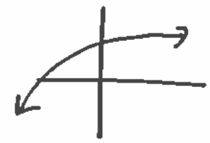
9.3 continued

<u>Concavity</u> $\frac{d^2y}{dx^2}$ indicates

pos, concave up

neg, concave down



How to tell with param. curves: $x = f(t)$
$y = g(t)$

<u>Ex</u> $x = 2t^3 - 3t^2 \longrightarrow \frac{dx}{dt} = 6t^2 - 6t$

$y = t^2 - 2t \longrightarrow \frac{dy}{dt} = 2t - 2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-2}{6t^2-6t} = \frac{2(t-1)}{6t(t-1)}$$

$$= \frac{1}{3t}$$

But how to ged $\frac{d^2y}{dx^2}$?

Chain Rule
$\frac{d}{dx}[h] = \frac{d}{dt}[h] \cdot \frac{dt}{dx}$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{1}{3t}\right]$$

$$= \frac{d}{dt}\left[\frac{1}{3t}\right] \cdot \frac{dt}{dx}$$

$$= \frac{d}{dt}\left[\frac{1}{3t}\right] \Big/ \frac{dx}{dt} = \frac{-\frac{1}{3t^2}}{6t^2-6t}$$

So $\dfrac{d^2y}{dx^2} = \dfrac{-1}{18t^3(t-1)}$

So where is this curve concave up or down?

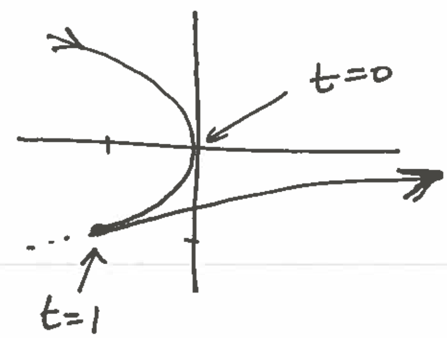
Rational Function

- - - DNE + + + DNE - - -

t: 0, 1

concave up: $[0,1]$

down: $(-\infty, 0]$ and $[1, \infty)$



Ex $x = \sqrt{t} = t^{1/2}$ for t in $[0,\infty)$

$y = \sin(t)$

Where is this concave up & down?

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[2\sqrt{t}\cos(t)\right]}{\frac{1}{2\sqrt{t}}}$$

$\dfrac{dy}{dt} = \cos(t)$

$\dfrac{dx}{dt} = \dfrac{1}{2t^{1/2}} = \dfrac{1}{2\sqrt{t}}$

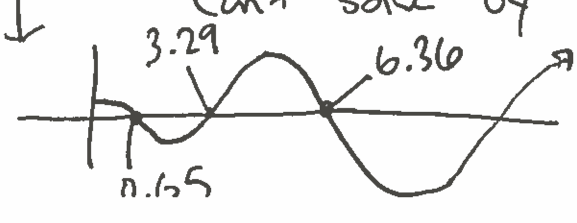
~~To be fixed~~

$$= \frac{\frac{1}{\sqrt{t}}\cdot\cos(t) - 2\sqrt{t}\sin(t)}{\frac{1}{2\sqrt{t}}}$$

In class, I felt there was a mistake here. But no mistake. These values: 0.65, 3.29, ... are t-values not x-values

Need: $\frac{1}{\sqrt{t}}\cos(t) - 2\sqrt{t}\sin(t) = 0$

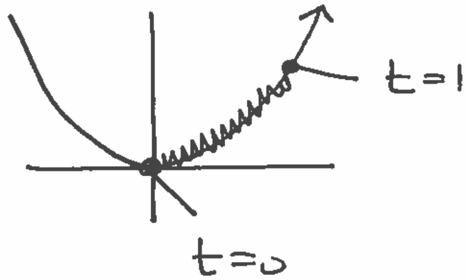
Can't solve by hand...



So for example, concave up on $[0, 0.65...]$

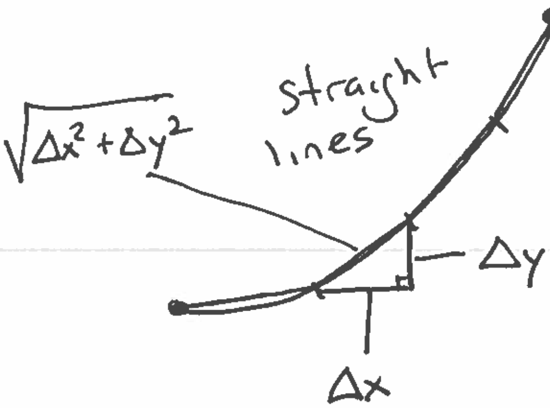
Arc Length

$x = t$
$y = t^2$



How long is this arc?

Zoom in



$$\text{Area} \approx \sum \sqrt{\Delta x^2 + \Delta y^2}$$

Make $\Delta x \to 0$

$$\boxed{\text{Area} = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt}$$

So... $\text{Area} = \int_0^1 \sqrt{1^2 + (2t)^2}\, dt = \int_0^1 \sqrt{1 + 4t^2}\, dt$

trig substitution...

$\approx 1.4789....$

$\underline{\text{Ex}}$ Length of one loop of

$x = \cos(t)\sin(4t)$

$y = \sin(t)\sin(4t)$

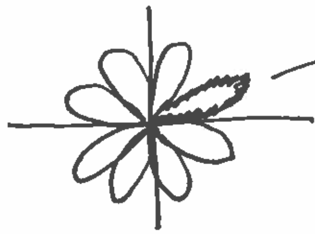
$\pi/2$ $\quad$ $\frac{\pi}{4}$ $\underline{\underline{\text{earliest}}}$

$\cos(t)\cdot\sin(4t) = 0$

&

$\sin(t)\cdot\sin(4t) = 0$

$\cancel{\pi}$



$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

?

start at 0

$$\int_0^{\pi/4} \sqrt{\left(-\sin(t)\sin(4t) + \cos(t)\cdot 4\cos(4t)\right)^2 + \cdots \left(\cos(t)\cdot\sin(4t) + \sin(t)\cdot 4\cos(4t)\right)^2}\,dt$$

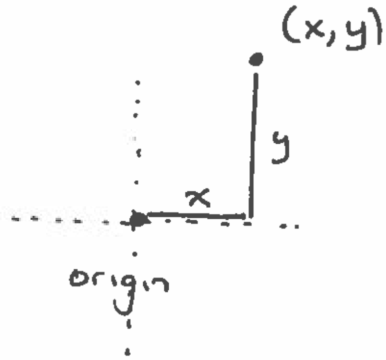
$=$ simplify ...

$$= \int_0^{\pi/4} \sqrt{16\cos^2(4t) + \sin^2(4t)}\,dt \approx 8.578\ldots$$

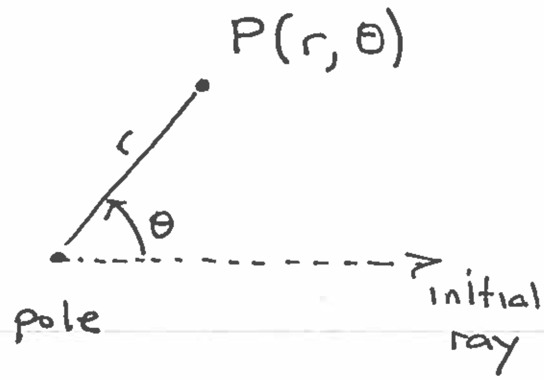
using calculator

9.4 Introduction to Polar Coordinates

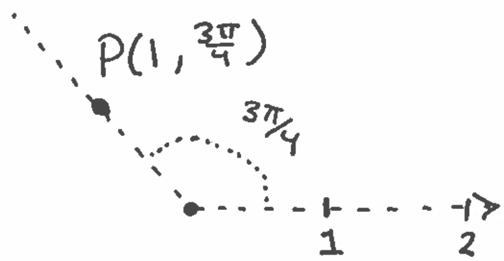
Cartesian:
(rectangular)



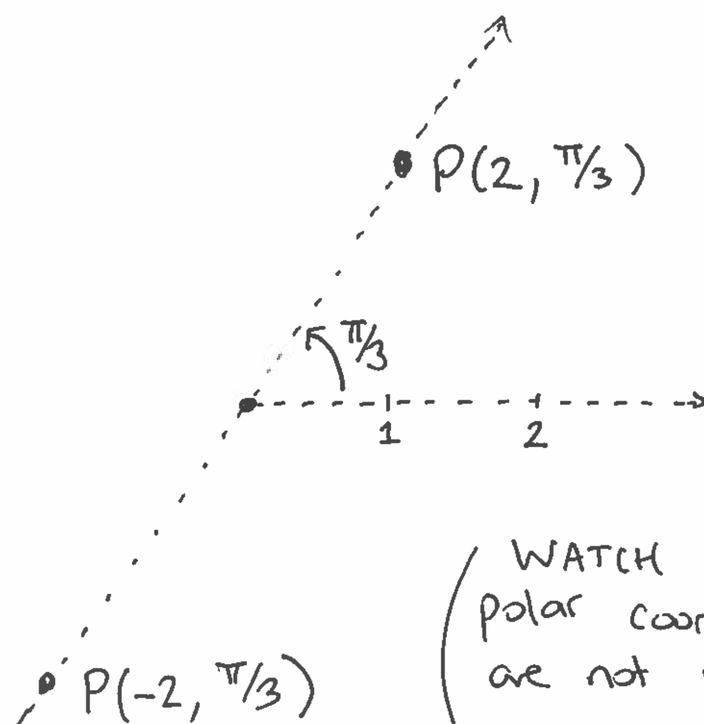
Polar:



Ex Plot $P(1, \frac{3\pi}{4})$.



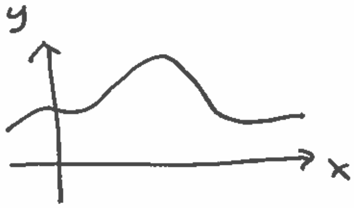
Ex Plot $P(-2, \frac{\pi}{3})$.



(WATCH OUT: polar coordinates are not unique!)

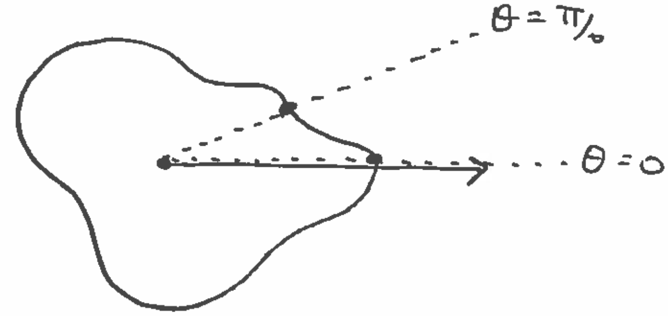
Rectangular functions

$y = f(x)$



Polar functions

$r = f(\theta)$

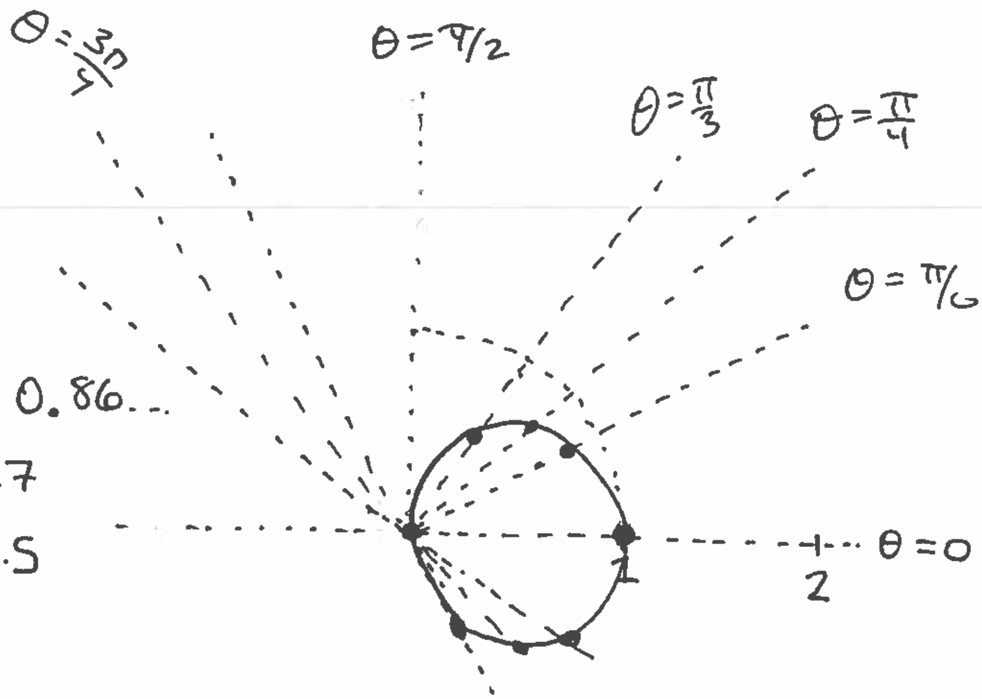


<u>Ex</u> $r = \cos\theta$

Plot this.

By hand:

θ	r
0	1
$\pi/6$	$\sqrt{3}/2 \approx 0.86\ldots$
$\pi/4$	$\sqrt{2}/2 \approx 0.7$
$\pi/3$	$1/2 = 0.5$
$\pi/2$	0
$2\pi/3$	$-1/2 = -0.5$
$3\pi/4$	$-\sqrt{2}/2 = -0.7\ldots$
$5\pi/6$	$-\sqrt{3}/2 \approx -0.86\ldots$
π	-1

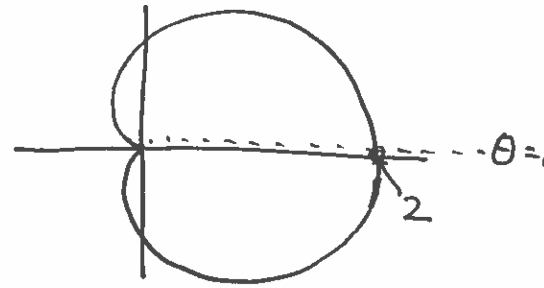


<u>Looks</u> like a circle

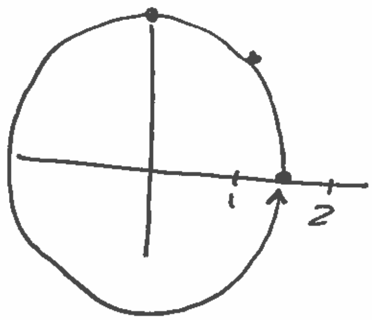
It <u>is</u> a perfect circle

Ex $r = 1 + \cos\theta$

Use Wolfram Alpha:



Ex $r = 1.5$



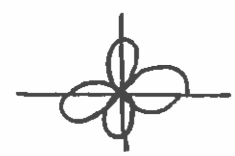
$$x = \overbrace{\cos(2\theta)}^{r} \cdot \cos(\theta)$$
$$y = \cos(2\theta) \cdot \sin(\theta)$$

Ex $r = \cos(2\theta)$

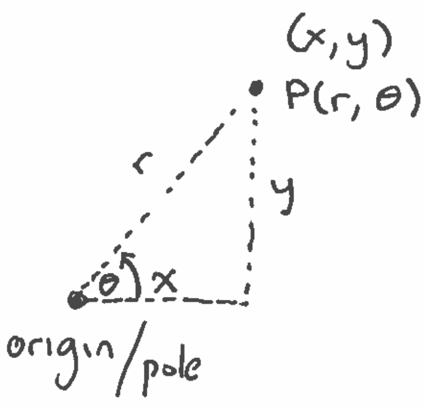
Use GeoGebra

(doesn't plot polar directly....

so change to parametric



Translate Cartesian ⟷ polar



Memorize!

$\tan\theta = \dfrac{y}{x}$	$x = r \cdot \cos\theta$
$r^2 = x^2 + y^2$	$y = r \cdot \sin\theta$
↓ Cartesian → polar	↓ polar → Cartesian

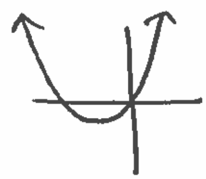
Gallery of Polar Curves on page 533.

$\underline{Ex}$ $r(\theta) = 1 + \underbrace{\tanh}_{\text{hyperbolic tangent.}}\left(100(0.9 - \cos\theta)\right)$

Pac-Man

Convert Rectangular Equations $\longleftrightarrow$ Polar Equations

$\underline{Ex}$ $y = x^2 + x$ $\qquad$ $y = x(x+1)$



$\downarrow$

$r \cdot \sin\theta = r^2\cos^2\theta + r\cos\theta$

isolate r

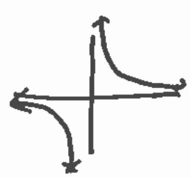
$\sin\theta = r \cdot \cos^2\theta + \cos\theta$

$\sin\theta - \cos\theta = r\cos^2\theta$

$r = \dfrac{\sin\theta - \cos\theta}{\cos^2\theta} = \dfrac{\tan\theta - 1}{\cos\theta}$

$r = \dfrac{\tan\theta - 1}{\cos\theta}$

$\underline{Ex}$ $y = \dfrac{1}{x}$



Same conversion

$r \cdot \sin\theta = \dfrac{1}{r \cdot \cos\theta}$

$r^2 = \dfrac{1}{\sin\theta\cos\theta}$

$\sqrt{r^2} = \underbrace{\sqrt{\dfrac{1}{\sin\theta \cdot \cos\theta}}}_{\text{only makes sense if } \theta \text{ is in quadrant I or quadrant III}}$

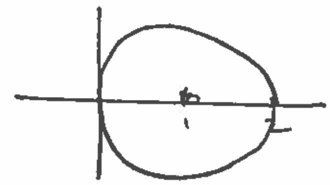
$|r|$ "

$r = \pm\sqrt{\dfrac{1}{\sin\theta\cos\theta}}$

Ex Convert Polar Eq $\longrightarrow$ Rectangular.

$r = 2 \cdot \cos\theta$

could memorized



$(x-1)^2 + y^2 = 1$

One good method, try to make $r \cdot \cos\theta$, $r \cdot \sin\theta$ appear....

Same if you play around ...

Here, mult by r: $r^2 = 2 \cdot r\cos\theta$

$x^2 + y^2 = 2 \cdot x \longrightarrow$ Done!

Ex $r = \dfrac{3}{\cos\theta + 2\sin\theta}$ Find rectangular version...

Divide by r on both sides

OR

Multiply by $\frac{r}{r}$ on right

$r = \dfrac{3r}{r \cdot \cos\theta + 2r\sin\theta}$

$r = \dfrac{3\cancel{r}}{x + 2y}$

$x + 2y = 3$

(Same as $y = \frac{-1}{2}x + \frac{3}{2}$)

Where do polar plots cross themselves?

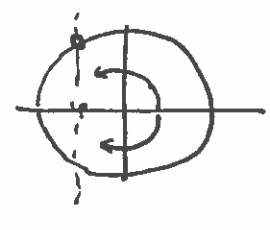
$$r = 1 + 2 \cdot \cos\theta$$

First consider crossing at pole.

$1 + 2\cos\theta = 0$

$2\cos\theta = -1$

$\cos\theta = -1/2$

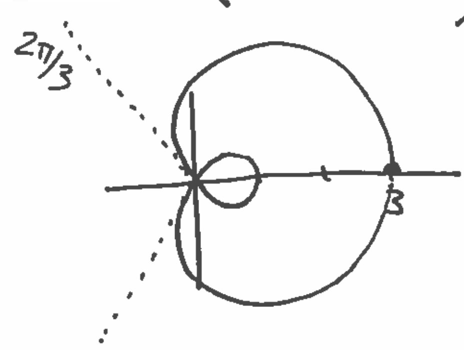


$\theta = 2\pi/3$

or

$-2\pi/3$ (or $4\pi/3$)

In fact:



$-2\pi/3 = 4\pi/3$

Self-crosses at pole

$\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$

has period 4π.

<u>Ex</u> $r = 1 + 2\sin\left(\frac{\theta}{2}\right)$ Where does it self-cross?

investigate pole crossings

third senario... See next page.

imagine $\theta_2 > \theta_1$ but $\theta_2 - \theta_1$ is an even multple of 2π. where r_2 and r_1 are equal.

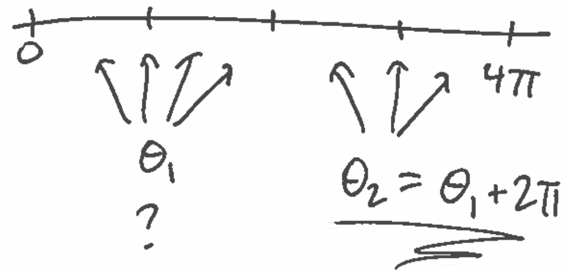
Two angles in $[0, 4\pi)$ do all this

$$r_2 = r_1$$

$$1 + 2\sin\left(\frac{\theta_2}{2}\right) = 1 + 2\sin\left(\frac{\theta_1}{2}\right)$$

$$\sin\left(\frac{\theta_2}{2}\right) = \sin\left(\frac{\theta_1}{2}\right)$$

Angles in $[0, 4\pi)$



$$\sin\left(\frac{\theta_1 + 2\pi}{2}\right) = \sin\left(\frac{\theta_1}{2}\right)$$

$$\sin\left(\frac{\theta_1}{2} + \pi\right) = \sin\left(\frac{\theta_1}{2}\right)$$

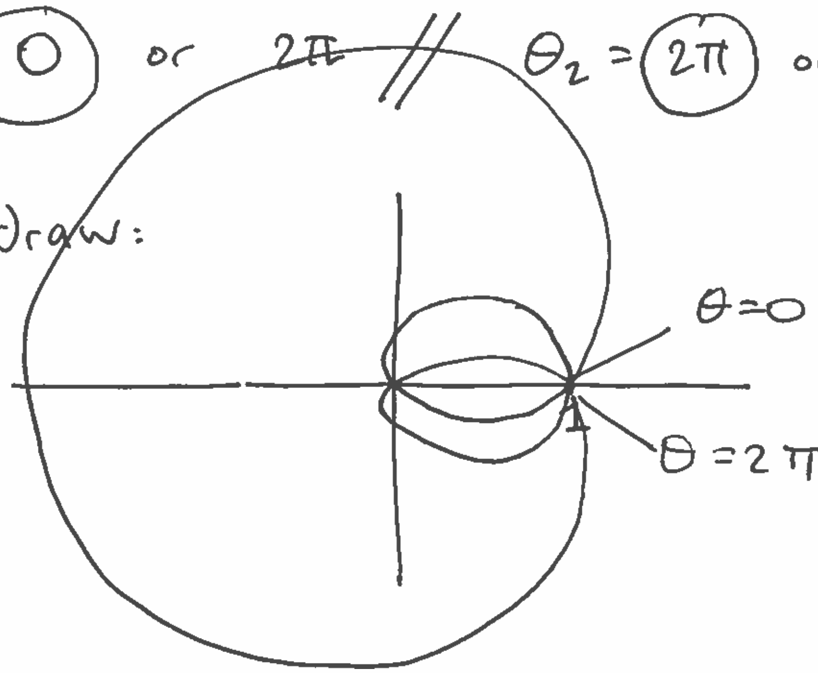
$$-\sin\left(\frac{\theta_1}{2}\right) = \sin\left(\frac{\theta_1}{2}\right)$$

$$\Rightarrow \quad \sin\left(\frac{\theta_1}{2}\right) = 0$$

$$\rightarrow \quad \frac{\theta_1}{2} = 0 \text{ or } \pi$$

$$\theta_1 = 0 \text{ or } 2\pi \quad // \quad \theta_2 = 2\pi \text{ or } 4\pi.$$

Now, draw:



So θ_1 is either π or 2π

And $\theta_2 = \theta_1 + \pi$.

θ_2 could be $2\pi + \pi$

$= 3\pi$.