2.8 & 4.3

$\mathbb{R}^n$ is an example of a <u>vector space</u>.

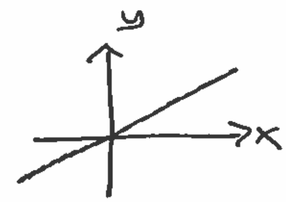
any set where
- addition has been defined,
- scalar multiplication has been defined
- and 8 other features that capture the essence of $\mathbb{R}^n$

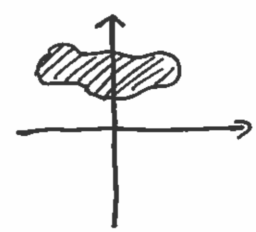
Ch 4 stuff not to worry about

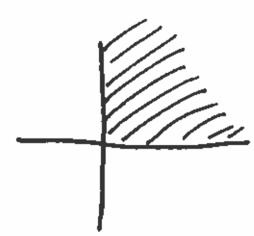
<u>Def</u> A <u>subspace</u> of $\mathbb{R}^n$ is a sub<u>set</u> H of $\mathbb{R}^n$..

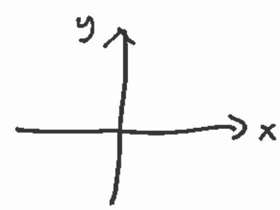
satisfying 3 properties

- $\vec{0}$ is inside H.
- for all $\vec{u}, \vec{v}$ in H, $\underbrace{\vec{u}+\vec{v}}_{\text{in } \mathbb{R}^n}$ is in H.
- for all $\vec{u}$ in H and for all c in $\mathbb{R}$, $\underbrace{c \cdot \vec{u}}_{\text{in } \mathbb{R}^n}$ is in H.









Fact Any Span is a subspace.

If $H = \text{Span}\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p\}$ H is a subspace

may or may not
be linearly independent

✓ $\bar{0} = 0 \cdot \bar{v}_1 + 0 \cdot \bar{v}_2 + \cdots + 0 \cdot \bar{v}_p \in \text{Span}\{\bar{v}_i\}$

✓ If $\vec{u}, \vec{w}$ are in H ...

$$\vec{u} = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_p\bar{v}_p$$

and $\vec{w} = d_1\bar{v}_1 + d_2\bar{v}_2 + \cdots + d_p\bar{v}_p$

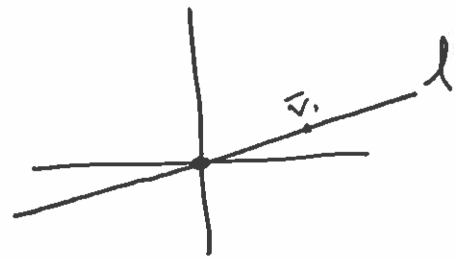
$$\Longrightarrow \vec{u}+\vec{w} = \underbrace{(c_1+d_1)\bar{v}_1 + (c_2+d_2)\bar{v}_2 + \cdots + (c_p+d_p)\bar{v}_p}_{\in \text{Span}\{\bar{v}_i\}}$$

✓ If $\vec{u}$ is in H & c is in $\mathbb{R}$,

$$\vec{u} = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_p\bar{v}_p$$

$$\Longrightarrow c \cdot \vec{u} = \underbrace{(cc_1)\bar{v}_1 + (cc_2)\bar{v}_2 + \cdots + (cc_p)\bar{v}_p}_{\in \text{Span}\{\bar{v}_i\}}$$

<u>Ex</u> Take a line in $\mathbb{R}^2$ passing through the origin.



This is a subspace ... because this is $\text{Span}\{\vec{v}\}$ where $\vec{v}$ is any nonzero vector on that line.

So it's a subspace ✓

<u>Ex</u> Take a plane in $\mathbb{R}^3$ passing through $\vec{0}$. That is a subspace because its $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Trivial subspaces: Is $\{\vec{0}\}$ a subspace of $\mathbb{R}^n$?

yes!

Is $\mathbb{R}^n$ a subspace of $\mathbb{R}^n$?

yes!

Given an $m \times n$ matrix A, two meaningful subspaces we look at are:

the column space of A	the null space of A
$\text{Col}(A)$	$\text{Nul}(A)$
is a subspace of $\mathbb{R}^m$	is a subspace of $\mathbb{R}^n$.

$$\text{Col}(A) := \text{Span}\{A\text{'s columns}\}$$

Ex $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

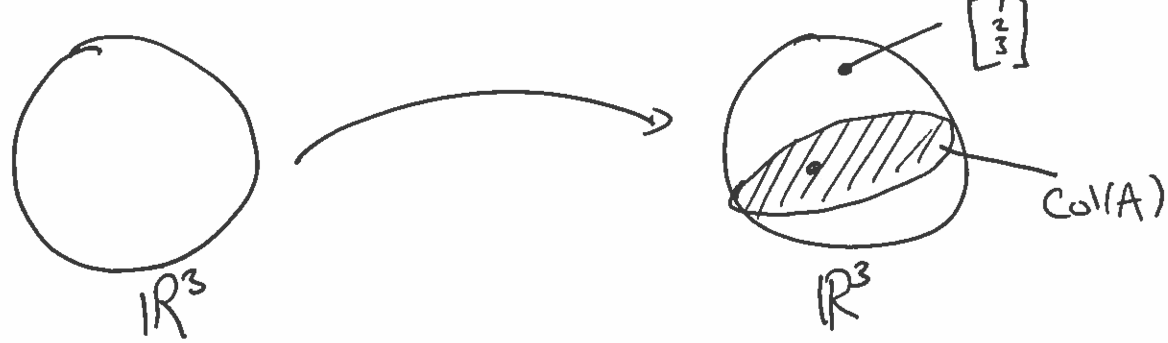
Questions like: Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\text{Col}(A)$?

Translation: Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a lin. combo of A's columns?

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{start RREF...}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

no solution.

No, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not in $\text{Col}(A)$.



Same

Ex Describe Col(A) as a Span.

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Describe Col(A) as a Span of linearly independent vectors.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

linearly independent

linearly dependent

these relationships hold in exactly the same way.

So $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(we can just take the columns of A corresponding to where RREF of A showed us the pivots.)

<u>Define</u> $\text{Nul}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n \mid A \cdot \vec{x} = \vec{0} \}$

$\uparrow$ domain

(Nul(A) is the collection of all vectors that A turns into $\vec{0}$.)

✓ Is $\vec{0}$ in Nul(A)? well,

$A \cdot \vec{0} = \vec{0}$

$\checkmark$

✓ If $\vec{u}, \vec{v}$ are in Nul(A) ...

$\Longrightarrow A \cdot \vec{u} = \vec{0}$ & $A \cdot \vec{v} = \vec{0}$

$\Longrightarrow A\underbrace{(\vec{u} + \vec{v})}_{\text{is in Nul}(A)} = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$

✓ If $\vec{u}$ is in Nul(A), c is in $\mathbb{R}$

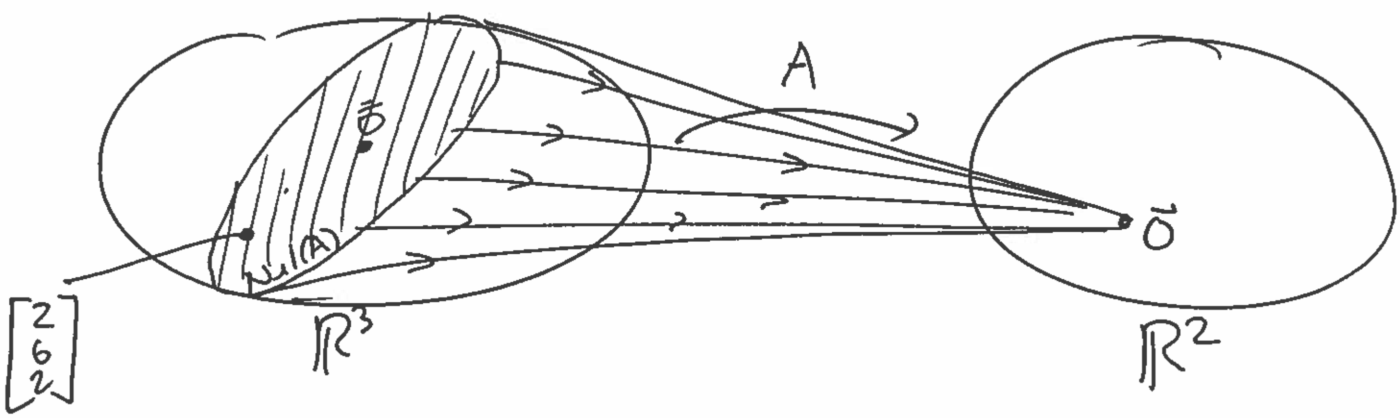
$\Longrightarrow A \cdot \vec{u} = \vec{0}$

$\Longrightarrow A\underbrace{(c\vec{u})}_{\text{is in Nul}(A)} = c \cdot (A \cdot \vec{u}) = c \cdot \vec{0} = \vec{0}$

Ex $A = \begin{bmatrix} 2 & 1 & -5 \\ -3 & 1 & 0 \end{bmatrix}$. Is $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$ in Nul(A)?

Well, $A \cdot \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -5 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

So yes, $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$ is in Nul(A).



Now describe Nul(A) as a Span....

"$\vec{x}$ is in Nul(A)" $\iff$ "$A \cdot \vec{x} = \vec{0}$"

$\left[A \mid \vec{0} \right]$

Ex $\left[\begin{array}{ccc|c} 2 & 1 & -5 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right]$

$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \implies \vec{x} = \begin{bmatrix} x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

x_3 free

$\vec{x}$ is in Nul(A) $\iff$ $\vec{x}$ is in Span$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$

$\Uparrow$

$\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$

<u>Definition</u> When you have a subspace H, a <u>basis</u> of H is a list of vectors (it's an <u>ordered</u> list) where

- it's linearly independent.
- the span of those vectors is H.

<u>Ex</u> The standard basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

<u>Ex</u> Another basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}^{A} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot in every column $\Longrightarrow$ these are linearly indep.

pivot in every row $\Longrightarrow$ "$A\vec{x} = \vec{b}$" has a solution no matter what $\vec{b}$ is...

$\Longrightarrow$ Span of these columns is $\mathbb{R}^3$

Ex $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ Find a basis for Col(A).

$$\mathrm{Col}(A) = \mathrm{Span}\underbrace{\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix} \right\}}_{\text{these 5 do span all of Col(A).}}$$

Using RREF, $\mathrm{Col}(A) = \mathrm{Span}\underbrace{\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}}_{\text{these 2 do span Col(A).}}$.

<u>and</u> they are linearly independent

A basis for Col(A) is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$.

$\Rightarrow$ Find a basis for Nul(A). Based on A's RREF,

"$\vec{x}$ is in Nul(A)" $\Longleftrightarrow$ $\vec{x} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$

$$\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

these 3 vectors do span Nul(A).

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Let $W = \begin{bmatrix} \overline{w}_1 & \overline{w}_2 & \cdots & \overline{w}_q \end{bmatrix} = V \cdot \begin{bmatrix} c_{ij} \end{bmatrix}$

$n \times q$

So $\begin{bmatrix} \overline{w}_1 & \overline{w}_2 & \cdots & \overline{w}_q \end{bmatrix} = \begin{bmatrix} \overline{v}_1 & \cdots & \overline{v}_p \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pq} \end{bmatrix}$

using c's from prev. page

So $W = V \cdot C$

C is a $p \times q$ matrix $\Rightarrow$ C has more columns than rows

$\Rightarrow$ At least one <u>free</u> <u>column</u> in C

$\Rightarrow$ $C\vec{x} = \vec{0}$ has a nontrivial solution $\overline{x}$.

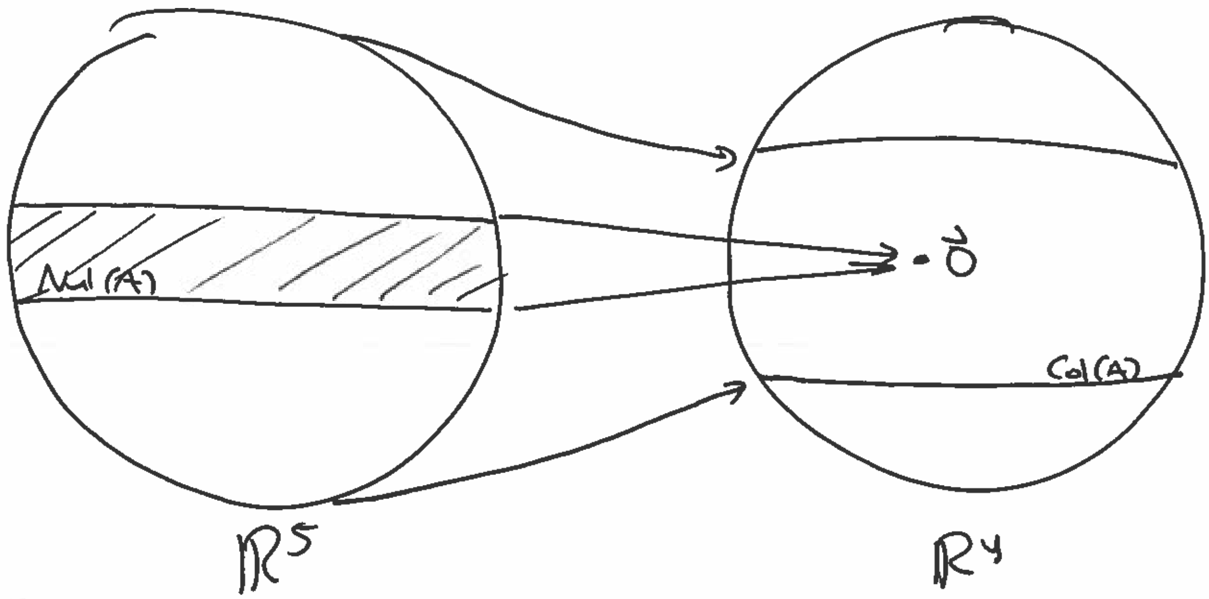
$\Rightarrow W\vec{x} = V \cdot C \cdot \vec{x} = \vec{0}$

nontrivial...

$\Rightarrow$ W's columns are linearly <u>dependent</u>...

This is a logical contradiction since W was a basis.

It means "$q > p$" is an impossibility. So $p = q$.



Imagine a subspace H.

$\mathcal{B} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p\}$

$\mathcal{C} = \{\bar{w}_1, \bar{w}_2, \ldots \bar{w}_q\}$

Maybe these are two valid bases... and $p \neq q$.

Turns out p must equal q.

Suppose $q > p$. Each $\bar{w}_i$ can be written as

$$c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_p\bar{v}_p$$

better: $c_{1i}\bar{v}_1 + c_{2i}\bar{v}_2 + \cdots + c_{pi}\bar{v}_p = \vec{w}_i$

(justified because $\mathcal{B}$ is a basis...)

To check these three are independent,

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot in every column
$\Downarrow$
the three are l.n. independent.

So $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of Nul(A).

Ex $A = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for Col(A).

A is already RREF.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Find a basis for Nul(A).

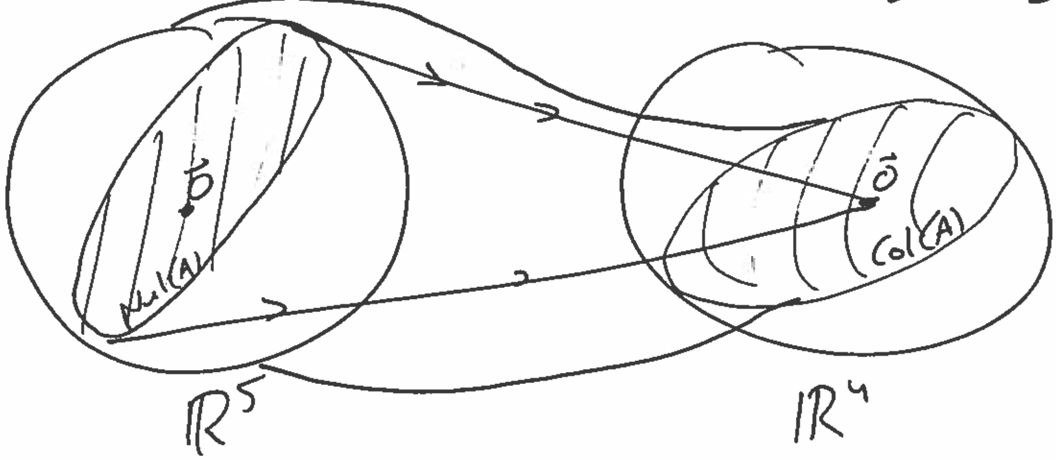
"$A\vec{x} = \vec{0}$"

$\Downarrow$

$$\vec{x} = \begin{bmatrix} 3x_3 - 5x_4 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Basis for Nul(A) is

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$



Definition Given a subspace H, the **dimension of H** is the number of vectors in a **basis** for H.

Recall: $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

Col(A) has basis

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

Nul(A) has a basis

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of Col(A) is 2.

The dimension of Nul(A) is 3.

rank(A) := is the number of pivots in A

So rank(A) = dim Col(A).

nullity(A), aka null(A) is 3.

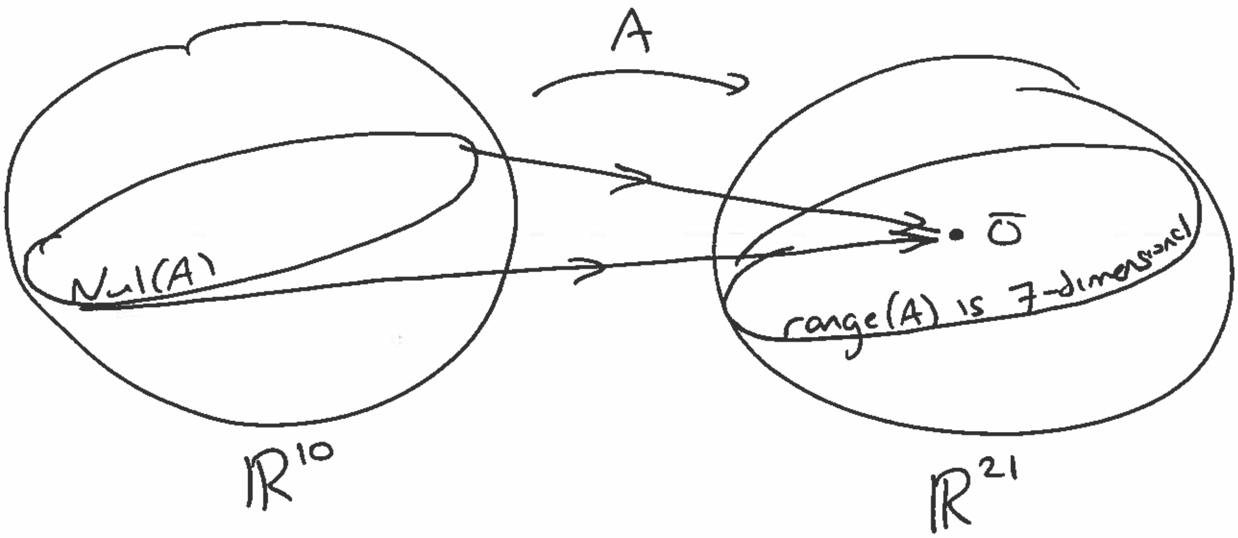
null(A) = dim Nul(A).

Theorem: With an $m \times n$ matrix:

$$\begin{aligned} \#\text{pivot cols} + \#\text{free cols} &= n \\ \dim \operatorname{Col}(A) + \dim \operatorname{Nul}(A) &= n \\ \operatorname{rank}(A) + \operatorname{null}(A) &= n \end{aligned}$$

~~m×n~~ matrix A

21×10 matrix A



what is the dimension of Nul(A)?

$$\dim \operatorname{Col}(A) + \dim \operatorname{Nul}(A) = n$$
$$7 + \dim \operatorname{Nul}(A) = 10$$
$$\dim \operatorname{Nul}(A) = 3$$

Facts: Let H be a p-dimensional subspace of $\mathbb{R}^n$.

- *Any* set of p linearly independent vectors in H is automatically a basis for H.
- Any set of p vectors that span H is automatically a basis for H.

Invertible Matrix Theorem

(a) A is invertible

$\vdots$

(l) ... A^T is invertible

(m) columns of A form a basis for $\mathbb{R}^n$

(n) $\operatorname{Col}(A) = \mathbb{R}^n$

(o) $\dim \operatorname{Col}(A) = n$

(p) $\operatorname{rank}(A) = \cancel{\dim \operatorname{Col}(A)} = n$

(q) $\operatorname{Nul}(A) = \{\vec{0}\}$

(r) $\operatorname{nullity}(A) = 0$
(aka $\dim \operatorname{Nul}(A)$)

(s) $\det(A) \neq 0$