3.1 & 3.2 Determinants

Given a matrix, A, "the determinant of A" is a number that tells us about A.

Notations: det(A) |A|

$\det\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\right)$ $\begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix}$

- How do we calculate det(A)?
- What is it good for?

Well, det(A) has many equivalent definitions. We'll see several. No need to prove they are equivalent; just be familiar with them.

One definition: Think of det(A) as a function of n column vectors.

$$\det(A) = \det(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$$

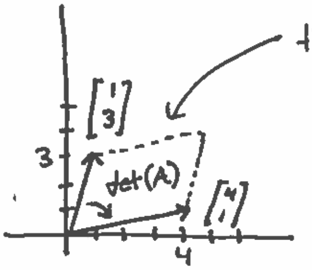
$$\det\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\right) = \det\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right)$$

Then det(A) is the "volume" of the n-dimensional "parallelogram" (parallelepiped, hyperparallelepided)
$n=2$ $n=3$ $n=4$

with edges defined by $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$.

Except det(A) is "signed" according to the order of vectors

Ex $\det\left(\begin{bmatrix}1 & 4\\ 3 & 1\end{bmatrix}\right)$

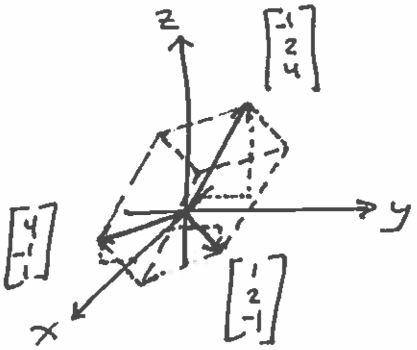


this area... except negative because

the vectors are oriented clockwise

This definition might not be so helpful to calculate det(A)....

Ex $\det\left(\begin{bmatrix}4 & 1 & -1\\ -1 & 2 & 2\\ 1 & -1 & 4\end{bmatrix}\right)$



Volume of this parallelepiped. positive because $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are ordered in agreement with the right-hand rule.

Again, so far not helpful to calculate det(A).

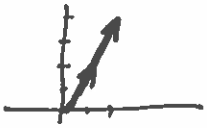
Observation: If columns are linearly dependent

$\Longrightarrow$ Span$\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ is not all of $\mathbb{R}^n$; its some (hyper) plane within $\mathbb{R}^n$

↑ generic prefix for when the dimensions are > 3.

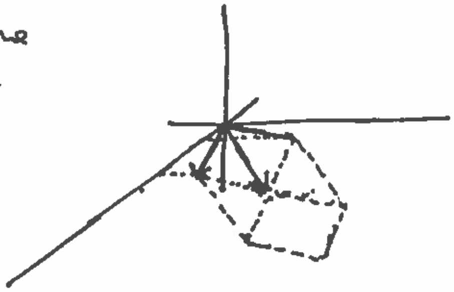
$\Longrightarrow$ So the "volume" will be ... 0.

Ex $\det\left(\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\right)$ = Area of



= 0.

Ex $\det\left(\begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}\right)$ = Volume of



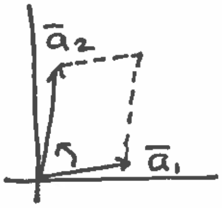
Note linearly dependent since $\vec{a}_1 + \vec{a}_2 - \vec{a}_3 = \vec{0}$

$= 0$

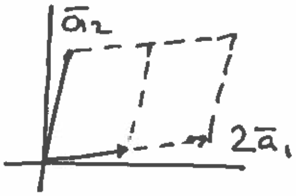
This "box" is flattened in the xy-plane. Its volume is 0.

Fact $\det(c\vec{a}_1, \bar{a}_2, \ldots, a_n) = c \cdot \det(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$

because scaling <u>one</u> edge of the (hyper) parallelepiped scales "volume" by the same amount



Area = $\det(\bar{a}_1, \bar{a}_2)$



$\det(2\bar{a}_1, \bar{a}_2) = 2\det(\bar{a}_1, \bar{a}_2)$

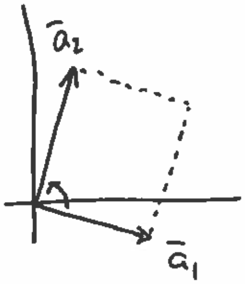
So we can say things like

$$\det\left(\begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}\right) = 2 \cdot \det\left(\begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}\right)$$

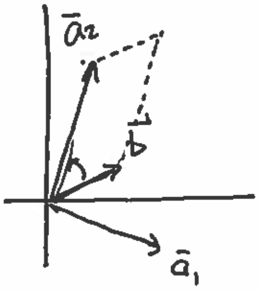
↑ both divisible by 2

Fact: $\det(\bar{a}_1, \underbrace{\bar{a}_2, \dots, \bar{a}_n}) + \det(\vec{b}, \underbrace{\vec{a}_2, \dots, \vec{a}_n})$

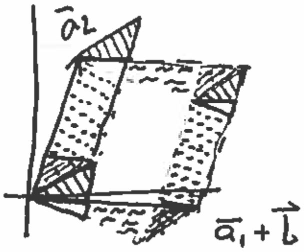
$$= \det(\bar{a}_1 + \vec{b}, \underbrace{\bar{a}_2, \dots, \bar{a}_n}_{\text{don't change.}})$$



$\det\left(\begin{bmatrix} \bar{a}_1 & \bar{a}_2 \end{bmatrix}\right)$



$\det\left(\begin{bmatrix} \vec{b} & \bar{a}_2 \end{bmatrix}\right)$



$\det\left(\begin{bmatrix} \bar{a}_1 + \vec{b} & \bar{a}_2 \end{bmatrix}\right)$

→ By cutting up the first two parallelograms in the right way, they can be rearranged to make the third. So areas add up.

These two facts let us calculate a formula for det. of a 2×2 matrix.

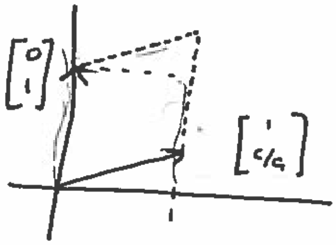
$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \underset{\substack{\text{let's assume} \\ a \neq 0}}{=} \det\left(\begin{bmatrix} a \cdot 1 & b \\ a \cdot \frac{c}{a} & d \end{bmatrix}\right) \overset{\text{used first Fact}}{=} a \cdot \det\left(\begin{bmatrix} 1 & b \\ \frac{c}{a} & d \end{bmatrix}\right)$$

$$= a \left[\det\left(\begin{bmatrix} 1 & b \\ \frac{c}{a} & d \end{bmatrix}\right) + \underbrace{\det\left(\begin{bmatrix} 1 & -b \cdot 1 \\ \frac{c}{a} & -b \cdot \frac{c}{a} \end{bmatrix}\right)}_{\substack{\text{this is 0 because} \\ \text{columns are parallel}}} \right]$$

$$\overset{\text{used 2nd Fact}}{=} a \cdot \det\left(\begin{bmatrix} 1 & 0 \\ \frac{c}{a} & d - b\frac{c}{a} \end{bmatrix}\right) \underset{\text{[illegible]}}{=} a \cdot \left(d - b\frac{c}{a}\right) \cdot \det\left(\begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix}\right)$$

Now, $\det\left(\begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix}\right)$ is definitely 1,

because it's the area of :



So $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a\cdot\left(d - b\frac{c}{a}\right) = \underline{\underline{a\cdot d - bc}}$

Are there formulas like this for a 3×3 det? 4×4?
<u>we'll get to that.</u>

Some observations:
(with 2×2)

- $\det\left(\begin{bmatrix} t\cdot a & b \\ t\cdot c & d \end{bmatrix}\right) = t\cdot\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

  ↑
  scaling one column by some number scales the determinant too

- $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc = \det\left(\underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{\text{transpose of } A}\right)$

  [A matrix and its transpose have the same determinant.

- $\det\left(\underbrace{\begin{bmatrix} a & b \\ \underline{ta} + c & \underline{tb} + d \end{bmatrix}}_{\text{adding } t \text{ times one row to another row...}}\right) = a(tb+d) - b(ta+c) = \cancel{atb} + ad - \cancel{bta} - cd$

  $= ad - bc$

  $= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

  doesn't change the determinant.

- $$\det\left(\underbrace{\begin{bmatrix} c & d \\ a & b \end{bmatrix}}_{\text{swapping rows...}}\right) = cb - ad = -(ad-bc) = -\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

negates the determinant.

For higher n than 2×2, here is the <u>plan</u>.

Take $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and RREF it...

Track the row operations. We know how each one affects determinant. Determinant of the RREF is easy (either 0 or 1). So we can engineer what $\det(A)$ is.

<u>Ex</u> $\det\left(\begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}\right) = ?$

$$\begin{bmatrix} \boxed{1} & 3 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix} \xrightarrow[-R_1+R_2]{2R_1+R_2} \begin{bmatrix} \boxed{1} & 3 & 1 \\ 0 & 7 & 4 \\ 0 & -3 & 3 \end{bmatrix} \xrightarrow[\text{multiplies det by } -\frac{1}{3}]{-\frac{1}{3}R_3 \to R_3} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 7 & 4 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[\text{negates determinant}]{R_2 \leftrightarrow R_3}$$

(this kind of row op <u>doesn't</u> affect det.)

$$\begin{bmatrix} \boxed{1} & 3 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 7 & 4 \end{bmatrix} \xrightarrow[\text{(no effect)}]{-7R_2+R_3} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 11 \end{bmatrix} \xrightarrow[\text{mult. det by } 1/11]{\frac{1}{11}R_3} \begin{bmatrix} \boxed{1} & 3 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

row reductions that add multiples of rows to other rows don't affect determinant

$$\longrightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

has determinant 1
since it represents
a 1×1×1 cube

So $\det(A)\cdot\underbrace{\left(\frac{-1}{3}\right)\cdot(-1)\cdot\left(\frac{1}{11}\right)}_{\text{tracking effects of row operations}} = 1$

So $\det(A) = 1\cdot 11\cdot(-1)\cdot(-3)$
$= 33$

The above method is what computers do, especially with large matrices like 42×42, etc.

Here's another way: expansion by a column (or row)

$$\det\left(\begin{bmatrix}1&3&1\\-2&1&2\\1&0&4\end{bmatrix}\right) = \underbrace{\det\left(\begin{bmatrix}1&3&1\\0&1&2\\0&0&4\end{bmatrix}\right)}_{\text{this equals } \det\left(\begin{bmatrix}1&2\\0&4\end{bmatrix}\right)} + \underbrace{\det\left(\begin{bmatrix}0&3&1\\-2&1&2\\0&0&4\end{bmatrix}\right)}_{\text{this equals } -(-2)\det\left(\begin{bmatrix}3&1\\0&4\end{bmatrix}\right)} + \underbrace{\det\left(\begin{bmatrix}0&3&1\\0&1&2\\1&0&4\end{bmatrix}\right)}_{\text{this equals } \det\begin{bmatrix}3&1\\1&2\end{bmatrix}}$$

break up as

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-2\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}$$

because the area of that 2D parallelogram would be multiplied by height 1 to get volume

similarly, but the "height" is (−2) and now the vectors are not respecting the right-hand-rule

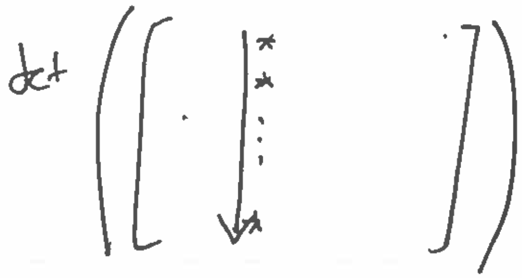
$$= 1\cdot\det\left(\begin{bmatrix}1&2\\0&4\end{bmatrix}\right) + 2\cdot\det\left(\begin{bmatrix}3&1\\0&4\end{bmatrix}\right) + 1\cdot\det\left(\begin{bmatrix}3&1\\1&2\end{bmatrix}\right)$$

⟨reduced 3×3 to smaller cases...⟩

$$= 4 + 2\cdot(12) + 5$$

$$= 33$$

Generally pick a column (or row)



and move down (or across)

with each element, take that element, negate according to

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix} \quad \left(\text{aka take } (-1)^{i+j}\right)$$

and multiply by the determinant of the smaller matrix that comes from blocking out that row & column. Add them all up.

Wisdom: Pick a column (or row) with lots of 0's.

Ex $\det\left(\begin{bmatrix} -2 & 1 & -2 & 3 \\ 6 & 0 & 4 & 0 \\ -5 & 8 & 1 & -2 \\ \underline{0} & \underline{0} & \underline{3} & \underline{0} \end{bmatrix}\right)$ lots of 0s →

$$= 0\cdot\det\left(\text{who cares}\right) + 0\cdot\det\left(\text{who cares}\right)$$

$$+\ 3\cdot\underset{\text{why "4+3"?}}{(-1)^{4+3}}\cdot\det\left(\begin{bmatrix} -2 & 1 & 3 \\ 6 & 0 & 0 \\ -5 & 8 & -2 \end{bmatrix}\right)$$

$$+\ 0\cdot\det\left(\text{who cares}\right)$$

$$= -3\cdot\det\left(\begin{bmatrix} -2 & 1 & 3 \\ \cancel{6} & \cancel{0} & \cancel{0} \\ -5 & 8 & -2 \end{bmatrix}\right) = -3\cdot 6\cdot(-1)^{2+1}\cdot\det\left(\begin{bmatrix} 1 & 3 \\ 8 & -2 \end{bmatrix}\right)$$

$$= -3(-6)\cdot(-2-24) = 18(-26) = -468$$

⑨

Ex Work through finding $\det\left(\begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 0 & 0 & 4 \\ 1 & 0 & 2 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}\right)$

One more method... useful for matrices with lots of 0's. (Permutation Based)

Ex $\det\left(\begin{bmatrix} 0 & 8 & 1 & 0 \\ 0 & 0 & 2 & 5 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 3 & 1 \end{bmatrix}\right)$

$$\begin{bmatrix} \blacksquare & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \blacksquare \\ \cdot & \blacksquare & \cdot & \cdot \\ \cdot & \cdot & \blacksquare & \cdot \end{bmatrix}$$

There are $4! = 24$ things like this one where the "blocks" don't share a row or column. Note this one would take two row swaps to become $\begin{bmatrix} \blacksquare & & & \\ & \blacksquare & & \\ & & \blacksquare & \\ & & & \blacksquare \end{bmatrix}$.

For each such thing, multiply corresponding entries in A, multiply by $(-1)^{\#\text{ of row swaps}}$, and add up.

Pay attention to 0's since such terms don't contribute

Ex $\begin{bmatrix} \cdot & \blacksquare & \cdot & \cdot \\ \cdot & \cdot & \blacksquare & \cdot \\ \blacksquare & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \blacksquare \end{bmatrix}, \begin{bmatrix} \cdot & \blacksquare & \cdot & \cdot \\ \cdot & \cdot & \blacksquare & \cdot \\ \cdot & \cdot & \cdot & \blacksquare \\ \blacksquare & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & \blacksquare & \cdot \\ \cdot & \cdot & \cdot & \blacksquare \\ \cdot & \blacksquare & \cdot & \cdot \\ \blacksquare & \cdot & \cdot & \cdot \end{bmatrix}$ ← the only such things that don't land on 0 slots

two row swaps

$(-1)^2 \cdot 8 \cdot 2 \cdot 1 \cdot 1 \qquad (-1)^3 \cdot 8 \cdot 2 \cdot 2 \cdot 1 \qquad (-1)^3\, 1 \cdot 5 \cdot 1 \cdot 1$

$16 \quad - \quad 32 \quad - \quad 5 \qquad = \boxed{-21}$

Special 3x3 case is worth memorizing

$$\begin{bmatrix} \blacksquare & \cdot & \cdot \\ \cdot & \blacksquare & \cdot \\ \cdot & \cdot & \blacksquare \end{bmatrix} + \begin{bmatrix} \cdot & \blacksquare & \cdot \\ \cdot & \cdot & \blacksquare \\ \blacksquare & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \cdot & \cdot & \blacksquare \\ \blacksquare & \cdot & \cdot \\ \cdot & \blacksquare & \cdot \end{bmatrix} - \begin{bmatrix} \blacksquare & \cdot & \cdot \\ \cdot & \cdot & \blacksquare \\ \cdot & \blacksquare & \cdot \end{bmatrix} - \begin{bmatrix} \cdot & \blacksquare & \cdot \\ \blacksquare & \cdot & \cdot \\ \cdot & \cdot & \blacksquare \end{bmatrix} - \begin{bmatrix} \cdot & \cdot & \blacksquare \\ \cdot & \blacksquare & \cdot \\ \blacksquare & \cdot & \cdot \end{bmatrix}$$

<u>Ex</u>

$$\det\left(\begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}\right) = 1\cdot1\cdot4 + 3\cdot2\cdot1 + 1(-2)\cdot0 - 1\cdot2\cdot0 - 3(-2)(4) - 1\cdot1\cdot1$$

$$= 4 + 6 + 0 - 0 + 24 - 1$$

$$= 33$$

⟨Now Offer "Determinant Computation" Worksheet⟩

Applications

Cross-product of two vectors in $\mathbb{R}^3$

$$(1,3,-2) \times (4,1,8)$$

$$= \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ 4 & 1 & 8 \end{bmatrix}$$

$\bar{e}_1, \bar{e}_2, \bar{e}_3$

$\hat{\imath} \quad \hat{\jmath} \quad \hat{k}$

$$24\hat{\imath} - 8\hat{\jmath} + \hat{k} - (-2\hat{\imath}) - 8\hat{\jmath} - 12\hat{k}$$

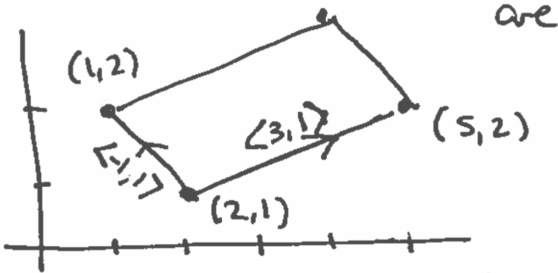
$$26\hat{\imath} - 16\hat{\jmath} - 11\hat{k}$$

$$= (26, -16, -11)$$

* * *

Parallelogram area:

area?



$$\text{area} = \left|\det\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\right| = |3-(-1)| = 4$$

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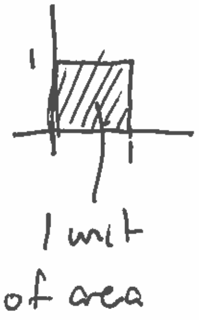
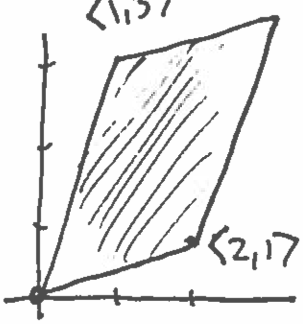
When $T(\vec{x}) = A \cdot \vec{x}$, $\det(A)$ tells you how T scales areas.

Ex



$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

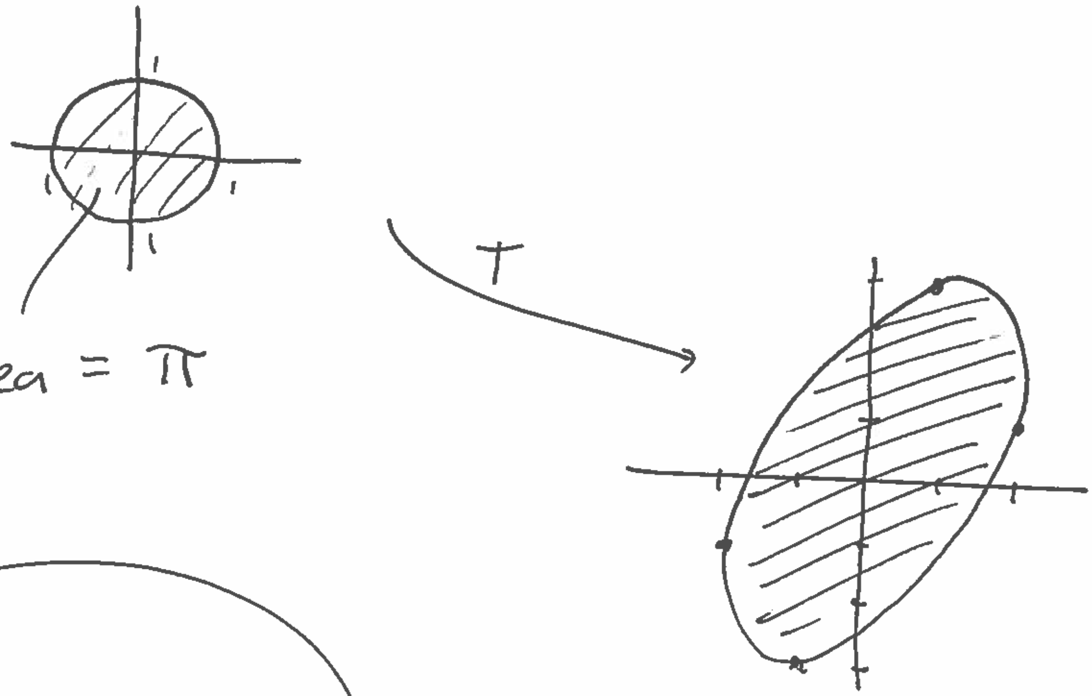
$$\xrightarrow{T(\text{square})}$$



area is $\det(A) = 5$

T scales areas by 5

Ex



area = π

area is 5π

use det(A) to scale area.

More Facts:

- $\det(A\cdot B) = \det(A)\cdot\det(B)$

T_A : scales area by det A.

T_B : scales area by det B.

T_{AB} : scales area by det(AB)

$$\begin{tikzcd}
\text{shape} \arrow[r, "T_B"] \arrow[rr, bend right, "T_{AB}", "\text{stretches area by } \det(AB)"'] & \text{stretches area by } \det B \arrow[r, "T_A"] & \text{stretch area by } \det A
\end{tikzcd}$$

So $\det(AB) = \det(A)\cdot\det(B)$

- A is invertible if and only if $\det(A) \neq 0$

$$(A \cdot A^{-1} = I \implies \det(AA^{-1}) = 1$$

$$\implies \underbrace{\det(A)}_{\text{cannot be } 0!!!!} \cdot \det(A^{-1}) = 1$$

- $\det(A^{-1}) = \dfrac{1}{\det(A)}$

- $\det(A^{T}) = \det(A)$

- $\det(k \cdot A) = k^{n} \cdot \det(A)$