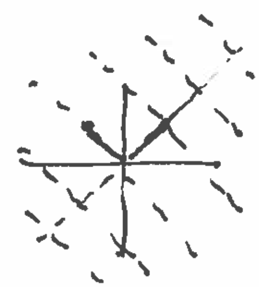
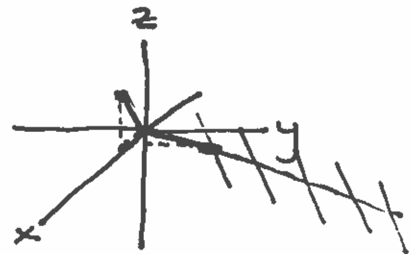
1.3 review

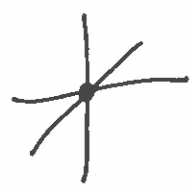
Notation: $\text{Span}\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p\}$ means the collection of all linear combinations of $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p$

Ex $\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\} =$  $= \mathbb{R}^2$

Ex $\text{Span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\} =$ 

$=$ a plane in $\mathbb{R}^3$

Ex $\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\} =$  $=$ the line $y = x$.

Ex $\text{Span}\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\} =$  $=$ the origin (only)

Ex $\text{Span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} = \mathbb{R}^4$

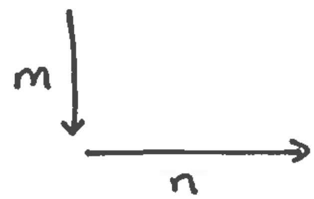
Any $\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = a\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$.

So every vector in $\mathbb{R}^4$ IS a lin. combo of those four vectors

1.4 Matrix Equation $A\vec{x} = \vec{b}$.

A matrix M is said to be $m \times n$ if

- m rows
- n columns



<u>Ex</u> $2\downarrow \underset{\xrightarrow{\qquad 3 \qquad}}{\begin{bmatrix} 1 & 3 & 8 \\ 2 & 0 & 1 \end{bmatrix}}$ is a 2×3 matrix

Well: $\begin{cases} 3x + 2y - z = 8 \\ x + y + 2z = 5 \end{cases}$ A system of linear equations

$$x\begin{bmatrix} 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$ now... a vector equation.

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$ now... a matrix equation.

In general: $\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}\begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}$ means use these numbers as weights applied to these columns

and make a linear combination.

Ex
$$\underset{2\times 3}{\begin{bmatrix} 3 & 2 & 1 \\ 8 & 0 & -2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}} = 5\begin{bmatrix} 3 \\ 8 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$= \begin{bmatrix} 15 \\ 40 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$
$$= \begin{bmatrix} 15 \\ 32 \end{bmatrix}$$

Ex
$$\underset{3\times 2}{\begin{bmatrix} 7 & 1 \\ 1 & 7 \\ 3 & 2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 1 \end{bmatrix}$$

Ex
$$\underset{2\times 3}{\begin{bmatrix} 3 & 1 & 8 \\ 4 & 2 & 9 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} = \text{undefined!}$$

don't match!

Everything from 1.3 can be reinterpreted with a matrix equation.

"Is $\vec{b}$ a linear combo of $\bar{c}_1, \bar{c}_2, \dots, \bar{c}_p$?"

"Is there a solution to $x_1\bar{c}_1 + x_2\bar{c}_2 + \dots + x_p\bar{c}_p = \vec{b}$?"

"Is there a solution to $\begin{bmatrix} \vdots & \vdots & & \vdots \\ \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_p \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \vec{b}$?"

Ex Does $\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ have a solution for all b_1, b_2, b_3?

"Is the system $\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right]$ consistent for all b_1, b_2, b_3?"

"Is the span of $\left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix} \right\}$ equal to $\mathbb{R}^3$?"

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \xrightarrow[3R1+R3 \to R3]{4R1+R2 \to R2} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & -\frac{1}{2}(4b_1 + b_2) + 3b_1 + b_3 \end{array}\right]$$

Only has a solution when $-\frac{1}{2}(4b_1 + b_2) + 3b_1 + b_3 = 0$.

$$b_1 - \frac{1}{2}b_2 + b_3 = 0$$

With this $A\vec{x} = \vec{b}$, there's only a solution when $\vec{b}$ is in a certain plane defined by

Fact The following things are all equivalent for a given $m \times n$ matrix A. (all are true, or all are false, depending on what A is...)

a) The equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ in $\mathbb{R}^m$.

b) Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c) The columns of A span $\mathbb{R}^m$.
($\text{Span}\{\vec{a}_1, \vec{a}_2, \dots, a_n\} = \mathbb{R}^m$)

d) The matrix A has a pivot position in every row.

Prove d) $\Longrightarrow$ a). Assume d), that the matrix <u>A has a pivot position in every row</u>. Imagine for some $\vec{b}$, the equation $A\vec{x} = \vec{b}$. You set up $[A \mid \vec{b}]$. You RREF: $\left[\begin{array}{ccc|c} 1 & & & * \\ & 1 & & * \\ & & 1 & * \end{array}\right]$. In other words, $\left| \underbrace{0\ 0\ \cdots\ 0}_{\text{there should have been a pivot.}} \middle| \begin{smallmatrix}\text{not}\\ \text{zero}\end{smallmatrix} \right|$ is not possible.

So the system is consistent (no matter what $\vec{b}$ is.)

Ex Do $\left\{ \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix} \right\}$ span $\mathbb{R}^3$?

(item (c) ~~says~~ is about columns of A spanning $\mathbb{R}^m$.)

Equivalent to whether or not

$$\begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 6 \\ 8 & 1 & 9 \end{bmatrix}$$ has a pivot pos. in every row.

$\downarrow$ RREF

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ It does not! So by this theorem, those 3 vectors don't span $\mathbb{R}^3$.

One Last Thing!

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 8 \end{bmatrix} + z \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Alternatively:

$$\begin{array}{cc} \text{1st row} & \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 5 \end{bmatrix} \end{array} \begin{array}{c} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \text{1st column} \end{array} \qquad 1x + 2y + 4z$$

$$\begin{array}{cc} \text{2nd row} & \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 5 \end{bmatrix} \end{array} \begin{array}{c} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \text{1st col.} \end{array} \qquad 3x + 8y + 5z$$

So...

$$\begin{bmatrix} x + 2y + 4z \\ 3x + 8y + 5z \end{bmatrix} \begin{array}{l} \text{1,1 entry} \\ \text{2,1 entry} \end{array}$$

Ex

$$\begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3+1+4 \\ 0+4+12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 16 \end{bmatrix}$$

Ex

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{identity matrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Facts

$$A(\vec{u} + \vec{v}) = A \cdot \vec{u} + A\vec{v}$$

$$A(c\vec{u}) = c(A\vec{u})$$

Sol. Sets of Linear Systems (1.5)

A system of lin eq.

is equivalent to

$$A \cdot \vec{x} = \vec{b}$$

matrix

known vector

unknown variable vector.

When $\vec{b} = \vec{0}$ ....

when we try to solve $A\vec{x} = \vec{0}$

we call the system <u>homogeneous</u>.

<u>Ex</u>
$$\begin{cases} 2x + 5y - 3z = 0 \\ x - 3y + 2z = 0 \end{cases}$$
is homogeneous.

<u>Ex</u>
$$\begin{cases} 5x - 8y = 3 \\ 2x - 3y = 1 \\ 3x - 5y = 2 \end{cases}$$
is not homogeneous.

A homogeneous system is always <u>consistent</u>.
because setting all variables to 0 is an obs. solution.

With a homogeneous system, ask "are there nontrivial solutions?"

Ex $\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \vec{x} = \vec{0}$.

(A matrix equation... and it's homogeneous.)

$\vec{x} = \vec{0}$ is a solution .... are there more?

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

presence of free col in RREF of the matrix indicates there are nontrivials solutions!

What do they "look like"?

$$\left.\begin{array}{l} x - \frac{4}{3}z = 0 \\ y = 0 \\ z \text{ is free.} \end{array}\right\} \Longrightarrow \begin{array}{l} x = \frac{4}{3}z \\ y = 0 \\ z = z \end{array}$$

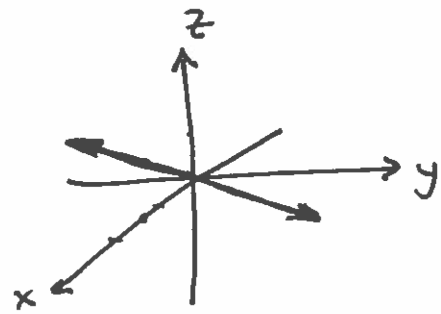
$$\vec{x} = \begin{bmatrix} \frac{4}{3}z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

So... solutions are some free variable times $\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$.

So... solutions are the span of $\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$.

So the solution set is a line.

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \vec{x} = \vec{0}$$



$\mathbb{R}^3$

this line is the sol. set

Report: the solution set is $\left\{ t \cdot \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$

neutral parameter...

<u>Ex</u> "System" $\{ 3x - 8y + 2z = 0$

homogeneous ✓

nontrivial sols ?

$$\left[\begin{array}{ccc|c} 3 & -8 & 2 & 0 \end{array}\right]$$

RREF

$$\left[\begin{array}{ccc|c} 1 & -8/3 & 2/3 & 0 \end{array}\right]$$

free columns!

we do have nontrivial sols.

y, z are free

$$x - \frac{8}{3}y + \frac{2}{3}z = 0 \longrightarrow \begin{aligned} x &= \frac{8}{3}y - \frac{2}{3}z \\ y &= y \\ z &= z \end{aligned}$$

So $$\vec{x} = \begin{bmatrix} \frac{8}{3}y - \frac{2}{3}z \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{3}y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{2}{3}z \\ 0 \\ z \end{bmatrix}$$

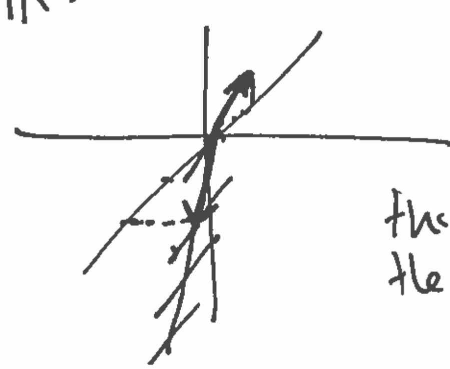
$$\vec{x} = y \begin{bmatrix} 8/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix}$$

free!

So the solution set is $\text{Span}\left\{ \begin{bmatrix} 8/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

And this is a plane in $\mathbb{R}^3$.

$\{3x - 8y + 2z = 0$



this plane is the sol. set.

Ex $\begin{cases} 3x + 5y - 4z = 0 \\ -3x - 2y + 4z = 0 \\ 6x + y - 8z = 0 \end{cases}$

Describe the solution set geometrically.

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Short version: since 1 free col. the sol set is a 1-dimensional shape... it's a line inside $\mathbb{R}^3$...

Nonhomogeneous systems...

Ex $\begin{cases} 3x + 5y - 4z = 7 \\ -3x - 2y + 4z = -1 \\ 6x + y - 8z = -4 \end{cases}$

Geom. describe the sol. set.

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Consistent!

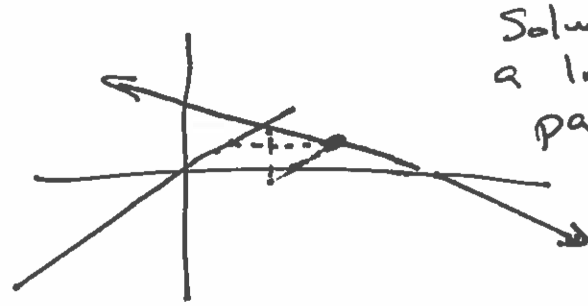
$x_1 = \frac{4}{3}x_3 - 1$

$x_2 = 2$

x_3 is free

$$\vec{x} = \begin{bmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

So solutions $\vec{x} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

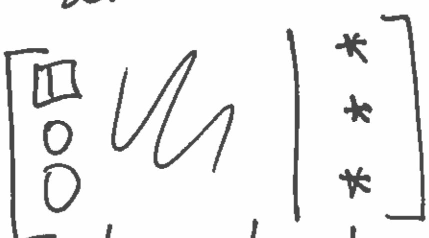


Solution set is a line <u>not</u> passing through the origin.

The solution set is a line parallel to $\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$ passing through $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

The sol set is $\underbrace{\left\{ t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}}_{\text{Not a span!}}$

To solve a nonhomogeneous system, RREF it:

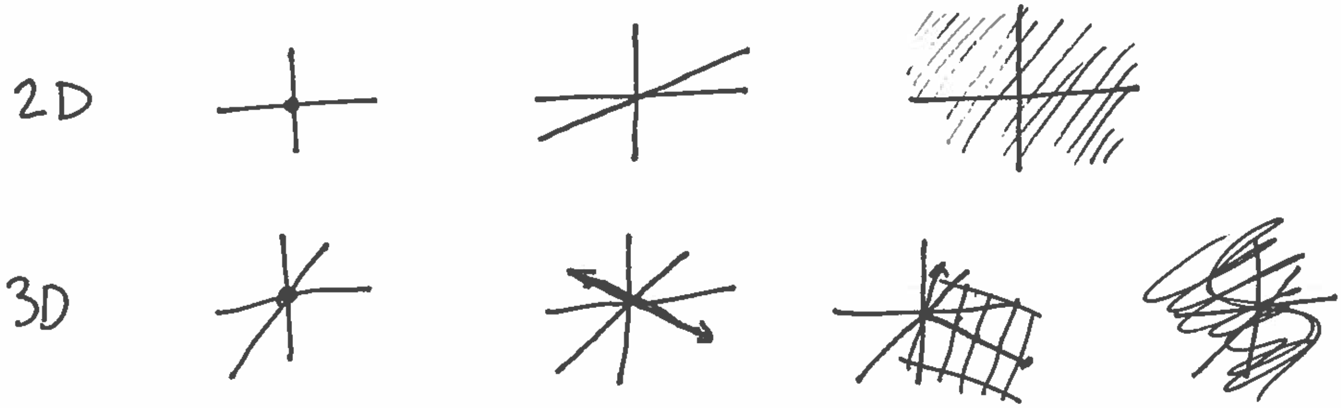


count free columns...

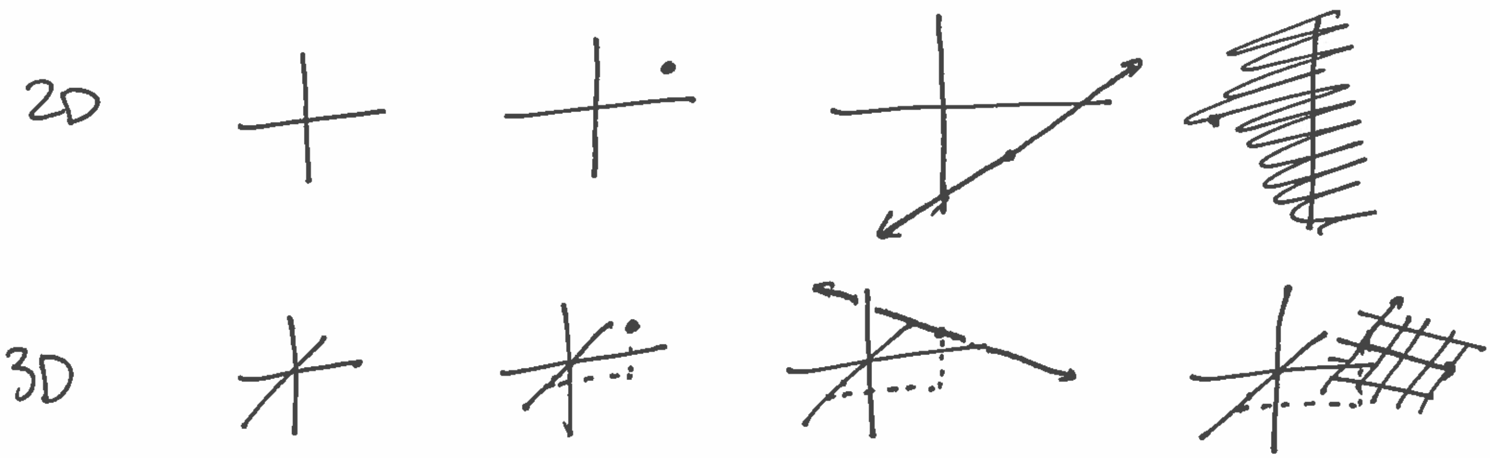
one specific solution.

the solution set is a point/line/plane/hyperplane passing through that one solution...

Homogeneous System Solution Sets

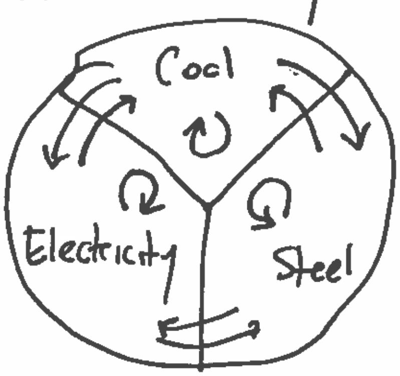


Nonhomogeneous System Sol sets



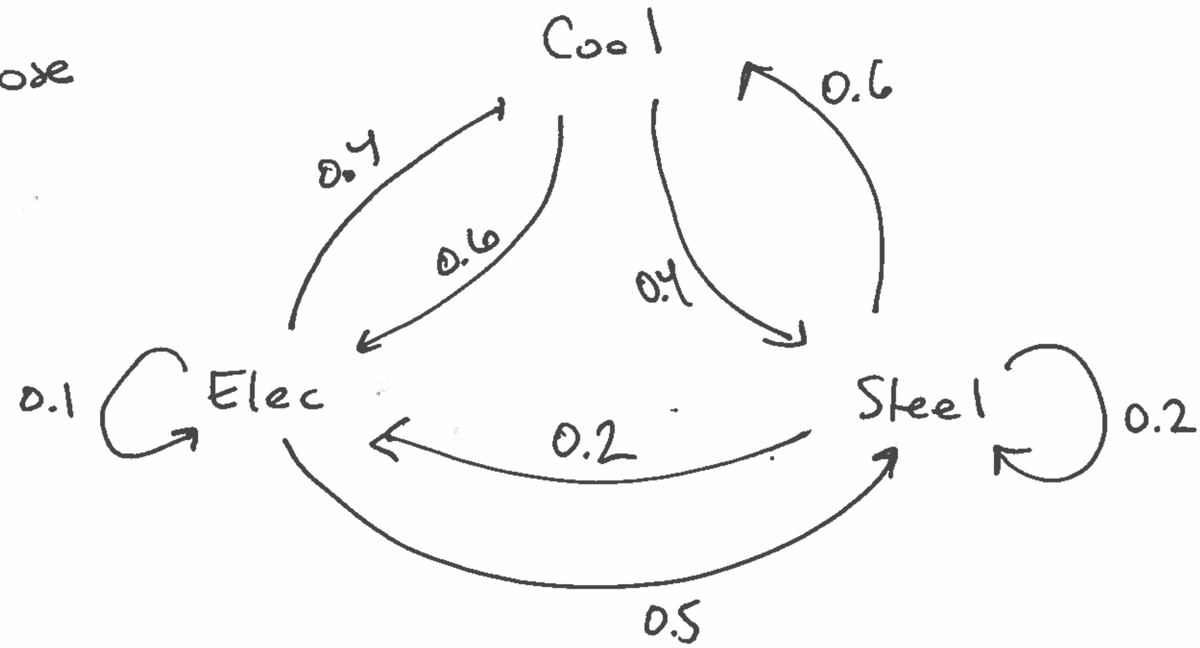
$\boxed{1.6}$ Applications

An "economy" has "sectors"



On any given day P_C, P_E, P_S are the valuations of these sectors.

Suppose



Electricity starts with p_E of value $\rightarrow$ $0.4p_E$ becomes coal's value

$\rightarrow$ $0.5p_E$ becomes steel's

$\rightarrow$ $0.1p_E$ stays.

On day 1 (following day 0)

$$\begin{cases} p_C = 0.4p_E + 0.6p_S \\ p_E = 0.6p_C + 0.1p_E + 0.2p_S \\ p_S = 0.4p_C + 0.5p_E + 0.2p_S \end{cases}$$

$$\begin{aligned} 0 &= -p_C + 0.4p_E + 0.6p_S \\ 0 &= 0.6p_C - 0.9p_E + 0.2p_S \\ 0 &= 0.4p_C + 0.5p_E - 0.8p_S \end{aligned}$$

$$
\begin{array}{ccc|c}
p_C & p_E & p_S & \\
\end{array}
$$

$$
\left[\begin{array}{ccc|c}
-1 & 0.4 & 0.6 & 0 \\
0.6 & -0.9 & 0.2 & 0 \\
0.4 & 0.5 & -0.8 & 0
\end{array}\right]
$$

$$
\xrightarrow{\text{RREF}}
\left[\begin{array}{ccc|c}
1 & 0 & -0.94 & 0 \\
0 & 1 & -0.85 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

p_S is free $(p_S = p_S)$

$p_E = 0.85 p_S$

$p_C = 0.94 p_S$