2.1 (cont'd)

How to multiply $\underset{m\times n}{A}$, $\underset{n\times p}{B}$ (two matrices)

- We know if we have a linear transformation T we can build a matrix:

$$A_T = \begin{bmatrix} \vdots & \vdots & & \vdots \\ T(\bar{e}_1) & T(e_2) & \cdots & T(\bar{e}_n) \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

- we know $A, B \longrightarrow T_A, T_B$
- we can build $T_A \circ T_B$.
- We can make a matrix for $T_A \circ T_B$.
- So define $A \bullet B$ as the matrix for $T_A \circ T_B$.

Ex $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix}$ Task: Calculate $A \cdot B$.

$B = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$

Calculate $(T_A \circ T_B)(\bar{e}_1)$

$$= A(B \cdot \bar{e}_1)$$

$$= A \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

Now $(T_A \circ T_B)(\bar{e}_2) = A(B \cdot \bar{e}_2)$

$$= A\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 10 & 6 \\ -2 & 6 \end{bmatrix}$$

Method 2

Alternatively, to calculate $A \cdot B$,

$$[A]\cdot[B] = [A]\left[\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_p\right]$$

multiply A by vectors to get columns of $[AB]$.

Ex
$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}+3\begin{bmatrix}3\\-2\end{bmatrix} & 3\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}+1\begin{bmatrix}3\\-2\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 6 \\ -2 & 6 \end{bmatrix}$$

Method 3

To find $[AB]_{ij}$, dot product A's ith row with B's jth column.

Ex
$$\downarrow\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix}\underset{\longrightarrow}{\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}} = \begin{bmatrix} \overbrace{1\cdot1+0\cdot2+3\cdot3}^{\text{a certain dot product}} & 1\cdot3+0\cdot2+3\cdot1 \\ 2\cdot1+1\cdot2-2\cdot3 & 2\cdot3+1\cdot2-2\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 6 \\ -2 & 6 \end{bmatrix}$$

can do in your head.... if comfortable.

$$\left([AB]_{ij} = \sum_{k=1}^{n} A_{ik}\cdot B_{kj} \right)$$

Facts

$\underset{m\times n}{A}$

$$A(BC) = (AB)C$$

$$A(B+C) = AB + AC$$

$$(B+C)A = BA + CA$$

$$r(AB) = (rA)\cdot B = A\cdot(rB)$$

$$I_m \cdot A = A$$

$$A \cdot I_n = A$$

WARNING: In general $A\cdot B \neq B\cdot A$

$$\begin{array}{cc} A & B \\ \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \end{array} = \underset{AB}{\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}}$$

$$\begin{array}{cc} B & A \\ \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \end{array} = \underset{BA}{\begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}}$$

very different!

WARNING: $(A+B)^2 \neq A^2 + 2AB + B^2$

instead: $(A+B)^2 = A^2 + AB + BA + B^2$

Transpose

Turn rows to columns
columns to rows

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix}, \quad \underbrace{A^t}_{\text{"A transpose"}} = \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$$

$$\left(A^t_{ij} = A_{ji} \right)$$

So.. when A is $m \times n$, A^t is $n \times m$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "A\cdot"] & \mathbb{R}^m \arrow[l, bend left, "A^t\cdot"] \end{tikzcd}$$

Facts

$$(A^t)^t = A$$

$$(A + B)^t = A^t + B^t$$

$$(rA)^t = r \cdot A^t$$

$$(\underset{m\times n}{A} \cdot \underset{n\times p}{B})^t = \underset{p\times n}{B^t} \cdot \underset{n\times m}{A^t}$$

2.2 The Inverse of a Matrix

We know I is a special matrix $\left(\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\right)$ where $I \cdot \vec{x} = \vec{x}$.

(the $n \times n$ identity matrix).

Suppose you have a square matrix A...
Often (but not always) you can find another matrix A^{-1} where $A \cdot A^{-1} = I$ and $A^{-1} \cdot A = I$.

Ex If $A = \begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix}$, it turns out we can find $A^{-1} = \begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix}$,

and $$A \cdot A^{-1} = \begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1} \cdot A = \begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -9 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \checkmark$$

For 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

then *if* $a \cdot d - b \cdot c \neq 0$, $A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(swap diagonal entries & negate off-diagonals).

Check: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ba \\ cd-cd & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Discuss where $ad-bc = 0 \implies ad = bc$

$$\implies \begin{bmatrix} ad \\ cd \end{bmatrix} = \begin{bmatrix} bc \\ cd \end{bmatrix}$$

$$\implies d \begin{bmatrix} a \\ c \end{bmatrix} = c \begin{bmatrix} b \\ d \end{bmatrix}$$

columns of A

So A's columns are parallel... so Span{A's columns} is a line in $\mathbb{R}^2$.

So... for some $\vec{b}$, "$A\vec{x} = \vec{b}$" has no solutions. (Thm 4)

If an A^{-1} existed, then

$$A\vec{x} = \vec{b}$$

$$A^{-1}\cdot A\vec{x} = A^{-1}\cdot\vec{b}$$

$$\vec{x} = \underbrace{A^{-1}\cdot\vec{b}}_{\text{there's \underline{always} a solution}}$$

So $ad-bc = 0$

$$\Downarrow$$

A^{-1} does not exist.

[With 2×2 matrices A, A^{-1} exists if and only if $ad-bc \neq 0$, and $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad-bc$ is called its "determinant".
We'll study these in Ch 3)

Ex $A = \begin{bmatrix} 3 & 8 \\ -1 & 4 \end{bmatrix}$. Find A^{-1}.

$$A^{-1} = \frac{1}{\underset{3\cdot 4 - 8(-1)}{20}}\begin{bmatrix} 4 & -8 \\ 1 & 3 \end{bmatrix}$$

Ex $\begin{cases} 2x+3y = 5 \\ -4x+y = 2 \end{cases}$

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -1 \\ 24 \end{bmatrix}$$

$$x = -\frac{1}{14} \qquad y = \frac{24}{14}$$

$n \times n$ Matrices...

Facts

- If A is invertible, $(A^{-1})^{-1} = A$.
- If A, B are both invertible, then $A \cdot B$ is invertible too.

$$\left((AB)\underset{\checkmark}{(B^{-1}A^{-1})} = (A(BB^{-1}))A^{-1} = A \cdot A^{-1} = I \right)$$

<u>And</u> $(AB)^{-1} = B^{-1} \cdot A^{-1}$

- $(A^t)^{-1} = (A^{-1})^t$

Elementary Row Operations

- interchange rows →

$$\overbrace{\begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 5 \\ 0 & 4 & 2 \end{bmatrix}}^{A} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \\ 0 & 4 & 2 \end{bmatrix}$$

Do the same row op. to $I_{3\times3}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{1,2}}$$

Take $E_{1,2} \cdot A$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 5 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \\ 0 & 4 & 2 \end{bmatrix} \longleftarrow \text{same!}$$

Any elementary row operation applied to A is the <u>same</u> as left-multiplying A by an <u>elementary matrix</u>, where an elementary matrix is the result of a ~~STAND~~ elementary row operation applied to I.

Now: $$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \\ 0 & 4 & 2 \end{bmatrix} \xrightarrow{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -13 \\ 0 & 4 & 2 \end{bmatrix} \checkmark$$

Instead $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_2 \to R_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{-3\cdot 1,2} \text{ or } E_{-3;\,2,1}}$$

And: $$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -13 \\ 0 & 4 & 2 \end{bmatrix}$$

✓ —— same

Well... $$\overbrace{A \to \to \to \cdots \to \to}^{\text{row ops...}} I$$

$$\underbrace{E \cdots\cdots E \cdot E \cdot E \cdot}_{\text{mult by elem matrices...}} A = I$$

What we see: if A row reduces to I, then A <u>is</u> invertible. And A^{-1} is the product of the elementary matrices that row reduced A to I.

Ex Find A^{-1}, where $A = \begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\cancel{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}$

repeat row operations...

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix}$$

is A^{-1}!!!...

well, $\overbrace{\begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}}^{A} \begin{bmatrix} 2 & -1 \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

✓

Alternately

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 3 & 8 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 2 & -\frac{3}{2} & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1/2 & 0 \\ 0 & 1 & -3/4 & 1/2 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3/4 & 1/2 \end{array}\right]$$

$$\left[\, A \mid I \,\right] \xrightarrow[\text{ops}]{\text{row}} \left[\, I \mid A^{-1} \,\right]$$

Ex Find $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}^{-1}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

I

$$\xrightarrow{-4R_1 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \xrightarrow{3R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \xrightarrow[-3R_3 + R_1 \to R_1]{-2R_3 + R_2 \to R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

So $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

2.3 The Invertible Matrix Theorem

The following are equivalent for a square matrix A $(n \times n)$.

(a) A is invertible. (A^{-1} exists) (there is a matrix you can left-multiply or right-multiply A by and get I.)

sec 2.2

(b) A row reduces to I.

square

(c) A has n pivot positions.

since A square and no free columns...

(d) "$A\vec{x} = \vec{0}$" has only the trivial solution

Suppose (c). And you try to solve $A\vec{x} = \vec{0}$.

Well, $[A \mid \vec{0}] \xrightarrow{RREF} [\underset{\downarrow\downarrow\downarrow\downarrow}{I} \mid \vec{0}]$

says each variable <u>must</u> be 0.

This implies (d)

(e) The columns of A are linearly independent.

Asking if $A\vec{x} = \vec{0}$ has nontrivial sols...

Assuming (f), then when $T_A(\vec{x}) = \vec{0}$, $\vec{0}$ is the <u>only</u> solution.

(f) The linear transformation T_A is one-to-one

If $T_A(\vec{u}) = T_A(\vec{v}) \Longrightarrow T_A(\vec{u}) - T_A(\vec{v}) = \vec{0} \Longrightarrow T_A(\vec{u} - \vec{v}) = \vec{0}$

$\Longrightarrow A \cdot (\vec{u} - \vec{v}) = \vec{0} \xrightarrow{(e)} \vec{u} - \vec{v} = \vec{0} \Longrightarrow \vec{u} = \vec{v}$

$$\begin{tikzcd}
(c) \arrow[r, Leftrightarrow, "\text{by Thm 4}"] & (g) \arrow[d, Rightarrow] \\
& (h) \arrow[d, Rightarrow] \\
& (i) \arrow[uu, Rightarrow, bend left] \\
(a) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (j) \arrow[uu, Rightarrow, bend left] \\
(g) & (k) \arrow[l, Rightarrow]
\end{tikzcd}$$

(g) The equation "$A\vec{x} = \vec{b}$" has at least one solution for all $\vec{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$

(means for <u>any</u> $\vec{b}$ in $\mathbb{R}^n$ some lin. combo of A's columns equals $\vec{b}$. So If $A\vec{x} = \vec{b}$ has a solution $\vec{x}$, $\vec{x}$'s entries are your weights...)

(i) The transformation T_A is onto.

(j) There exists a $\overset{n \times n}{\vee}$ matrix C with $A \cdot C = I$

($j \Rightarrow h$ ... because let $\vec{b} \in \mathbb{R}^n$

item j $\Longrightarrow$ $A \cdot \underbrace{C \cdot \vec{b}}_{\text{is a vector}} = \vec{b}$

So A's columns lin. combine to make $\vec{b}$... That is what (h) demands.

(k) There exists an $n \times n$ matrix D with $DA = I$.

Assume item (k). Examine $A\vec{x} = \vec{b}$

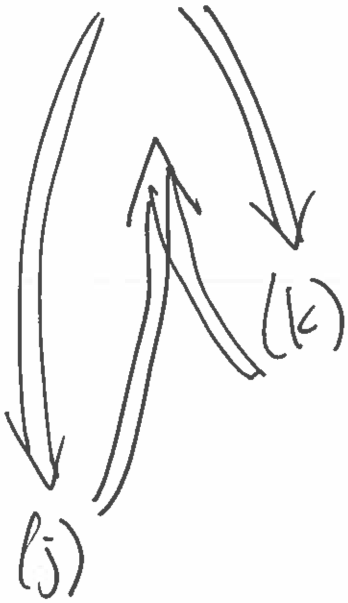
$\Longrightarrow \underline{DA}\,\vec{x} = D\vec{b}$

$\Longrightarrow \vec{x} = D\vec{b}$

$\Longrightarrow$ for all $\vec{b}$ "$A\vec{x} = \vec{b}$" has a solution

this is item (g).

(l) A^T is invertible.



Assume l: $(A^T)(A^T)^{-1} = I$

<transpose both sides>

$$\left((A^T)\cdot(A^T)^{-1}\right)^T = I^T$$

$$\left(\left(A^T\right)^{-1}\right)^T \cdot \left(A^T\right)^T = I$$

$$\underbrace{\left((A^T)^{-1}\right)^T}_{} \cdot A = I$$

$\Longrightarrow$ A has a left-inverse (item k...)