 + 5\lambda^2 + 6\lambda = 0$

$-\lambda^3 + 5\lambda^2 + 8\lambda - 2 = 0$

$\lambda^3 - 5\lambda^2 - 8\lambda + \underline{2} = 0$

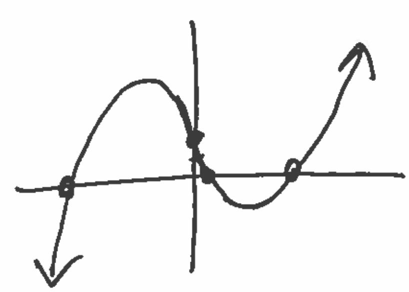
cubic?
guess roots.

only possibilities
divide 2

-1, 1, -2, 2

In this example
none are roots.... :

rely on tech
to solve for λ.



Definitions The "characteristic polynomial of A"

is $\det(A - \lambda \cdot I)$

λ: a variable.

The "characteristic equation of A"

$$\det(A - \lambda I) = 0$$

E'values of A $\Longleftrightarrow$ roots of A's characteristic polynomial.

Ex $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Find A's eigenvalues.

$$\det\left(\begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

$$(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$$

$\Rightarrow \lambda = 5, 3, \text{or } 1$ are the only eigenvalues.

"A is <u>similar</u> to B" (for square matrices)

means that there is some invertible matrix P such that

$$A = P \cdot B \cdot P^{-1}.$$

<u>Theorem</u>: If A is similar to B, A, B have the same characteristic polynomial. (so A, B will have the same eigenvalues).

char(A)

$$
\begin{aligned}
&= \det(A - \lambda I) = \det(P \cdot B \cdot P^{-1} - \lambda I) \\
&= \det\left(\underline{P} \cdot B \cdot P^{-1} - \lambda \cdot \underline{P} \cdot I P^{-1}\right) \\
&= \det\left(P\left(BP^{-1} - \lambda \cdot I \cdot P^{-1}\right)\right) \\
&= \det\left(P(B - \lambda \cdot I) \cdot P^{-1}\right) \\
&= \det(P) \cdot \det(B - \lambda I) \cdot \det(P^{-1}) \\
&= \det(B - \lambda I) \cdot \underbrace{\det(P) \cdot \det(P^{-1})}_{=1} \\
&= \text{char}(B)
\end{aligned}
$$