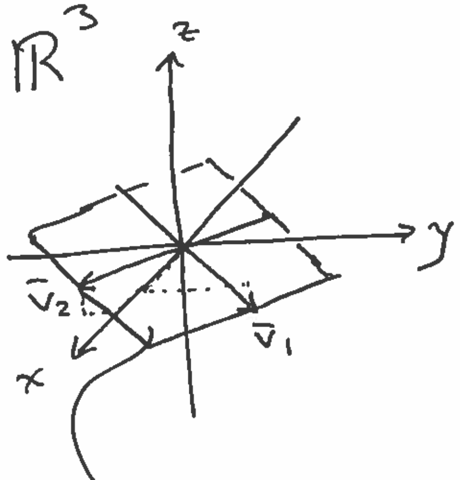
4.4 &

When you $\underline{\underline{\underline{a}}}$ basis for a subspace H of $\mathbb{R}^n$....

$$\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p\}$$

(span H & be linearly independent)

$\mathbb{R}^3$



H : one basis for H is $\{\vec{v}_1, \vec{v}_2\}$

each of these is in $\mathbb{R}^3$, so has three entries

Exs

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^2$

$$C = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$

Ex

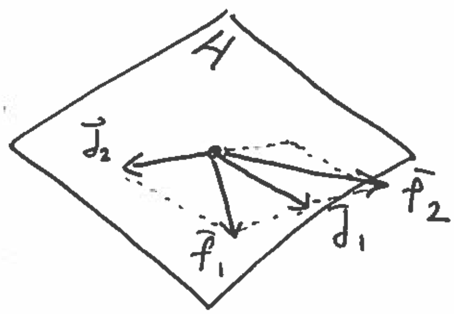
Suppose $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

A basis for H could be $\mathcal{D} = \left\{ \underset{\vec{d}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{d}_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right\}$

But note:

$$\left.\begin{aligned} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \end{aligned}\right\} \quad \mathcal{E} = \left\{ \overset{\vec{f}_1}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}, \overset{\vec{f}_2}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \right\}$$

<u>could</u> be another basis for H.



In fact $\mathcal{E}$ <u>is</u> another basis for H.

Given H, a subspace of $\mathbb{R}^n$, p-dimensional, with basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$ ... well, <u>any</u> vector $\vec{x}$ inside H can be written:

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_p\vec{b}_p$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

gather these weights for the basis vectors from $\mathcal{B}$..

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

the $\mathcal{B}$-coordinate vector for $\vec{x}$.

Ex

We had H, with basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \beta$.

take $\bar{x} = 5\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} \in H$.

well, $[\vec{x}]_\beta = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Ex Let $H = \mathbb{R}^3$. Consider $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

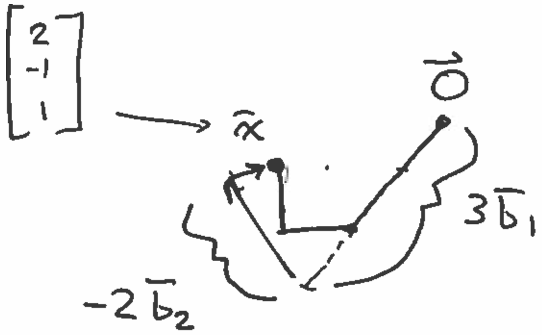
as a basis for H.

Consider $\bar{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. $\longleftarrow 2\bar{e}_1 + (-1)\bar{e}_2 + (1)\bar{e}_3$

Find $[\vec{x}]_\beta$. Is asking us to find c_1, c_2, c_3 such that $\underset{?}{c_1}\bar{b}_1 + \underset{?}{c_2}\bar{b}_2 + \underset{?}{c_3}\bar{b}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = 3 \\ c_2 = -2 \\ c_3 = 1 \end{array} \Longrightarrow [\vec{x}]_\beta = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$



$$\vec{x} = 3\underset{\begin{bmatrix}1\\0\\0\end{bmatrix}}{\bar{b}_1} - 2\underset{\begin{bmatrix}1\\1\\0\end{bmatrix}}{\bar{b}_2} + 1\cdot\underset{\begin{bmatrix}1\\1\\1\end{bmatrix}}{\vec{b}_3}$$

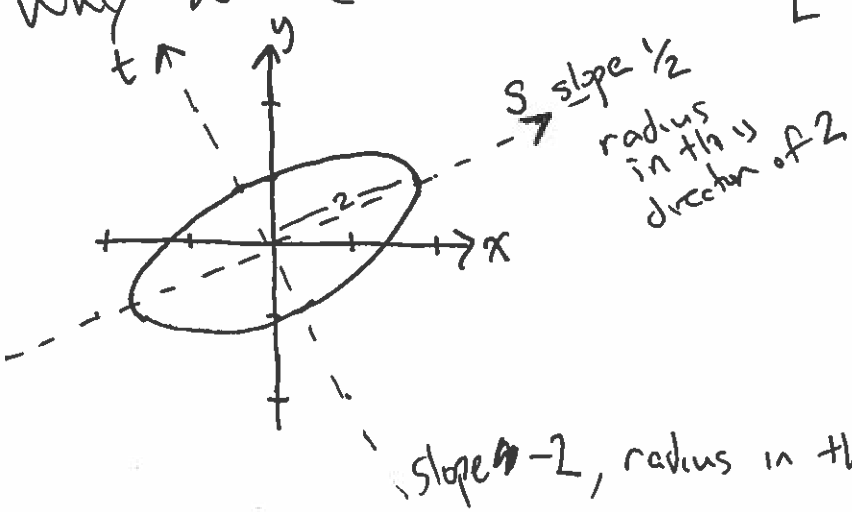
Note we set up $\begin{bmatrix}\beta\text{'s vectors for columns}\end{bmatrix}\begin{bmatrix}\bar{x}\end{bmatrix}_\beta = \begin{bmatrix}\vec{x}\end{bmatrix}$

Motivates: Put together vectors from a basis β as columns of a matrix, call that the "change of basis matrix from β to standard coordinates".

$$P_\beta \quad \text{or} \quad {}_{\text{std.}\leftarrow\beta}P$$

So $${}_{\text{std}\leftarrow\beta}P\,[\bar{x}]_\beta = [\bar{x}]$$

Why do we care about $[\vec{x}]_\beta$?



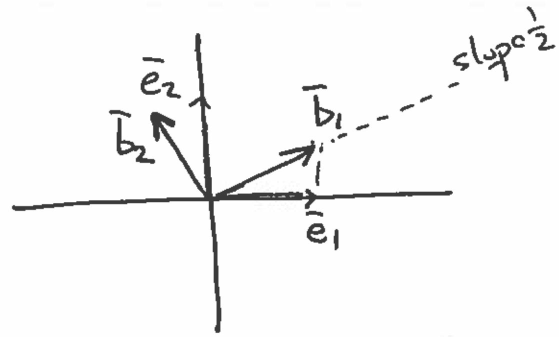
Observe upper right vertex $(x,y) = \left(\frac{4}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

but $(s,t) = (2,0)$ nice!

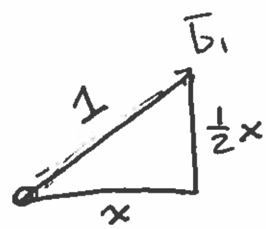
Suppose (x,y) lives on that ellipse...

and I want $[(x,y)]_{\beta}$ coordinates (which respect the (s,t) axes)...

$$\beta = \left\{ \ ? \ , \ ? \ \right\}$$



$$\beta = \left\{ \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \right\}$$



$$x^2 + \left(\tfrac{1}{2}x\right)^2 = 1$$
$$\tfrac{5}{4}x^2 = 1$$
$$x^2 = \tfrac{4}{5}$$
$$x = \tfrac{2}{\sqrt{5}}$$

(and so height is $\frac{1}{\sqrt{5}}$)

$$\Downarrow$$

$$P_{\beta} = P_{\text{std} \leftarrow \beta} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}_{\beta} = \begin{bmatrix} x \\ y \end{bmatrix}$$

In (s,t)-coordinates ellipse equation is:

$$\frac{s^2}{4} + \frac{t^2}{1} = 1$$

$$\Longrightarrow \quad \begin{bmatrix} s \\ t \end{bmatrix}_{\beta} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y \\ \frac{-1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y \end{bmatrix}$$

So $\quad s = \frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y \qquad t = \frac{-1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y$

$$\frac{s^2}{4} + \frac{t^2}{1} = 1$$

$$\frac{\left(\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right)^2}{4} + \left(\frac{-1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)^2 = 1$$

$$\frac{1}{5}x^2 + \frac{1}{5}xy + \frac{1}{20}y^2 + \frac{1}{5}x^2 - \frac{4}{5}xy + \frac{4}{5}y^2 = 1$$

$$\frac{2}{5}x^2 - \frac{3}{5}xy + \frac{17}{20}y^2 = 1$$