1.9 continued

Given a linear transformation, T (from $\mathbb{R}^n$ to $\mathbb{R}^m$), it's always possible to create a matrix A_T such that

$(T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
&
$T(c\vec{u}) = cT(\vec{u}))$

$$T(\vec{x}) = A_T \cdot \vec{x}$$

To build A_T, $A_T = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

(an $m \times n$ matrix)

<u>Ex</u> Suppose $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is defined ~~to~~ to be $\begin{bmatrix} x-2y \\ 5x-y \\ 8x+y \end{bmatrix}$. (This is in fact a linear transformation.)

Find A_T. Since the domain of T is $\mathbb{R}^2$, look at $T(\vec{e}_1)$ & $T(\vec{e}_2)$.

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ & $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

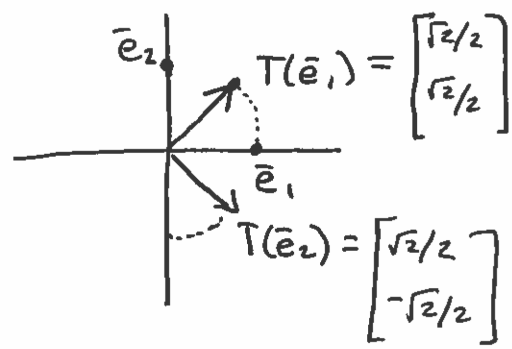
$\begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}$ $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ $\Longrightarrow A_T = \begin{bmatrix} 1 & -2 \\ 5 & -1 \\ 8 & 1 \end{bmatrix}$.

So $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 5 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ 5x-y \\ 8x+y \end{bmatrix}$

Ex $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear...

T reflects over the x-axis, and then rotates c.c.w. about the origin by $\frac{\pi}{4}$.

Find the matrix for T.



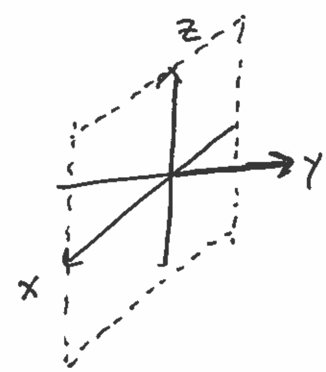
$$A_T = \begin{bmatrix} \uparrow & \uparrow \\ T(\bar{e}_1) & T(\bar{e}_2) \\ \downarrow & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

Ex $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear...

T reflects through xz-plane.

Find A_T.



$$A_T = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Def's A transformation can be onto and/or one-to-one.

- T is onto if every $\vec{b}$ in $\mathbb{R}^m$ has an $\vec{x}$ in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$.
  $\mathbb{R}^n \to \mathbb{R}^m$

- T ($\mathbb{R}^n \to \mathbb{R}^m$) is one-to-one if whenever $\underbrace{T(\vec{x}) = T(\vec{y})}_{\text{in } \mathbb{R}^m}$ it means $\underbrace{\vec{x} = \vec{y}}_{\text{in } \mathbb{R}^n}$.

If T is linear, then we have A_T.

Facts: T is onto if and only if "$A_T \cdot \vec{x} = \vec{b}$" has a solution for all $\vec{b}$ in $\mathbb{R}^m$.

T is one-to-one if and only if "$A_T \vec{x} = \vec{0}$" only has the trivial solution.

$\underline{\text{Ex}}$ $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ 5x + 7y \\ x + 3y \end{bmatrix}$$

(This is linear...)

Is T onto?

Is T one-to-one?

Well $A_T = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$

iff "$A\vec{x} = \vec{b}$" has a sol for all $\vec{b}$

Thm 4

iff A has a pivot pos on every row...

With a 3×2 matrix it's impossible for 3rd row to have a pivot

So T is not onto.

iff "$A_T\vec{x} = \vec{0}$" only has the trivial solution

iff there are no free columns.

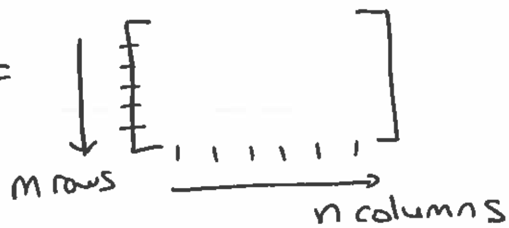
$$A_T \xrightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

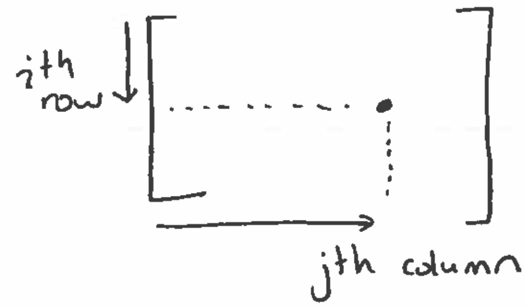
So T is one-to-one.

2.1 Matrix Operations

if A is an $m \times n$ matrix,

$A =$





the (i,j)th entry of A is denoted a_{ij}

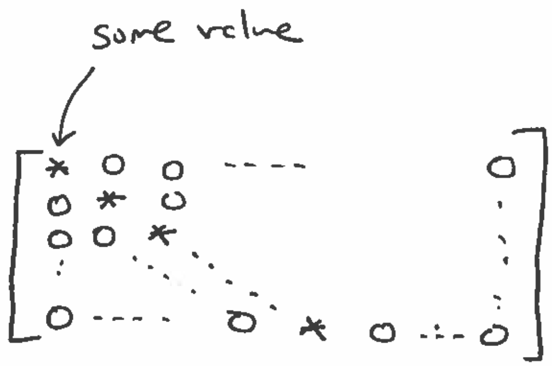
new notation

Also new notation

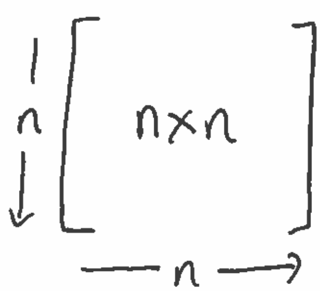
$$A = [a_{ij}]$$

$$a_{ij} = (A)_{ij}$$

A diagonal matrix: is a matrix A where $a_{ij} = 0$ wherever $i \neq j$.



A square matrix: has $m = n$.



The <u>zero matrix</u> has $a_{ij} = 0$ for all (i,j).

~~The~~ An <u>identity matrix</u> is a square, diagonal matrix where every diagonal entry is 1.

(An $n \times n$ matrix where $a_{ij} = 0$ when $i \neq j$ and $a_{ii} = 1$ for all i.)

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Called "identity" because...

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

its identity not altered...

We can add/subtract matrices... of equal dimensions.

<u>Ex</u> $\begin{bmatrix} 3 & 1 & 0 \\ 4 & 8 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 1 & -3 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 & -3 \\ 2 & 9 & 3 \end{bmatrix}$

Symbolically, $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$

<u>Ex</u> $\begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 1 & 8 \end{bmatrix} + \begin{bmatrix} 5 & 8 \\ 2 & 9 \end{bmatrix}$ is undefined.

We can scale matrices...

<u>Ex</u> $$30 \begin{bmatrix} 1 & 2 & 1 \\ 8 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 30 & 60 & 30 \\ 240 & 30 & 90 \end{bmatrix}$$

Symbolically: $(cA)_{ij} = c \cdot (A)_{ij}$

We can make linear combinations of matrices that are of the same dimensions...

<u>Ex</u> $$2 \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 8 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -18 \\ 14 & -3 \end{bmatrix}$$

<u>Facts</u>

$$A + B = B + A$$

$$(A+B)+C = A+(B+C)$$

$$A + O = A$$

$$r(A+B) = rA + rB$$

$$(r+s)A = rA + sA$$

$$r(sA) = (rs)A$$

Multiplication of Matrices

Given $\underset{(m \times n)}{A}$ and $\underset{(n \times p)}{B}$, matrices ...

T_A (defined by $T_A(\vec{x}) = \underset{m \times n}{A} \cdot \vec{x}$)

$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

T_B (defined by $T_B(\vec{x}) = \underset{n \times p}{B} \cdot \vec{x}$)

$T_B : \mathbb{R}^p \longrightarrow \mathbb{R}^n$

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "T_B"] \arrow[rr, bend right, "T_A \circ T_B"'] & \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^m \end{tikzcd}$$

(So $(T_A \circ T_B)(\vec{x}) = T_A(T_B(\vec{x}))$)

a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^m$ exists... $\rightsquigarrow$ Build $A_{(T_A \circ T_B)}$

$$\text{Define } A \cdot B = A_{(T_A \circ T_B)}$$