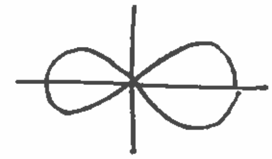
9.3 # 30 Find $\frac{d^2y}{dx^2}$ for $x = \cos(t)$, $y = \sin(2t)$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{\cos(2t)\cdot 2}{-\sin(t)}\right]$$

If you have calculator



$\longrightarrow$ Chain Rule

$$= \frac{\frac{d}{dt}\left[\frac{\cos(2t)\cdot 2}{-\sin(t)}\right]}{\frac{dx}{dt}}$$

$$= \frac{\frac{-\sin(t)\cdot(-\sin(2t)4) - \cos(2t)\cdot 2(-\cos(t))}{\sin^2(t)}}{-\sin(t)}$$

calculator - free

$$= \frac{-4\sin(t)\sin(2t) - 2\cos(t)\cos(2t)}{\sin^3(t)}$$

where is up/down? in $[0, 2\pi)$

$\frac{d^2y}{dx^2}$ changes sign: num $= 0$; den $= 0$ $\Downarrow$ $t = 0, \pi$

$\sin(2t) = 2\sin(t)\cdot\cos(t)$

$\cos(2t) = 1 - 2\sin^2(t)$

numerator $= 0$

$$-4\sin(t)\cdot\sin(2t) - 2\cos(t)\cos(2t) = 0$$

$$-4\sin(t)\cdot 2\sin(t)\cos(t) - 2\cos(t)\cdot\cos(2t) = 0$$

$$-2\cos(t)\left[4\sin^2(t) + \cos(2t)\right] = 0$$

$$-2\cos(t)\Big[\underbrace{1 + 2\sin^2(t)}_{\text{can't be } 0}\Big] = 0$$

$\hookrightarrow t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Potentially change concavity at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

$[0, \frac{\pi}{2}], [\frac{\pi}{2}, \pi]$ ....

I'd let you use calc to determine up/down

Ex from last class

$x = \sqrt{t}$
$y = \sin(t)$

Find where $\frac{d^2y}{dx^2} = 0$ or undefined

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = 0$$

$$\frac{d}{dx}\left[\frac{\cos(t)}{\frac{1}{2\sqrt{t}}}\right] = 0$$

$$\frac{d}{dx}\left[2\sqrt{t}\cos(t)\right] = 0$$

$$\frac{\frac{d}{dt}\left[2\sqrt{t}\cos(t)\right]}{\frac{dx}{dt}} = 0$$

$$\frac{\frac{1}{\sqrt{t}}\cdot\cos(t) - 2\sqrt{t}\sin(t)}{\frac{1}{2\sqrt{t}}} = 0 \quad \text{or undef}$$

$\downarrow$

$t = 0$

$\Longrightarrow$ Graphical solving ...

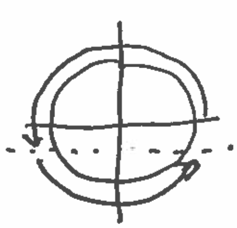
$t = 0.65\ldots,\ 3.29\ldots,\ 6.36\ldots$

challenging to look for self crossings elsewhere...

Ex $r = 1 + 2\underbrace{\sin\left(\frac{\theta}{2}\right)}$ Where does it self-cross?

period is $\frac{2\pi}{1/2} = 4\pi$.

easy if you look for crossings at pole.



$$0 = 1 + 2\sin\left(\tfrac{\theta}{2}\right)$$

$$-\frac{1}{2} = \sin\left(\tfrac{\theta}{2}\right)$$

$$\frac{\theta}{2} = \ldots, \frac{-\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \ldots$$

$\theta = \ldots, \frac{-\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{19\pi}{3}, \frac{23\pi}{3}, \ldots$

$\theta = \frac{7\pi}{3}$ or $\frac{11\pi}{3}$ in $[0, 4\pi)$

So tangent line: $y = \frac{\sqrt{2}}{-4-\sqrt{2}}\left(x - \left(\frac{\sqrt{2}}{2}+1\right)\right) + \frac{\sqrt{2}}{2} + 1$

b) Find places with horizontal tangents.

$$\frac{dy}{dx} = 0$$

$$\frac{-2\sin^2\theta + (1+2\cos\theta)\cos\theta}{-2\sin\theta\cos\theta - (1+2\cos\theta)\sin\theta} = 0$$

$$\Longrightarrow \quad -2\sin^2\theta + (1+2\cos\theta)\cos\theta = 0$$

$$-2\sin^2\theta + \cos\theta + 2\cos^2\theta = 0$$

<strategy: make it all about either $\cos\theta$ or $\sin\theta$>

$$-2(1-\cos^2\theta) + \cos\theta + 2\cos^2\theta = 0$$

$$-2 + \cos\theta + 4\cos^2\theta = 0$$

$$4x^2 + x - 2 = 0$$

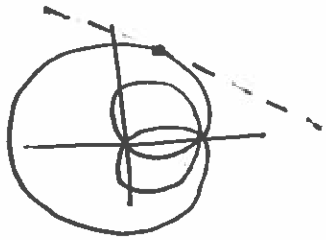
$$x = \frac{-1 \pm \sqrt{1^2 - 4(4)(-2)}}{2\cdot 4} = \frac{-1 \pm \sqrt{33}}{8}$$

$$\cos\theta = \frac{-1+\sqrt{33}}{8} \quad \text{or} \quad \cos\theta = \frac{-1-\sqrt{33}}{8}$$

$$\cos\theta \approx .5931\ldots \qquad \cos\theta \approx -.8431$$

$\theta = \arccos\left(\frac{-1+\sqrt{33}}{8}\right)$, $\theta = 2\pi - \arccos\left(\frac{-1+\sqrt{33}}{8}\right)$, $\theta = \arccos\left(\frac{-1-\sqrt{33}}{8}\right)$, $\theta = 2\pi - \arccos\left(\frac{-1-\sqrt{33}}{8}\right)$

9.5 Calculus with Polar Curves



$\frac{dy}{dx}$ when curve is $r = f(\theta)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$y = r \cdot \sin\theta$

$y = f(\theta) \cdot \sin\theta$

$\frac{dy}{d\theta} = f'(\theta) \cdot \sin\theta + f(\theta) \cdot \cos\theta$

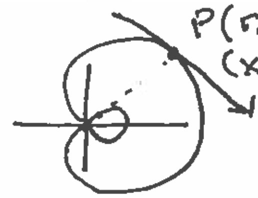
$$\longrightarrow \frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$x = r \cdot \cos\theta$

$x = f(\theta) \cos\theta$

$\frac{dx}{d\theta} = f'(\theta) \cdot \cos\theta - f(\theta)\sin\theta$

Ex $r = 1 + 2\cos\theta$



$\underbrace{\text{Go to } \theta = \frac{\pi}{4},}$ tangent line equation...

$(y = m(x - x_0) + y_0)$

$r = 1 + 2\cos(\frac{\pi}{4})$

$r = 1 + \sqrt{2}$

$P(1+\sqrt{2}, \frac{\pi}{4})$

$x_0 = (1+\sqrt{2}) \cdot \cos(\frac{\pi}{4})$

$= \frac{\sqrt{2}}{2} + 1$

$y_0 = (1+\sqrt{2}) \cdot \sin(\frac{\pi}{4})$

$= \frac{\sqrt{2}}{2} + 1$

$(\frac{\sqrt{2}}{2} + 1, \frac{\sqrt{2}}{2} + 1)$

Find slope:

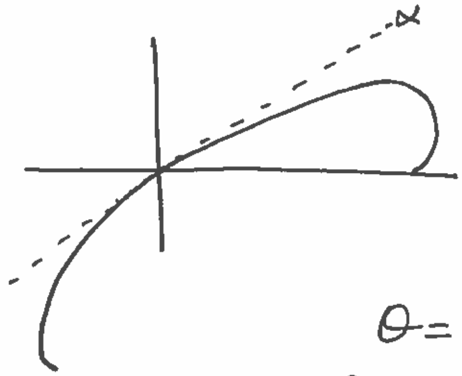
$$\frac{dy}{dx} = \frac{-2\sin\theta \cdot \sin\theta + (1+2\cos\theta)\cos\theta}{-2\sin\theta \cdot \cos\theta - (1+2\cos\theta)\sin\theta}$$

Use $\theta = \frac{\pi}{4}$

$$= \frac{-2(\frac{1}{2}) + (1+\sqrt{2}) \cdot \frac{\sqrt{2}}{2}}{-2(\frac{1}{2}) - (1+\sqrt{2})\frac{\sqrt{2}}{2}}$$

$$= \frac{\sqrt{2}/2}{-2 - \sqrt{2}/2} = \frac{\sqrt{2}}{-4-\sqrt{2}}$$

Fact Imagine angle α causes you to pass through pole:



$$\frac{dy}{dx} = \frac{f'(\theta)\cdot\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cdot\cos(\theta) - f(\theta)\sin\theta}$$

when $\theta = \alpha$

$\theta = \alpha$

$\Longrightarrow r = 0$

$\Longrightarrow f(\alpha) = 0$

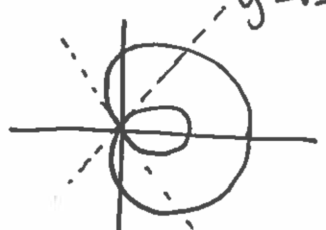
$\frac{dy}{dx}$ simplifies to $\tan(\alpha)$.

When you are at an angle passing through a pole, the tangent of α is that slope!

$\Longrightarrow$ Tangent line has equation $y = \tan(\alpha)\cdot x$

Ex $r = 1 + 2\cos\theta$

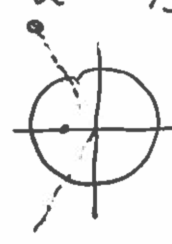
Find equations for tangent lines at the pole.



Need to solve $r = 0$

$1 + 2\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$



$$\tan\left(\frac{2\pi}{3}\right) = \frac{\sin(2\pi/3)}{\cos(2\pi/3)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$$

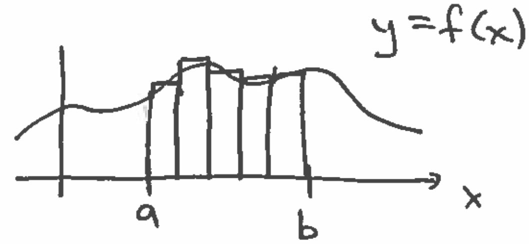
$$\tan(4\pi/3) = \cdots = \sqrt{3}$$

$y = -\sqrt{3}\,x$

$y = \sqrt{3}\,x$

Area with Polar Curves

In Calc II

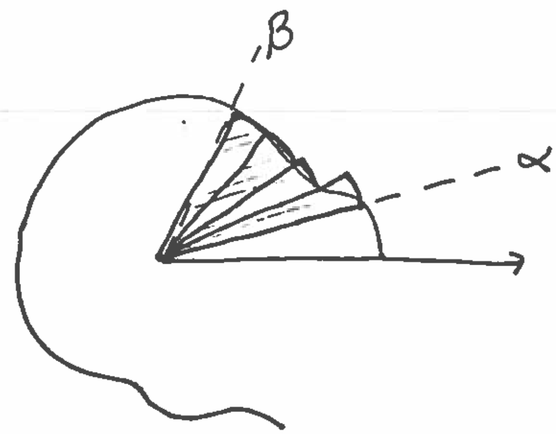


approximate a little piece of area with a rectangle..

$$\sum \underbrace{f(x_i^*)\cdot\Delta x}_{\text{one rectangle area}}$$

$$\to \int_a^b f(x)\cdot dx$$

With Polar



Approximate shaded area : $\sum \overbrace{\text{pie slice areas}}^{\text{sector}}$

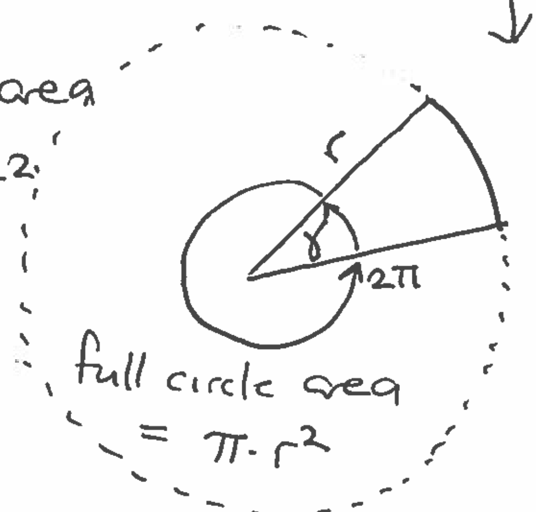
$$\sum \frac{1}{2}\left(f(\theta_i^*)\right)^2 \Delta\theta$$

$$A = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 \cdot d\theta$$

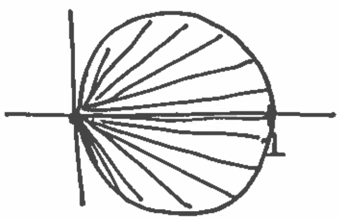
Sector area

$$= \frac{\gamma}{2\pi}\cdot \pi r^2$$

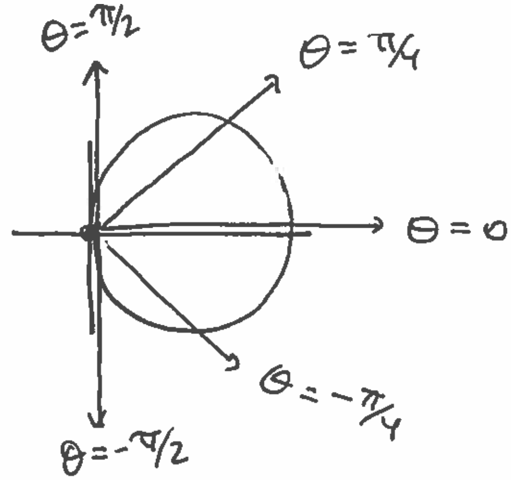
$$= \frac{1}{2}\gamma r^2$$



__Ex__ $r = \cos\theta$



(Area from grade school
$A = \pi\left(\frac{1}{2}\right)^2$
$A = \pi/4$)



drawing this...
can start at $-\pi/2$,
end at $\pi/2$.

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos^2\theta \cdot d\theta$$

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(\frac{1+\cos(2\theta)}{2}\right) d\theta$$

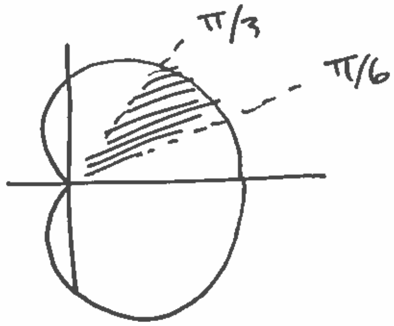
$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{4} + \frac{\cos(2\theta)}{4}\right) d\theta$$

$$= \left[\frac{1}{4}\theta + \frac{\sin(2\theta)}{8}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right) + \cancel{\frac{\sin\left(2\frac{\pi}{2}\right)}{8}} - \left[\frac{1}{4}\left(-\frac{\pi}{2}\right) + \cancel{\frac{\sin\left(2\frac{-\pi}{2}\right)}{8}}\right]$$

$$= \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4}$$

Ex $r = 1 + \cos\theta$



$$A = \int_{\pi/6}^{\pi/3} \frac{1}{2}\left(1 + \cos\theta\right)^2 \, d\theta$$

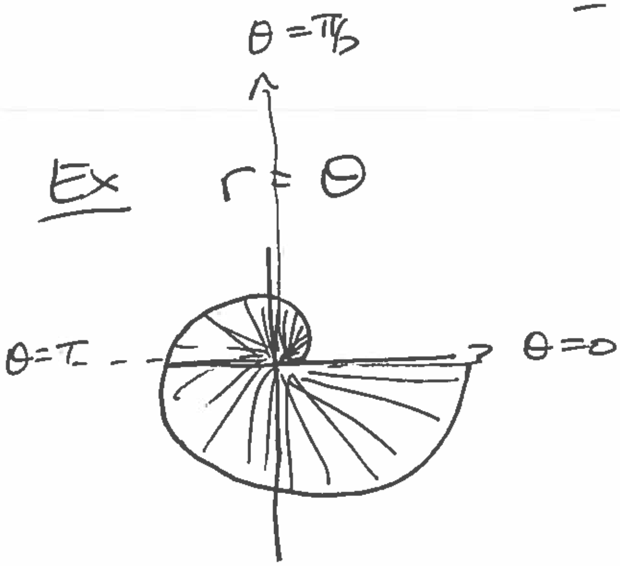
$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$\frac{1 + \cos(2\theta)}{2}$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_{\pi/6}^{\pi/3}$$

Ex $r = \theta$



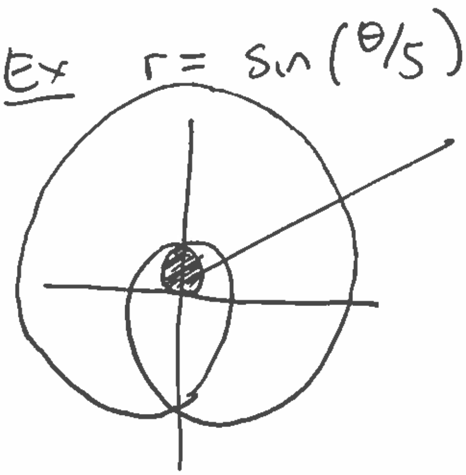
Area = ?

$$\int_0^{2\pi} \frac{1}{2}\theta^2 \, d\theta = \left[\frac{1}{6}\theta^3\right]_0^{2\pi}$$

$$= \frac{1}{6}(2\pi)^3$$

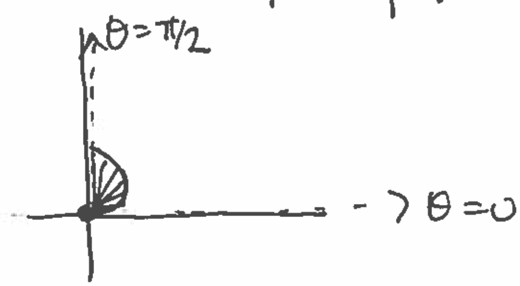
$$= \frac{4}{3}\pi^3$$

Ex $r = \sin(\theta/5)$



Area = ?

Where to start, stop?



$$A = 2\int_0^{\pi/2} \frac{1}{2}\sin^2(\theta/5) \, d\theta$$

$$A = \int_0^{\pi/2} \left(\frac{1 - \cos(2\theta/5)}{2} \right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos\left(\tfrac{2}{5}\theta\right) \right) d\theta$$

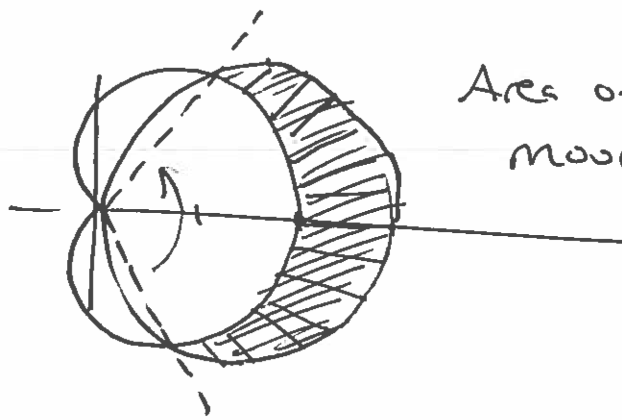
$$= \left[\frac{1}{2}\theta - \frac{5}{4}\sin\left(\tfrac{2}{5}\theta\right) \right]_0^{\pi/2}$$

$$= \frac{1}{2}(\pi/2) - \frac{5}{4}\sin\left(\frac{2}{5}\cdot\frac{\pi}{2}\right)$$

$$= \pi/4 - \frac{5}{4}\sin(\pi/5)$$

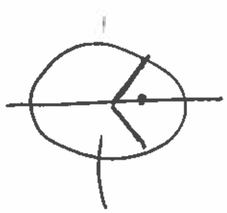
__Ex__ $r = 1 + \cos\theta$

$r = 3\cos\theta$

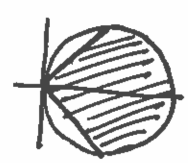
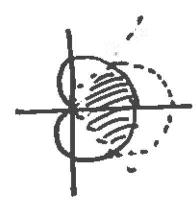


Area of moon?

__Will__ need to know where crossings happen.



$$1 + \cos\theta = 3\cos\theta$$
$$1 = 2\cos\theta$$
$$\tfrac{1}{2} = \cos\theta$$
$$\theta = \frac{-\pi}{3} \text{ or } \frac{\pi}{3}$$

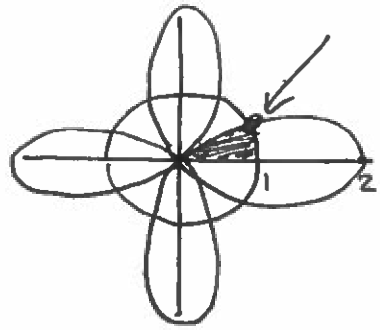
$A =$  $-$ 

$$\int_{-\pi/3}^{\pi/3} \frac{1}{2}(3\cos\theta)^2 d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2}(1+\cos\theta)^2 d\theta$$

$= \ldots$ compute!

Ex $r = 1$

$r = 2\cdot\cos 2\theta$

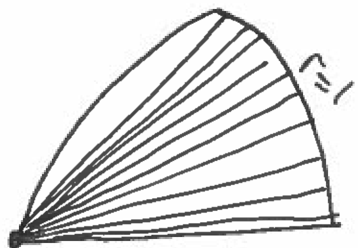


$1 = 2\cos 2\theta$

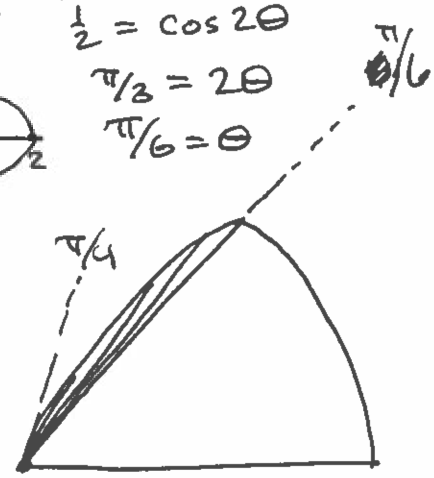
$\frac{1}{2} = \cos 2\theta$

$\pi/3 = 2\theta$

$\pi/6 = \theta$



pole



pole

$$\int_0^{\pi/6} \frac{1}{2}(1)^2\, d\theta \quad + \quad \int_{\pi/6}^{\pi/4} \frac{1}{2}(2\cos 2\theta)^2\, d\theta$$

$= \text{---}$ compute!

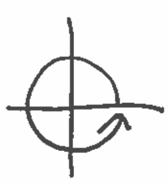
Arc Length with Polar

$r = f(\theta)$

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\; d\theta$$

Ex circle of radius 1 centered at origin : $r = 1$

what is its circumference?



$f(\theta) = 1$

$f'(\theta) = 0$

$$L = \int_0^{2\pi} \sqrt{0^2 + 1^2}\; d\theta$$

$$= \int_0^{2\pi} 1 \cdot d\theta$$

$$= [\theta]_0^{2\pi} = 2\pi$$

	Parametric	Polar
Self-crossings	Solve $\begin{cases} x(s) = x(t) \\ y(s) = y(t) \end{cases}$ for s,t.	crossing at pole, $r=0$ solve for θ. ~~#~~ rather difficult to find other self crossings...

Converting

rectangular → param.

Ex $y = x^2$

$t = \frac{dy}{dx}$

$t = 2x$

$x = t/2$

$y = (t/2)^2$

$x = r\cos\theta$

$y = r\sin\theta$

param. → rectangular

$x = \frac{1}{t^2+1}$

$y = \frac{t^2}{t^2+1}$

$x + y = 1$

Eliminate t

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

Slope

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{r'(\theta)\cdot\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

Chain Rule

$\square = t$

$$\frac{dy}{dx} = \frac{dy}{\square}\cdot\frac{d\square}{dx}$$

$\square = \theta$

Tangent & Normal

$$y = m(x - x_0) + y_0$$

m: $\frac{dy}{dx}$ for tangent or neg reciprocal for normal

x_0, y_0: point of tangency

Parametric | Polar

~~Curvature~~
Concavity

$$\frac{d^2y}{dx^2} = \cancel{\frac{d}{dx}\left[\frac{dy}{dx}\right]}$$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{\frac{d}{dt}\left[\text{known with t's}\right]}{\frac{dx}{dt}}$$

Arc Length

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$L = \int_\alpha^\beta \sqrt{r'(\theta)^2 + r(\theta)^2}\, d\theta$$

Area

$$A = \int_a^b \underbrace{x(t)}\cdot \underbrace{\frac{dy}{dt}(t)}\, dt$$

(Not tested on this)

$$A = \int_\alpha^\beta \frac{1}{2}\left(r(\theta)\right)^2 d\theta$$