1.1 Systems of Linear Equations

<u>Ex</u> Solve for x and y

$$\begin{cases} 2x + y = 3 \\ 3x - 2y = 8 \end{cases}$$

<u>Substitution</u>

1st $\Longrightarrow$ $y = 3 - 2x$

sub into second $\Longrightarrow$ $3x - 2(3-2x) = 8$

$$3x - 6 + 4x = 8$$
$$7x - 6 = 8$$
$$7x = 14$$
$$x = 2$$

back-sub: $y = 3 - 2x$

$$y = 3 - 2(2)$$
$$= -1$$

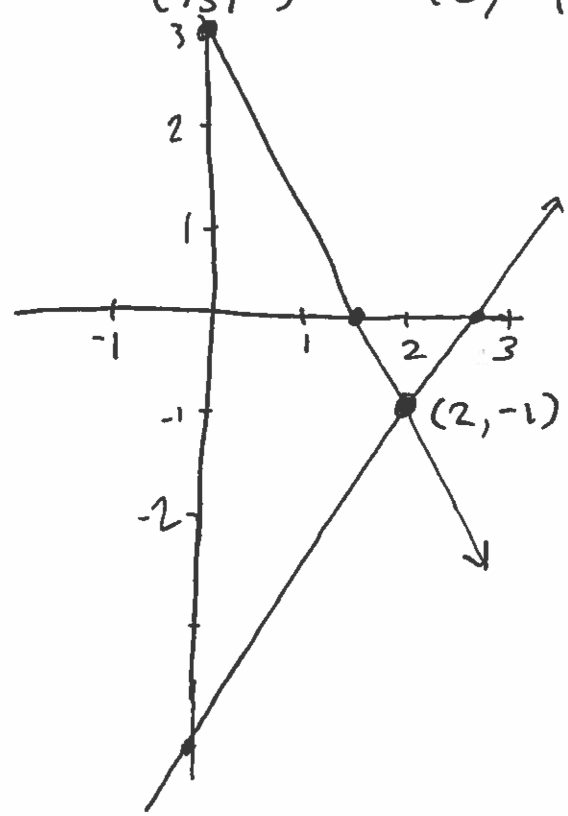
<u>Graphically</u>

$2x + y = 3$

x-intercept: $(\frac{3}{2}, 0)$ y-int: $(0, 3)$

$3x - 2y = 8$

x-int: $(8/3, 0)$ y-int: $(0, -4)$



BUT!!

$$\begin{cases} 2x + 3y - 8z + w = 12 \\ x - y \quad + 2w = 15 \\ x + 3y + z - w = -2 \\ 2x + y + 2z + w = 9 \end{cases}$$

the above methods are impractical!

Back to $\begin{cases} 2x + y = 3 \\ 3x - 2y = 8 \end{cases}$

manipulate equations so these will cancel using addition

3 first eq.

-2 second eq.

$$\begin{cases} 6x + 3y = 9 \\ -6x + 4y = -16 \end{cases} \xrightarrow[\text{together}]{\text{adding sides}} \{ \quad 7y = -7 \Downarrow y = -1$$

2 · first eq

second eq

$$\begin{cases} 4x + 2y = 6 \\ 3x - 2y = 8 \end{cases} \Longrightarrow \{ 7x = 14 \Downarrow x = 2$$

Again: $\begin{cases} 2x + y = 3 \\ 3x - 2y = 8 \end{cases} \xrightarrow[\text{Matrix for this system}]{\text{Augmented}} \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & -2 & 8 \end{array}\right]$

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & -2 & 8 \end{array}\right] \xrightarrow[-2 \cdot R2 \to R2]{3 \cdot R1 \to R1} \left[\begin{array}{cc|c} 6 & 3 & 9 \\ -6 & 4 & -16 \end{array}\right] \xrightarrow{R1 + R2 \to R2}$$

Goal: 0 here

$$\left[\begin{array}{cc|c} 6 & 3 & 9 \\ 0 & 7 & -7 \end{array}\right] \xrightarrow{\frac{1}{7} R2 \to R2} \left[\begin{array}{cc|c} 6 & \boxed{3} & 9 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{-3R2 + R1 \to R1}$$

$\Longrightarrow y = -1$

Next Goal:

$$\left[\begin{array}{cc|c} 6 & \boxed{0} & 12 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{\frac{1}{6}R1 \to R1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

says
$x = 2$
$y = -1$

Ex Solve $\begin{cases} x + 2y + 3z = 7 \\ 2x + y - z = -1 \\ x - 3y = 8 \end{cases}$

"pivot position"

Augmented Matrix:

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 7 \\ 2 & 1 & -1 & -1 \\ 1 & -3 & 0 & 8 \end{array}\right] \xrightarrow[-1R1+R3 \to R3]{-2R1+R2 \to R2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -3 & -7 & -15 \\ 0 & -5 & -3 & 1 \end{array}\right]$$

need to be 0

so that we can cancel middle...

$$\xrightarrow[-3R3 \to R3]{5R2 \to R2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -15 & -35 & -75 \\ 0 & 15 & 9 & -3 \end{array}\right] \xrightarrow{R2+R3 \to R3}$$

... entries without involving fractions

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -15 & -35 & -75 \\ 0 & 0 & -26 & -78 \end{array}\right] \xrightarrow{-\frac{1}{26}R_3 \to R3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -15 & -35 & -75 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \xrightarrow[-3R3+R1 \to R1]{35R3+R2 \to R2}$$

pivot!

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & -15 & 0 & 30 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{-\frac{1}{15}R2 \to R2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{-2R2+R1 \to R1}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \Longrightarrow \begin{cases} x = 2 \\ y = -2 \\ z = 3 \end{cases}$$

pivot positions

Check!

$$x + 2y + 3z \stackrel{?}{=} 7 \checkmark$$
$$2x + y - z \stackrel{?}{=} -1 \checkmark$$
$$x - 3y \stackrel{?}{=} 8 \checkmark$$

$$\begin{array}{rrrr} & -2(1 & 2 & 1) \\ + & 2 & -1 & 1 \\ \hline \end{array} \qquad \begin{array}{rrr} -2 & -4 & -2 \\ +\,2 & -1 & 1 \\ \hline 0 & -5 & -1 \end{array}$$

<u>Ex</u> Solve $\begin{cases} x + 2y + z = 3 \\ 2x - y + z = 4 \\ -4x + 7y - z = 2 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 1 & 4 \\ -4 & 7 & -1 & 2 \end{array}\right] \xrightarrow[4R1+R3 \to R3]{-2R1+R2 \to R2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & -1 & -2 \\ 0 & 15 & 3 & 14 \end{array}\right]$$

Attack!

Attack

$$\xrightarrow{3R2+R3 \to R3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & -1 & -2 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

$0x + 0y + 0z = 8$ has <u>no</u> solution.

We have an <u>inconsistent</u> system of linear equations.

there is <u>no</u> solution.

If your augmented matrix reduces to....

$$\left[\begin{array}{cccc|c} & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \text{not zero} \\ & & \vdots & & \vdots \end{array}\right] \Longrightarrow$$

no solutions to the corresponding system & "inconsistent" is the vocab.

We can have 1 solution, 0 solutions, ∞-ly many solutions...

<u>Ex</u> $\begin{cases} x + y + z = 8 \\ 2x + y = 4 \\ y + 2z = 12 \end{cases}$ Solve for x, y, z.

pivot pos.

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 8 \\ 2 & 1 & 0 & 4 \\ 0 & 1 & 2 & 12 \end{array}\right] \xrightarrow{-2R1+R2\to R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & -1 & -2 & -12 \\ 0 & 1 & 2 & 12 \end{array}\right] \xrightarrow{R2+R3\to R3}$$

attack

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & -1 & -2 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R2\to R2} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 8 \\ 0 & \boxed{1} & 2 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R2+R1\to R1}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & -4 \\ 0 & \boxed{1} & 2 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x - z = -4 \\ y + 2z = 12 \end{cases} \Longrightarrow \begin{cases} x = z - 4 \\ y = -2z + 12 \end{cases}$$

"free column"; z is "free"

$y = -2z + 12$

$x = z - 4$

We have ∞-ly many sols. ~~We say this system is~~

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z-4 \\ -2z+12 \\ z \end{bmatrix}$$

1.2

We've seen reducing an aug. matrix down to:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

you know:

$x = *$

$y = *$

$z = *$

✓

Some matrices are in (row) echelon form.

$$\begin{bmatrix} \boxed{2} & 1 & 3 \\ 0 & \boxed{1} & 8 \end{bmatrix} \qquad \begin{bmatrix} \boxed{8} & 1 \\ 0 & \boxed{3} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 3 & 5 & 9 \\ 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Rows have leading entries (first nonzero entry in that row) where below each such thing is all 0's

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix} \qquad \begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\blacksquare$ nonzero

$*$ anything

If, additionally, leading rowe entries are 1,

and entries above a leading entry are 0,

it's is in reduced (row) echelon form.

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

RREF is powerful.

If this is coming from an augmented matrix, then

$x = 2$
$y = 3$
$z = 4$

Solutions are spelled out

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & \boxed{1} & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is / is not in RREF?

It is!

These leading entries of a row in a matrix that is in RREF are called pivot positions.

Ex

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Can we use row operations to convert to RREF?

- swap rows
- scale a row
- substitution: scale a row & add to a diff. row

$$\xrightarrow{R1 \leftrightarrow R4} \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow[2R1+R3 \to R3]{R1+R2 \to R2} \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \longrightarrow \cdots \longrightarrow$$

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[9R_3+R_1 \to R_1]{3R_3+R_2 \to R_2} \begin{bmatrix} \boxed{1} & 4 & 5 & 0 & -7 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \boxed{1} & 0 & -3 & 0 & 5 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

No ~~path~~ matter what path we took, RREF is unique!

Original:

$$\begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

are this matrix's pivot positions.

<u>rank</u> := # pivot positions.

Here rank = 3.