__Ex__ System of 4 linear equations in 4 variables $\longrightarrow$ augmented matrix

$$\left[\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right]$$

Is there a solution?

If so, how many?

Can we describe them?

RREF

$$\xrightarrow{R1 \leftrightarrow R4} \left[\begin{array}{cccc|c} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right] \xrightarrow[2R1+R3 \to R3]{R1+R2 \to R2} \left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R2 \to R2} \left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right] \xrightarrow[3R2+R4 \to R4]{-5R2+R3 \to R3} \left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{array}\right]$$

$$\xrightarrow{R3 \leftrightarrow R4} \left[\begin{array}{cccc|c} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Is there a solution?
Yes because our echelon form has no $0\ 0\ 0\ 0 \,|\,$ not zero.

How many? There's a free column, so ∞-ly many.

$$\xrightarrow{-\frac{1}{5}R3 \to R3} \left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[3R3+R2 \to R2]{9R3+R1 \to R1} \left[\begin{array}{cccc|c} \boxed{1} & 4 & 5 & 0 & -7 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-4R2+R1\to R1}\quad \begin{array}{c} \begin{array}{ccccc} x & y & z & w & \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 0 & -3 & 0 & 5 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

RREF! Now we can describe solutions!

$w = 0$ (not free!)

z is free (no pivot in z's column)

$$y + 2z = -3$$
$$y = -2z - 3$$

$$x = 3z + 5$$

$w = 0$
z is free
$y = -2z - 3$
$x = 3z + 5$

Ex) $\begin{cases} x_2 - 3x_3 + x_4 = 1 \\ x_1 + x_2 + x_3 + 4x_4 = 2 \\ x_1 + 2x_2 - 2x_3 + 5x_4 = 1 \end{cases}$

Is this consistent? If so how many sols? Can we describe them?

$$\left[\begin{array}{cccc|c} 0 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & 4 & 2 \\ 1 & 2 & -2 & 5 & 1 \end{array}\right] \xrightarrow[\text{(tech.)}]{\text{RREF}} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 4 & 3 & 0 \\ 0 & \boxed{1} & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

Not consistent!
(no solutions to that system.)

Given any matrix

one sequence of row ops

a different sequence of row ops

row echelon form

might differ

row echelon form

all the way

~~RRE~~ all the way

RREF

RREF

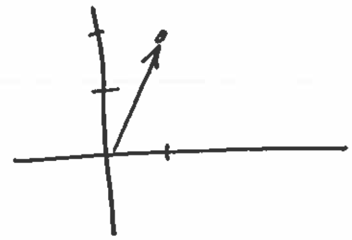
same!

A matrix's RREF is unique.

1.3 Vector Equations

In Linear Algebra, we use <u>column vectors</u>.

<u>Ex</u>: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ corresponds to $\langle 1, 2 \rangle$



Here, $\mathbb{R}^2$ means $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} : \underbrace{a, b \in \mathbb{R}} \right\}$

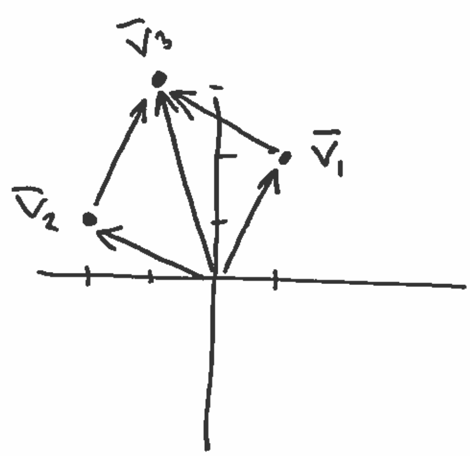
"the set of" — column vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ — such that — a and b are real numbers.

Column vectors can be added/subtracted, scaled

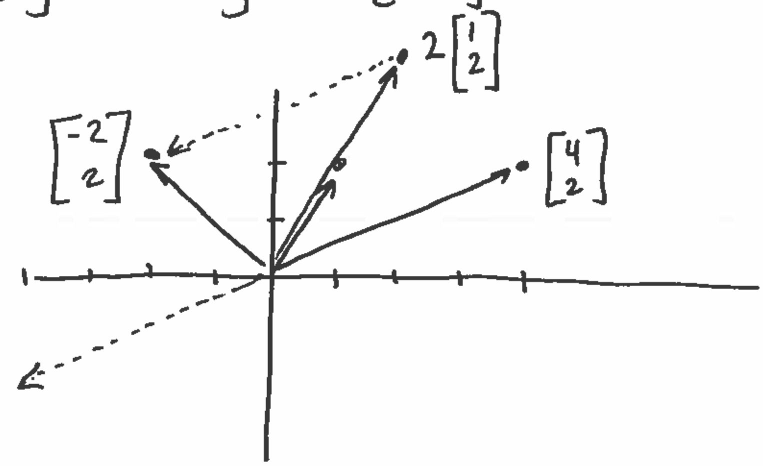
<u>Ex</u> $3 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 8 \end{bmatrix}$

In $\mathbb{R}^2$, pictures...

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$



Ex $2\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$



Facts Whenever $\vec{u}, \vec{v}, \vec{w}$ are in $\mathbb{R}^n$ and c, d are in $\mathbb{R}$, ...

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative prop. of addition)
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associative)
- $\vec{u} + \vec{0} = \vec{u}$, where $\vec{0} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$
- $\vec{u} + (-\vec{u}) = \vec{0}$, where $-\vec{u} = (-1)\cdot\vec{u}$
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (distributive...)
- $(c+d)\vec{u} = c\vec{u} + d\vec{u}$ (distributive...)
- $c(d\vec{u}) = (cd)\cdot\vec{u}$ (associativity of scalar mult.)

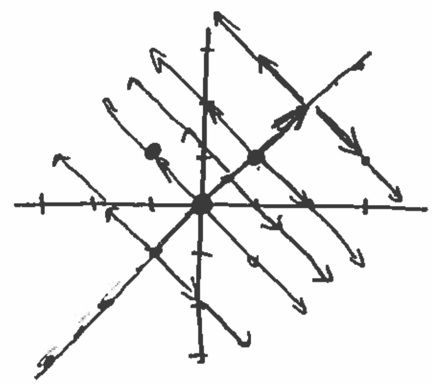
A <u>linear combination</u> of a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$

is something like: $\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p}$

a <u>linear combination</u> of $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$.

We call $c_1, c_2, \dots, c_p$ <u>weights</u>.

Try imagining the set of <u>all</u> linear combinations of $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.



$$0\cdot\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$0\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

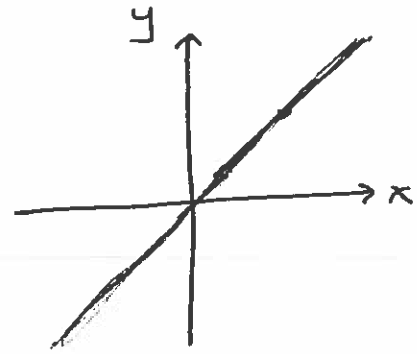
$$c_1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}c_1\\c_1\end{bmatrix}$$

$$2\begin{bmatrix}1\\1\end{bmatrix} + (-1)\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

The collection of linear combos of $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ is $\mathbb{R}^2$.

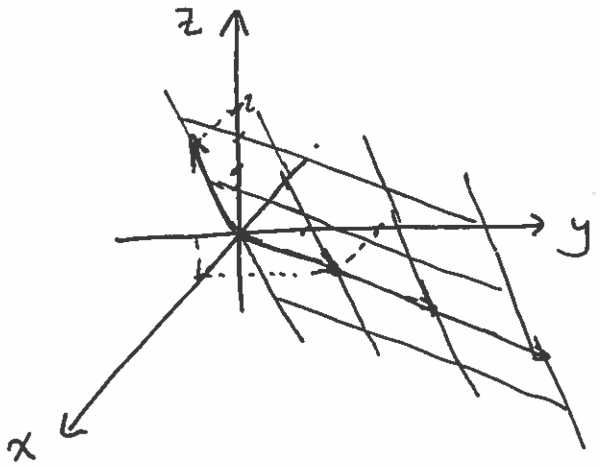
Ex Collection of linear combos of $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$.

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$



Here the collection of linear combos of $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ is a particular line in $\mathbb{R}^2$.

Ex Collection of linear combos of $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$?



Here, you get a certain plane in $\mathbb{R}^3$.

Define <u>the span</u> of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ as the collection of all linear combos of those vectors. Write $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$.

Ex Is $\begin{bmatrix}9\\11\\2\\7\end{bmatrix}$ in the span of $\left\{\begin{bmatrix}1\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\1\\1\end{bmatrix}\right\}$?

(Is $\begin{bmatrix}9\\11\\2\\7\end{bmatrix}$ a linear combo of $\left\{\begin{bmatrix}1\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\1\\1\end{bmatrix}\right\}$?)

Well, write: $c_1\begin{bmatrix}1\\3\\0\\1\end{bmatrix} + c_2\begin{bmatrix}2\\-2\\1\\1\end{bmatrix} = \begin{bmatrix}9\\11\\2\\7\end{bmatrix}$

Does this equation have a solution for c_1, c_2?

$$\begin{cases} c_1 + 2c_2 = 9 \\ 3c_1 - 2c_2 = 11 \\ c_2 = 2 \\ c_1 + c_2 = 7 \end{cases} \Longleftarrow \begin{bmatrix} c_1 + 2c_2 \\ 3c_1 - 2c_2 \\ c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix}9\\11\\2\\7\end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 9 \\ 3 & -2 & 11 \\ 0 & 1 & 2 \\ 1 & 1 & 7 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

there is a solution

$c_1 = 5$
$c_2 = 2$.

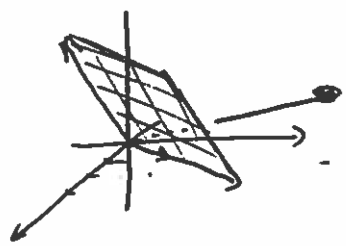
So $\begin{bmatrix}9\\11\\2\\7\end{bmatrix} = 5\begin{bmatrix}1\\3\\0\\1\end{bmatrix} + 2\begin{bmatrix}2\\-2\\1\\1\end{bmatrix}$ and ... the answer is yes!

<u>Ex</u> Is $\begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}$ in the span of $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \right\}$?

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 2 & 0 & 8 \\ 1 & 3 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \longrightarrow \text{no solutions!}$$

So, no, $\begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}$ is <u>not</u> in the span of those two.

Visually



$\begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}$ is not in this plane.

Facts:

- A span of one or more vectors always contains the zero vector.
- A span is always something "flat"
  - a point
  - a line
  - a plane
  - a hyperplane

Ex Beth & Carl live together.

Beth spends 15% of income on food, 20% on housing, 10% on clothing

$$\vec{b} = \begin{bmatrix} .15 \\ .20 \\ .10 \end{bmatrix}$$

Carl spends 20% of his income on food, 25% on housing, 5% on clothing

$$\vec{c} = \begin{bmatrix} .20 \\ .25 \\ .05 \end{bmatrix}$$

Collectively, they spend \$15.5K on food, \$20K on housing, \$7K on clothing every year.

What are their incomes?

$$\underbrace{x}_{\text{Beth's income}} \cdot \begin{bmatrix} .15 \\ .20 \\ .10 \end{bmatrix} + \underbrace{y}_{\text{Carl's income}} \begin{bmatrix} .20 \\ .25 \\ .05 \end{bmatrix} = \begin{bmatrix} 15.5 \\ 20 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} .15 & .20 & 15.5 \\ .20 & .25 & 20 \\ .10 & .05 & 7 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 50 \\ 0 & 1 & 40 \\ 0 & 0 & 0 \end{array}\right]$$

Beth's income is \$50K, Carl's is \$40K.