1.8 Intro to Linear Transformations

We understand that $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$

is the same as $A \cdot \vec{x} = \vec{b}$

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Prior to today, $\vec{x}$ is unknown, our job is to solve for $\vec{x}$.

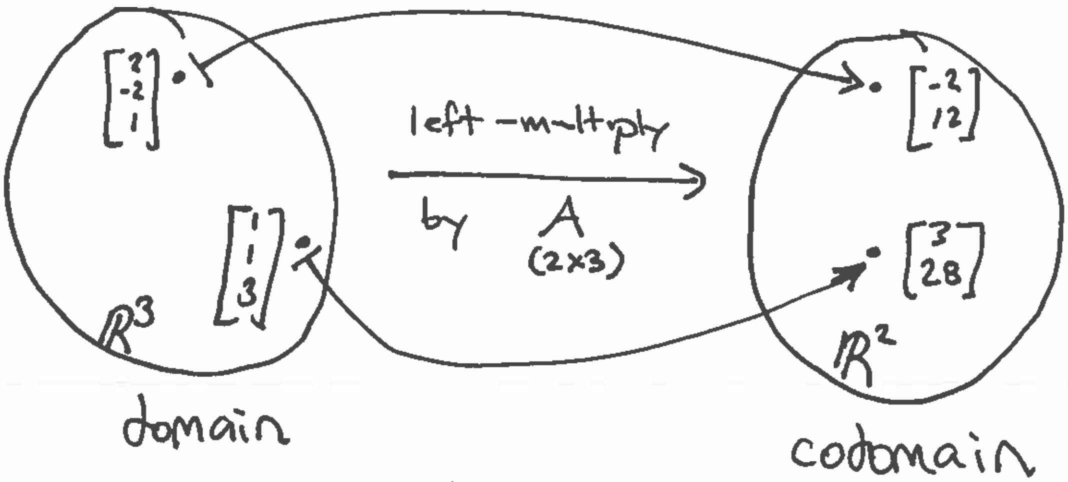
If $\vec{x}$ is a specific vector where $A\vec{x} = \vec{b}$, interpret this as "A transformed $\vec{x}$ into $\vec{b}$."

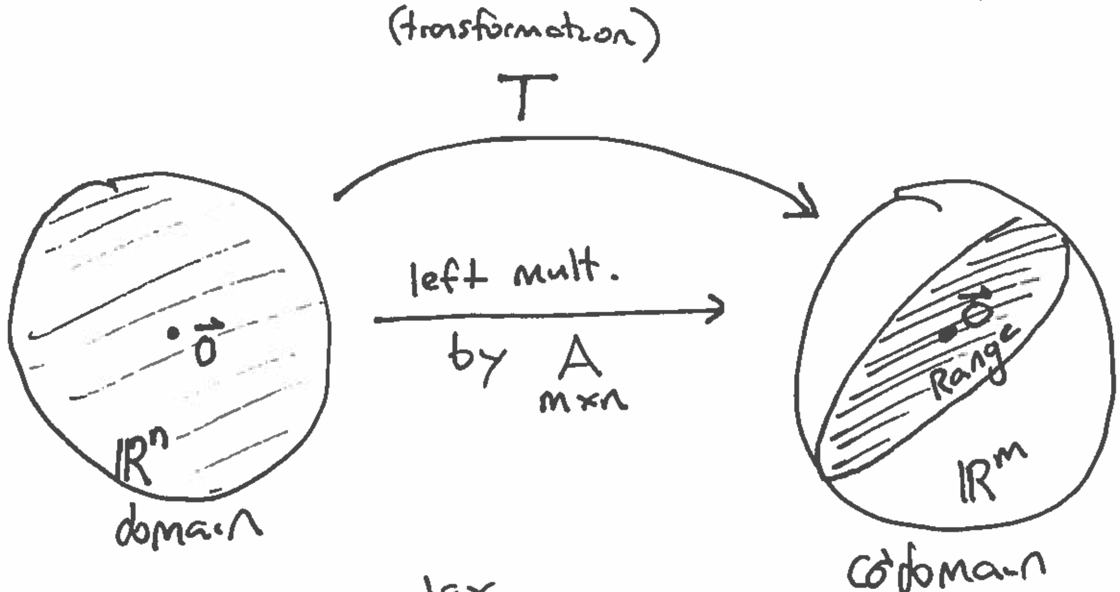
Left-multiplying a vector by a matrix produces another vector...

Ex $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 9 \end{bmatrix}$.

Calculate $A\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 12 \end{bmatrix}$

And: $A\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 28 \end{bmatrix}$





matrix

vector

Can discuss $A \cdot \vec{x}$

multiplication

function evaluation

or $T(\vec{x})$

transformation (like a function)

vector

A <u>transformation</u> T is any map/function that takes vectors as input and gives vectors as outputs.

A <u>matrix transformation</u> is a transformation T_A defined by $T_A(\vec{x}) = A \cdot \vec{x}$

So... if $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ 1 & 6 \end{bmatrix}$, then $T_A\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 6 \\ 24 \\ 10 \end{bmatrix}.$$

<u>Ex</u> $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 1 \end{bmatrix}$ with $T_A(\vec{x}) := A \cdot \vec{x}$

- What is $T_A\left(\begin{bmatrix} 3 \\ -4 \end{bmatrix}\right)$?

$$= A \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ -10 \end{bmatrix}$$

- Find $\vec{x}$ such that $T_A(\vec{x}) = \begin{bmatrix} 19 \\ 14 \\ 2 \end{bmatrix}$.

$$\Rightarrow A \cdot \vec{x} = \begin{bmatrix} 19 \\ 14 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 19 \\ 14 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 19 \\ 2 & 1 & 14 \\ -2 & 1 & 2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

- Is there more than one solution to $T_A(\vec{x}) = \begin{bmatrix} 19 \\ 14 \\ 2 \end{bmatrix}$?

  we saw $\left[\begin{array}{c|c} A & \begin{matrix} 19 \\ 14 \\ 2 \end{matrix} \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{array}\right]$

  No free columns $\Longrightarrow$ not more than one solution.

- Is $\begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}$ in the range of T_A?

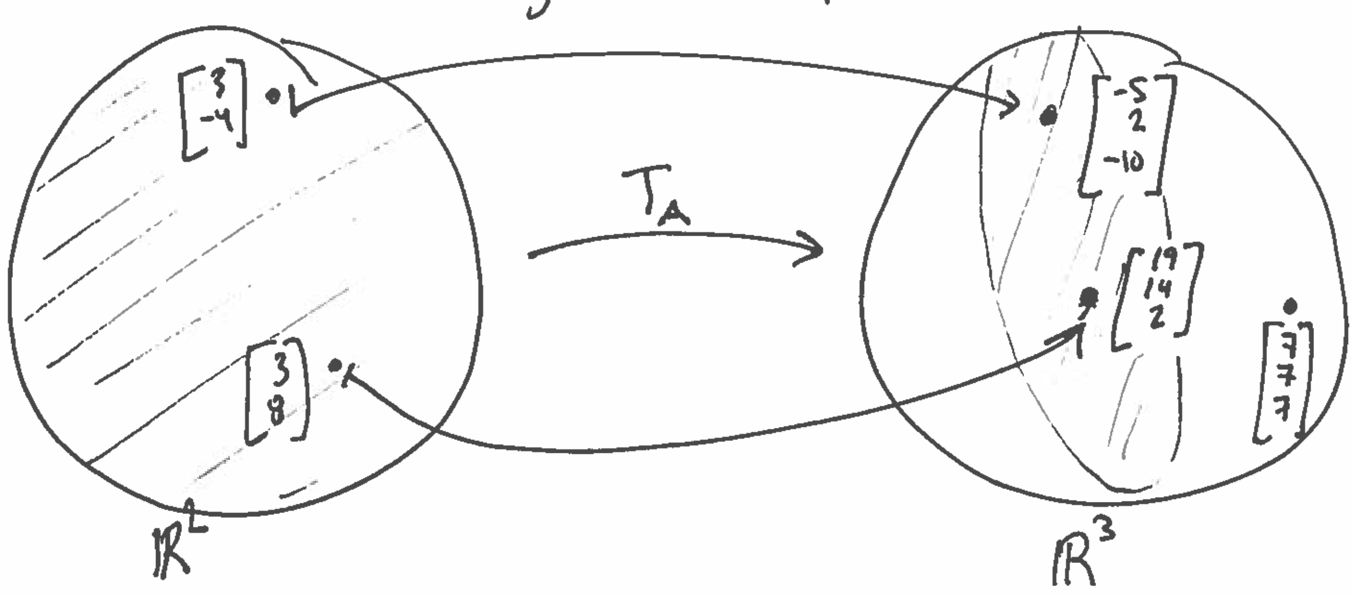
  the possible outputs from T.

  Is there a solution to $T_A(\vec{x}) = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}$?

  Set up $\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & 7 \\ -2 & 1 & 7 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

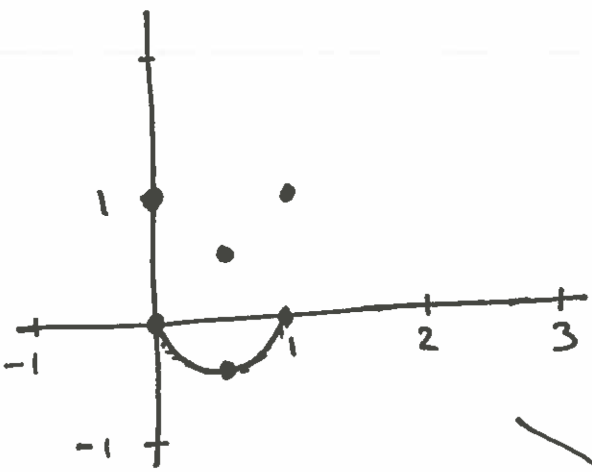
  inconsistent!

  So no, $\begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}$ is not in the range of T_A

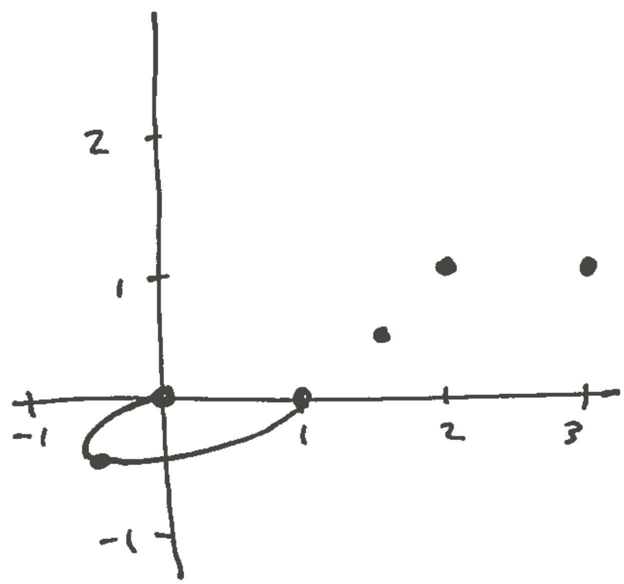


Ex Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and we work with T_A.

$$\Big(\underset{\substack{\text{name}\\ \text{of}\\ \text{transformation}}}{T_A} : \underset{\text{domain}}{\mathbb{R}^2} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^2} \Big)$$



$\mathbb{R}^2$ (domain) $\xrightarrow{T_A}$



$\mathbb{R}^2$ (codomain)

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$

$\downarrow$

$T_A\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

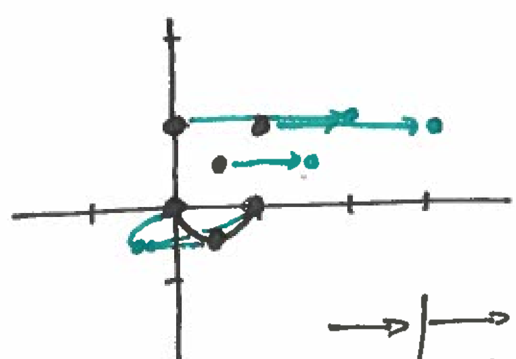
$T_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

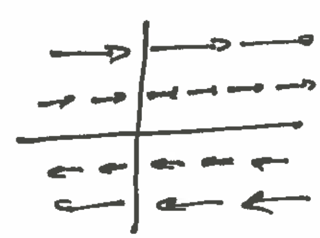
$T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$T_A\left(\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$

$T_A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \cdots = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

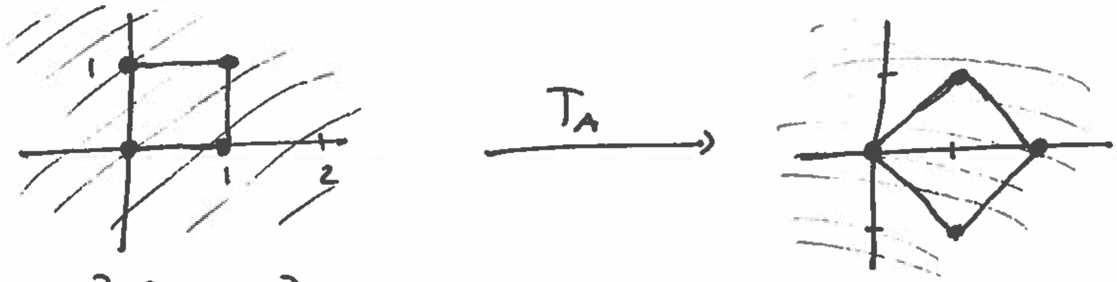
$T_A\left(\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix}$





Ex $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Work with T_A.

$(T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2)$

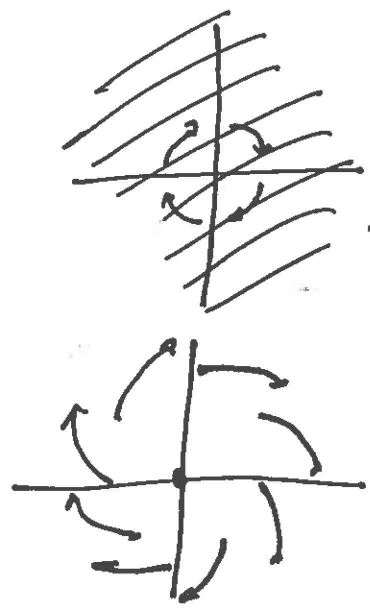


$\mathbb{R}^2$ (domain) $\qquad\qquad$ $\mathbb{R}^2$ (codomain)

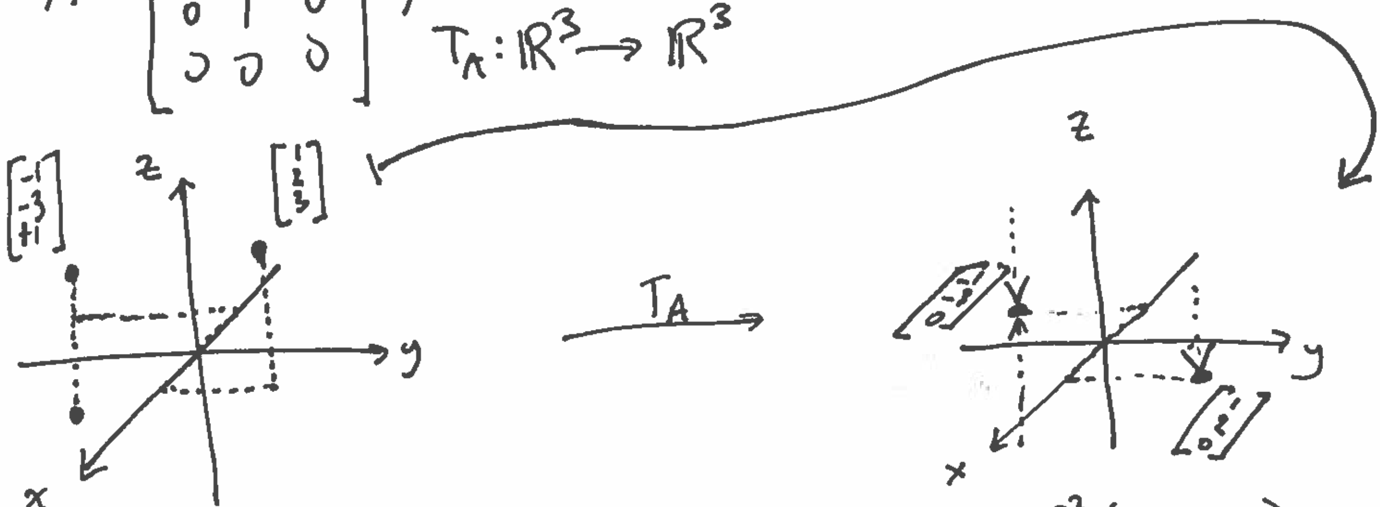
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow \end{matrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



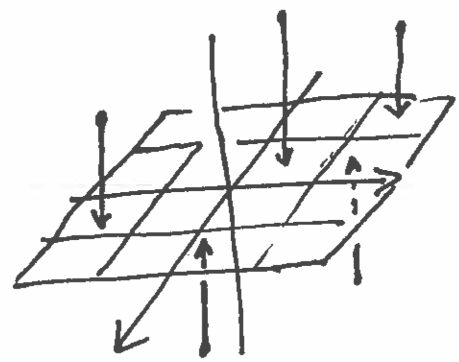
Ex $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, with $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$



$\mathbb{R}^3$ (domain) $\qquad\qquad$ $\mathbb{R}^3$ (codomain)

Observe $T_A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

this transformation <u>projects</u> $\mathbb{R}^3$ onto the xy-plane



Summary: Given $\underset{(m \times n)}{A}$, you get $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

And in low dimensions, visualizing T_A is possible.

Our examples: shearing,
rotated & dilated,
projection,

A transformation (in general) is any kind of function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

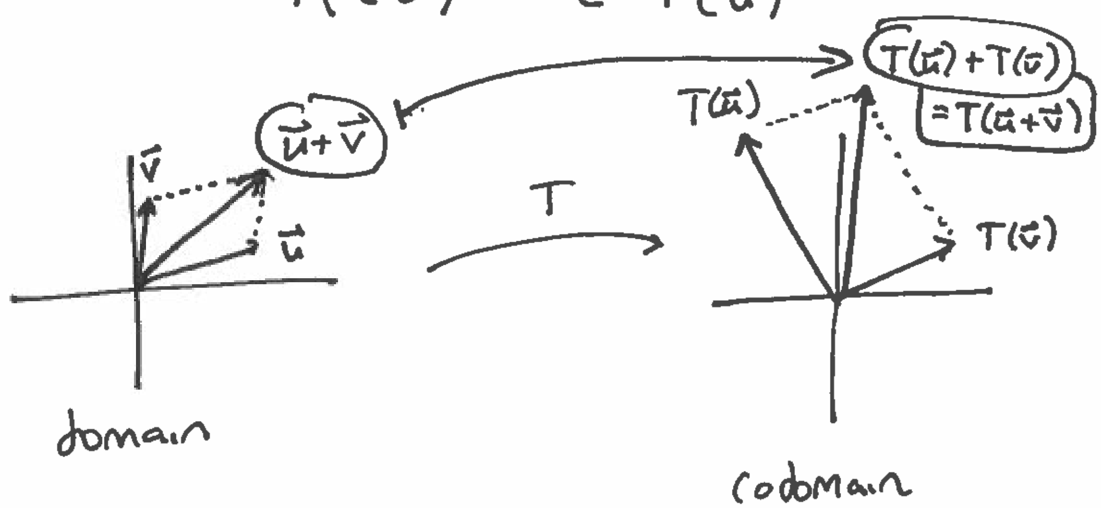
A matrix transformation is specifically a transformation defined by $T_A(\vec{x}) := A \cdot \vec{x}$

A <u>linear transformation</u> is a transformation T satisfying:

1) for all vectors $\vec{u}$ and $\vec{v}$ in the domain,
$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$

2) for all vectors $\vec{u}$ in the domain and for all scalars c,
$$T(c\vec{u}) = c \cdot T(\vec{u})$$

Visually:



Fact: every matrix transformation is a linear transformation.

Because: $T_A(\vec{u}+\vec{v}) = A\cdot(\vec{u}+\vec{v}) = A\cdot\vec{u} + A\cdot\vec{v} = T_A(\vec{u}) + T_A(\vec{v})$

$T_A(c\cdot\vec{u}) = A\cdot(c\vec{u}) = c\cdot A\cdot\vec{u} = c\cdot T_A(\vec{u})$ ✓

Ex T is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} x+y \\ x^2-y \\ 2+y \end{bmatrix} \qquad \text{E.g. } T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Is this linear?

If so, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

(no matter what $\vec{u}, \vec{v}$ are...)

well $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

$$= T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}$$

not the same!

This transformation is not linear!

$\underline{\text{Ex}}$ Suppose $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y-z \\ x \end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$.

Is T linear? Try randomly checking...

$$T\left(\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}$$

and...

$$T\left(\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}\right) + T\left(\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}$$

same!

start to <u>suspect</u> T <u>is</u> linear...

set out to prove it.

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1+x_2+2(y_1+y_2) \\ y_1+y_2-(z_1+z_2) \\ x_1+x_2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+2y_1 \\ y_1-z_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2+2y_2 \\ y_2-z_2 \\ x_2 \end{bmatrix}$$

same? Yes!

So condition 1) of linearity is met!

Also, $T\left(c\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = T\left(\begin{bmatrix}cx\\cy\\cz\end{bmatrix}\right)$

$$= \begin{bmatrix}cx+2cy\\cy-cz\\cx\end{bmatrix}$$

$$c\cdot T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = c\cdot\begin{bmatrix}x+2y\\y-z\\x\end{bmatrix}$$

Same? yes.

With both conditions verified, T <u>is linear!</u>

Shortcut: If T is linear, then $T(\vec{0}) = \vec{0}$.

$$\underset{\displaystyle T(\vec{0})}{\underset{\shortparallel}{T(\vec{0}+\vec{0})}} = \underset{\displaystyle 2T(\vec{0})}{\underset{\shortparallel}{T(\vec{0})+T(\vec{0})}} \implies T(\vec{0}) = \vec{0}.$$

Ex $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) := \begin{bmatrix}x+y\\x^2-y\\2+y\end{bmatrix}$. Well $T(\vec{0}) = \begin{bmatrix}0\\0\\2\end{bmatrix}$

not $\vec{0}$.

So T is not linear.

1.9 The Matrix of A Linear Transformation

Two Ideas

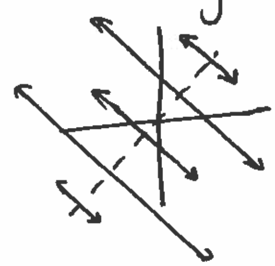
A matrix can be used via matrix multiplication to make a linear transformation

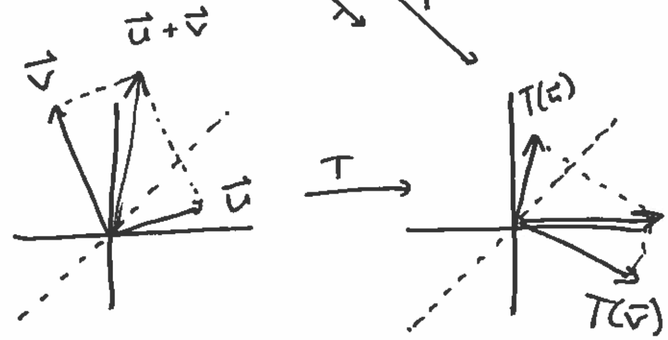
<u>Ex</u> $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Define $T_A(\vec{x}) = A \cdot \vec{x}$

A linear transformation in general need not "come with" a matrix

<u>Ex</u> Let $T: \mathbb{R}^2 \to \mathbb{R}$ reflect over the line $y = x$.





Ultimately these concepts are the same.

Any matrix $\rightleftarrows$ linear transformation

Some notation: in $\mathbb{R}^n$, $\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_2}, \ldots, \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\vec{e}_n}$ (the standard unit vectors)

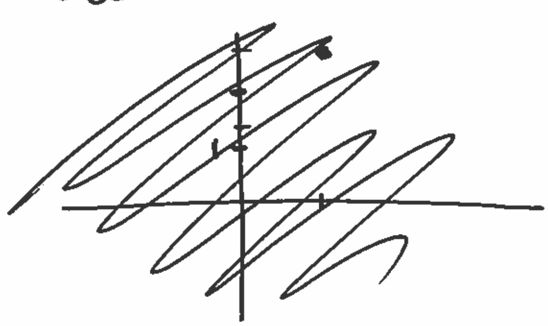
Imagine $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.

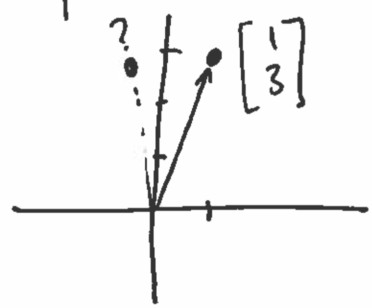
Well, $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T(x \cdot \bar{e}_1 + y \cdot \bar{e}_2)$$
$$= T(x\bar{e}_1) + T(y\bar{e}_2)$$
$$= x \cdot \underbrace{T(\bar{e}_1)}_{\text{vector}} + y \cdot \underbrace{T(e_2)}_{\text{vector}}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \uparrow & \uparrow \\ T(\bar{e}_1) & T(\bar{e}_2) \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This shows us that T leads to a special matrix, A_T, where $A_T = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ T(\bar{e}_1) & T(\bar{e}_2) & \cdots & T(\bar{e}_n) \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

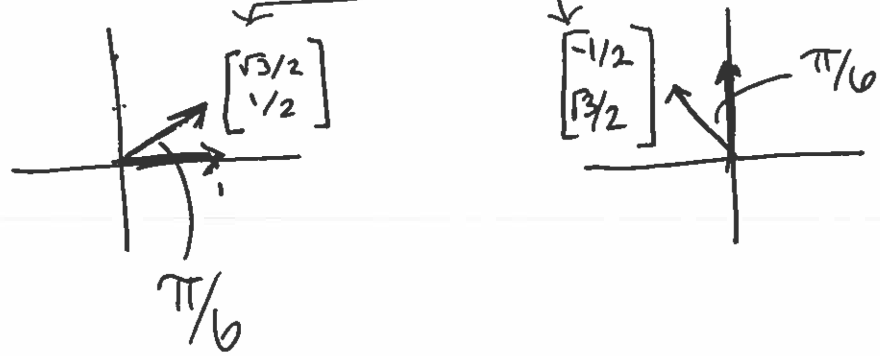
Ex Suppose T rotates counterclockwise by $\frac{\pi}{6}$ radians about $\vec{0}$. Ultimate question: Where does $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ land?





Rotation is linear ✓ Let's build A_T.

To do that, need $T(\vec{e}_1)$, $T(\vec{e}_2)$



So $A_T = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$.

So... where does $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ land?

$$T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 - 3/2 \\ 1/2 + 3\sqrt{3}/2 \end{bmatrix} \approx \begin{bmatrix} -.63 \\ 3.09 \end{bmatrix}$$