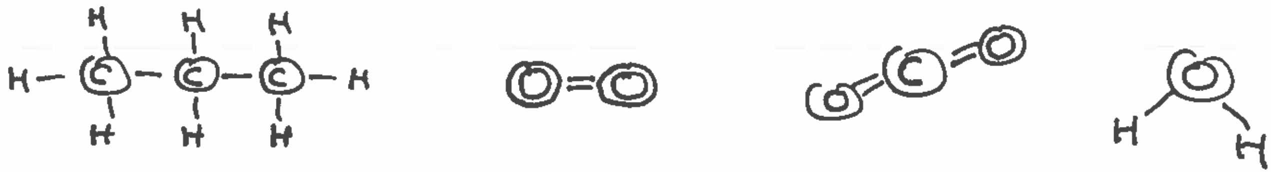
~~Ex~~ 1.6 (continued)

Ex Burning propane

$$n_1 \cdot C_3H_8 + n_2\, O_2 \longrightarrow n_3\, CO_2 + n_4\, H_2O$$



Finding acceptable values for n_1, n_2, n_3, n_4 is called balancing the equation.

Make equations:

Hydrogen: $8n_1 = 2n_4$

Carbon: $3n_1 = n_3$

Oxygen: $2n_2 = 2n_3 + n_4$

$$\begin{cases} 8n_1 - 2n_4 = 0 \\ 3n_1 - n_3 = 0 \\ \boxed{2n_2} - 2n_3 - n_4 = 0 \end{cases} \qquad \left[\begin{array}{cccc|c} 8 & 0 & 0 & -2 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} n_4 \text{ is free} \\ n_3 = \frac{3}{4} n_4 \\ n_2 = \frac{5}{4} n_4 \\ n_1 = \frac{1}{4} n_4 \end{array}\right\} \quad \begin{bmatrix} 1/4 \\ 5/4 \\ 3/4 \\ 1 \end{bmatrix} \cdot n_4$$

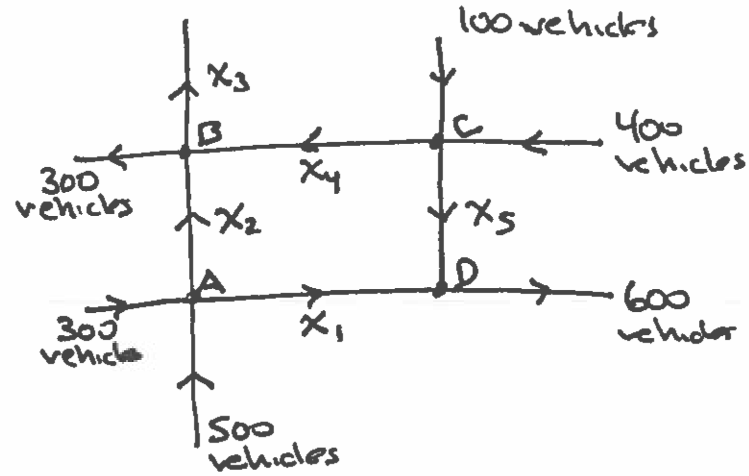
We choose $n_4 = 4$ to clear denominators:

$$\vec{n} = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 4 \end{bmatrix} \qquad C_3H_8 + 5O_2 \longrightarrow 3CO_2 + 4H_2O$$

Ex Network Flow

One-way streets

record traffic over one particular day at some locations



Goal: to understand x_1, x_2, x_3, x_4, x_5

Set up equations, based on in-flow = outflow.

at A: $300 + 500 = x_1 + x_2$

at B: $x_2 + x_4 = x_3 + 300$

at C: $100 + 400 = x_4 + x_5$

at D: $x_1 + x_5 = 600$

overall: $500 + 300 + 100 + 400 = 300 + x_3 + 600$

$$\begin{cases} x_1 + x_2 = 800 \\ x_2 - x_3 + x_4 = 300 \\ x_4 + x_5 = 500 \\ x_1 + x_5 = 600 \\ x_3 = 400 \end{cases} \qquad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{array}\right]$$

$\xrightarrow{\text{RREF}}$

$$\cancel{\begin{matrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{matrix}}$$

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 0 & 1 & 600 \\ 0 & \boxed{1} & 0 & 0 & -1 & 200 \\ 0 & 0 & \boxed{1} & 0 & 0 & 400 \\ 0 & 0 & 0 & \boxed{1} & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_5 is free

$x_4 = -x_5 + 500$

$x_3 = 400$

$x_2 = x_5 + 200$

$x_1 = -x_5 + 600$

Interpretation:

$x_3 = 400$

x_5 is free $\Longrightarrow$ $x_5 \geq 0$

$x_4 = -x_5 + 500 \Longrightarrow$ x_4 needs to be ≥ 0
So $x_5 \leq 500$

$x_2 = x_5 + 200 \Longrightarrow$ x_2 needs to be ≥ 0
So $x_5 \geq -200$

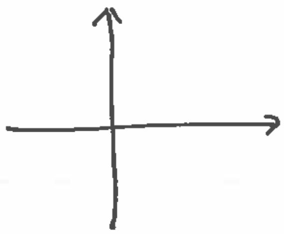
$x_1 = -x_5 + 600 \Longrightarrow$ x_1 needs to be ≥ 0
So $x_5 \leq 600$

x_5 must be in $[0, 500]$.

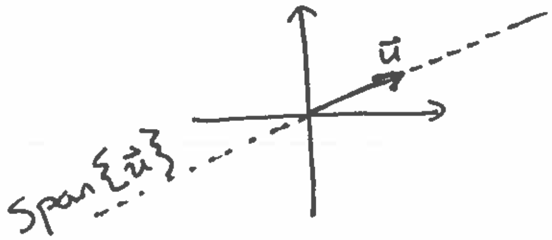
So x_1 in $[100, 600]$
x_2 in $[200, 700]$
$x_3 = 400$
x_4 ~~is~~ in $[0, 500]$

1.7 Linear Independence

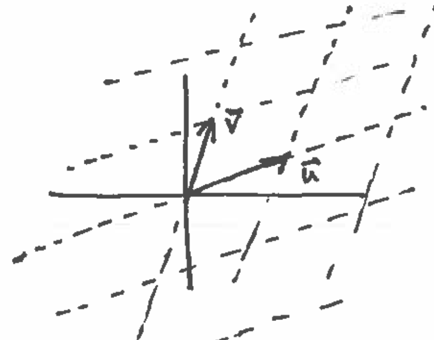
Consider $\mathbb{R}^2$



Now with a vector $\vec{u}$

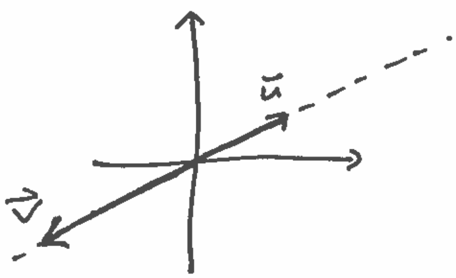


Now, also with $\vec{v}$



Span$\{\vec{u}, \vec{v}\}$ here, is $\mathbb{R}^2$

But could have gone down



Span$\{\vec{u}, \vec{v}\}$ is just a line.

Span$\{\vec{u}, \vec{v}\}$ is all of $\mathbb{R}^2$

OR

Span$\{\vec{u}, \vec{v}\}$ is just a line

in this case, one of $\vec{u}$, $\vec{v}$ is "redundant".

Span$\{\vec{u}, \vec{v}\}$ = Span$\{\vec{u}\}$

Span$\{\vec{u}, \vec{v}\}$ = Span$\{\vec{v}\}$

A vector in a set $\{\vec{v}_1, \ldots \vec{v}_p\}$ is "redundant" if Span$\{\vec{v}_1, \ldots, \vec{v}_p\}$ = Span$\{$with that one vector removed$\}$.

Ex $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

$\quad\vec{v}_1 \qquad \vec{v}_2 \qquad \vec{v}_3$

Are any of these "redundant"?

$\vec{v}_2 = 2 \cdot \vec{v}_1$.

Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ = Span$\{\vec{v}_1, \vec{v}_3\}$

$\vec{v}_2$ is "redundant"

Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ = Span$\{\vec{v}_2, \vec{v}_3\}$

$\vec{v}_1$ is "redundant"

$\Downarrow$

What mattered is $\vec{v}_2 = 2\vec{v}_1$.

$$\text{A.k.a.} \quad -2\vec{v}_1 + \vec{v}_2 = \vec{0}$$

$$\text{A.k.a.} \quad \underbrace{-2\vec{v}_1 + \vec{v}_2 + 0\vec{v}_3}_{\text{a nontrivial linear combination of } \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}} = \vec{0}$$

equal to $\vec{0}$.

nontrivial

Ex $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

What matters here is

$$-1 \cdot \vec{v}_1 + 2\vec{v}_2 = \vec{v}_3$$

$$\Updownarrow$$

$$\underbrace{-1 \cdot \vec{v}_1 + 2\vec{v}_2 - \vec{v}_3}_{\text{A nontrivial linear combination of } \vec{v}_1, \vec{v}_2, \vec{v}_3} = \vec{0}$$

that equals $\vec{0}$.

Def A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\}$ is __linearly dependent__ if there is a nontrivial linear combo of them that equals $\vec{0}$.

(an adjective that applies)

Ex $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is __linearly dependent__ because $-2\vec{v}_1 + \vec{v}_2 + 0\vec{v}_3 = \vec{0}$.

Ex $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is <u>linearly dependent</u>

because

$$-\bar{v}_1 + 2\bar{v}_2 - \bar{v}_3 = \vec{0}$$

A set $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p\}$ is <u>linearly independent</u>

if whenever a linear combo of these vectors is $\vec{0}$, it has to be the trivial linear combo.

$$\left(\sum_{i=1}^{p} c_i \vec{v}_i = \vec{0} \implies c_i = 0 \text{ for all } i \right)$$

Ex Is $\left\{ \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

$$\left[\begin{array}{ccc|c} 3 & -2 & -2 & 0 \\ 1 & 0 & 4 & 0 \\ -2 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(asking to explore solutions to $c_1 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$)

free column.
the system has nontrivial sols.
the set is linearly <u>de</u>pendent.

Further more... c_3 is free ... so choose $c_3 = 1$

$c_2 = -7c_3$ ... $c_2 = -7$

$c_1 = -4c_3$ ... $c_1 = -4$

$$\text{So } -4\underbrace{\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}}_{\vec{v}_1} - 7\underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2} + \underbrace{\begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}}_{\vec{v}_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = 4\vec{v}_1 + 7\vec{v}_2$$

$$\vec{v}_2 = \frac{1}{7}v_3 - \frac{4}{7}\vec{v}_1$$

$$\vec{v}_1 = -\frac{7}{4}v_2 + \frac{1}{4}\vec{v}_3$$

Linear Dependence $\Longrightarrow$ at least some of the $\vec{v}_i$ are "redundant".

<u>Shortcut</u> Columns of a matrix A are linearly dependent if and only if the matrix equation $A\vec{x} = \vec{0}$ has nontrivial solutions if and only if A has a free column.

Ex In $\mathbb{R}^3$, are $\{\hat{i}, \hat{j}, \hat{k}\}$ linearly independent?

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

well $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has no free columns, so the set is linearly <u>in</u>dependent.

Can a set of 2 vectors be linearly <u>dependent</u>?

Yes, if they are parallel.

Yes, if $\vec{v}_2 = k \cdot \vec{v}_1 \rightsquigarrow \underbrace{-k\vec{v}_1 + \vec{v}_2}_{\substack{\text{appropriate way} \\ \text{to tell someone, OK,} \\ \{\vec{v}_1, \vec{v}_2\} \text{ are} \\ \text{dependent.}}} = \vec{0}$

Could a set of <u>one</u> vector be linearly dependent?

Yes, if its the $\vec{0}$-vector.

$$\underbrace{1 \cdot \vec{0}}_{\substack{\text{nontrivial} \\ \text{linear combo!}}} = \vec{0}$$

With 3 or more vectors, linear dependency is NOT NOT NOT as simple as some vector being a scalar multiple of some other vector.

<u>Ex</u> $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$.

None of these are multiples of another.

And yet $2\vec{v}_1 - \vec{v}_2 - \vec{v}_3 = \vec{0}$.

So this is a linearly <u>dependent</u> set.

<u>Ex</u> Describe $\text{span}\left\{ \begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 3 \\ 1 \end{bmatrix} \right\}$ Geometrically.

Well, these vectors are independent...

because $\begin{bmatrix} 1 & -8 \\ 8 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ No free cols $\Downarrow$ Linearly Independent.

So we have two "truly different" directions

$\Longrightarrow$ The span is a <u>plane</u> in $\mathbb{R}^3$.

Ex Describe $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}5\\3\\0\end{bmatrix}, \begin{bmatrix}17\\\pi\\0\end{bmatrix}\right\}$

geometrically.

$$\begin{bmatrix}1 & 1 & 5 & 17\\ 0 & 1 & 3 & \pi\\ 0 & 0 & 0 & 0\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1 & 0 & * & *\\ 0 & 1 & * & *\\ 0 & 0 & 0 & 0\end{bmatrix}$$

free columns!

So we have a dependent set...

Last two cols are free...

I interpret last two vectors as "redundant".

So we have $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$.

So our span is a plane in $\mathbb{R}^3$.

Fact: If $\{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_p\}$ contains $\vec{0}$, then this set is linearly dependent.