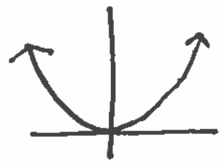
9.2 Parametric Equations

<u>Not</u> a parametric equation: $y = x^2$



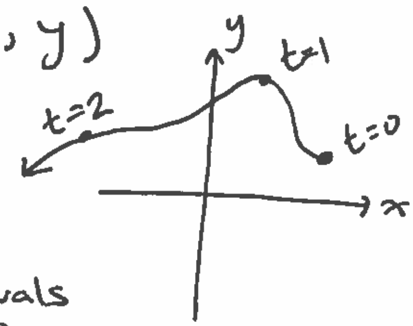
x-value
↓
$y = x^2$
↓
y-value

x	→	y
1	→	1
2	→	4
3	→	9

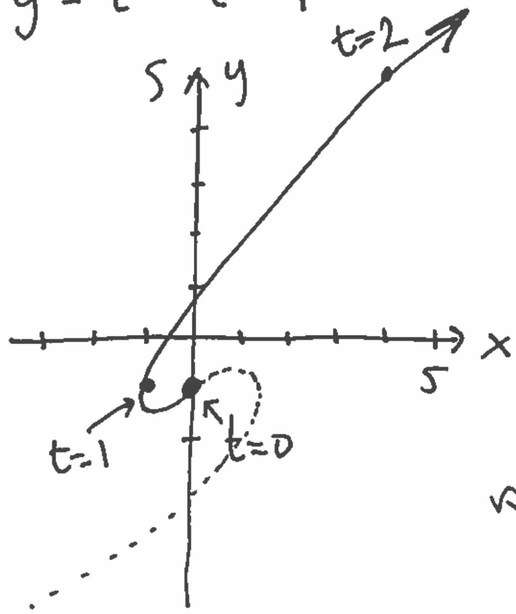
This is a rectangular equation.

Another way:

t → calculate x-value → (x, y)
t → calculate y-value → (x, y)



Ex $x = t^3 - 2t$
$y = t^3 - t - 1$



t-values	x-vals	y-vals	
0	0	−1	(0, −1)
1	−1	−1	(−1, −1)
2	4	5	(4, 5)

Completed using TI-89, GeoGebra

Ex $x = \sin^3(t) - 2\sin(t)$

$y = \sin^3(t) - \sin(t) - 1$

Familiar?

$x = \square^3 - 2\square$

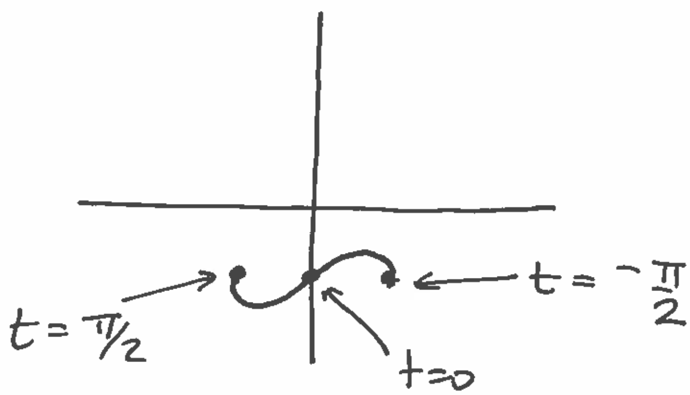
$y = \square^3 - \square - 1$

Just like previous... but not...

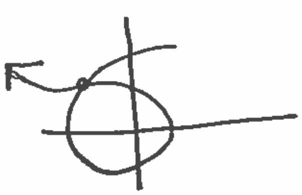
$\square$ could run $-\infty$ to ∞

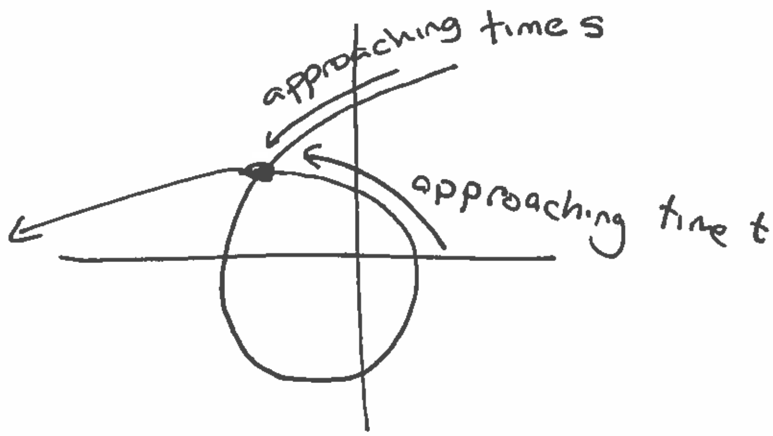
this time, given $\sin(t)$, $\square$ only will run -1 to 1.

Graph with GGB.

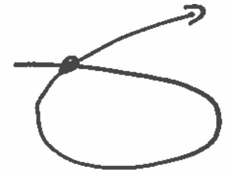


Parametric Curves can cross themselves





Ex $x = t^3 - 4t$
$y = t^2 + t$



First time, s, landed at crossing

$(s^3 - 4s,\ s^2 + s)$

Second Time, t

$(t^3 - 4t,\ t^2 + t)$

same location!

So $s^3 - 4s = t^3 - 4t$ & $s^2 + s = t^2 + t$

$\Rightarrow (-t-1)^3 - 4(-t-1) = t^3 - 4t$

$\Rightarrow -t^3 - 3t^2 - 3t - 1 + 4t + 4 = t^3 - 4t$

$\Rightarrow 0 = 2t^3 + 3t^2 - 5t - 3$

try to either solve for one variable in terms of other... or:

$s^2 - t^2 = t - s$

$\Rightarrow (s-t)(s+t) = -(s-t)$

$\Rightarrow \dfrac{(s-t)(s+t)}{s-t} = \dfrac{-(s-t)}{s-t}$

$\Rightarrow s + t = -1$

$s = -t - 1$

$s^3 - t^3 = 4s - 4t$

$(s-t)(s^2 + st + t^2) = 4(s-t)$

$s^2 + st + t^2 = 4$

$(-t-1)^2 + (-t-1)t + t^2 = 4$

$\cancel{t^2} + 2t + 1 - \cancel{t^2} - t + t^2 = 4$

$t^2 + t - 3 = 0$

$t = \dfrac{-1 \pm \sqrt{1 - 4(-3)}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{13}}{2}$

Tells me we reach the same point on plane at $-\frac{1}{2} - \frac{\sqrt{13}}{2}$, $-\frac{1}{2} + \frac{\sqrt{13}}{2}$

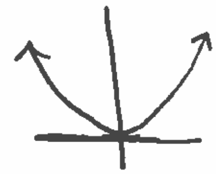
early s

late t

$(-3, 3)$ ← plug in to x, y

Familiar with rectangular curves like $y = x^2$

Can we find parametric equations to give same shape?



↓

<u>one</u> method is to set $t = \frac{dy}{dx}$.

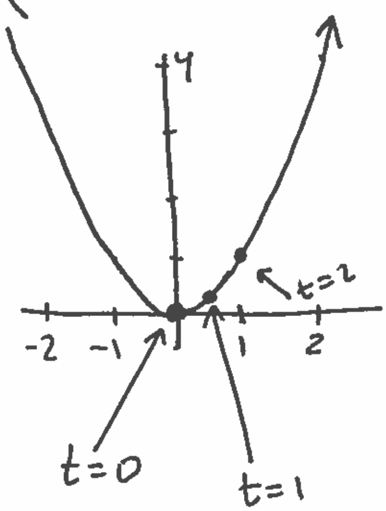
Like: $t = 2x \xrightarrow[\text{can solve for } x]{\text{after}} x = \frac{t}{2}$

↓ then use rectangular...

$y = \left(\frac{t}{2}\right)^2$

$y = \frac{t^2}{4}$

So $\begin{cases} x = \frac{1}{2}t \\ y = \frac{1}{4}t^2 \end{cases}$ ←



Cool byproduct: What time is it when $\frac{dy}{dx} = 2$?

Well, $t = \frac{dy}{dx}$, so at $t=2$. ✓

Verify: $t=2 \longrightarrow (1,1)$ is location

↓

$x=1$

$\frac{dy}{dx} = 2x \rightarrow 2 \cdot 1 = 2$ ✓

Going from parametric to rectangular can be more difficult.

<u>Ex</u> $x = \tan(t)$
$y = t^3$

Find a rectangular equation for shape... "eliminate t".

Easy approach
<u>can</u> solve for t in 2nd equation

$t = \sqrt[3]{y}$

So $x = \tan(\sqrt[3]{y})$. Done

<u>Ex</u> $x = \dfrac{1+t^2}{t^2+2}$

$y = \dfrac{-2+t^2}{t^2+2}$

Eliminate t...

Same denominators is a big hint...

yet to be determined

Try $a \cdot x + y = a\dfrac{1+t^2}{t^2+2} + \dfrac{-2+t^2}{t^2+2}$

$= \dfrac{a + at^2 - 2 + t^2}{t^2+2}$

$= \dfrac{t^2(a+1) + a-2}{t^2+2}$

$= (a+1)\dfrac{t^2 + \frac{a-2}{a+1}}{t^2+2}$

Want $\dfrac{a-2}{a+1} = 2$

$a-2 = 2a+2$
$-4 = a$

So
$-4x+y = (-4+1)$
$-4x+y = -3$

Ex $x = 4\cos(t) + 3$ Eliminate t.

$y = 2\sin(t) + 1$

Want to use $\cos^2(t) + \sin^2(t) = 1$

Try to isolate $\cos(t)$ & $\sin(t)$.

$x - 3 = 4\cos(t)$

$\frac{x-3}{4} = \cos(t)$

$y - 1 = 2\sin(t)$

$\frac{y-1}{2} = \sin(t)$

$$\Longrightarrow \cos^2(t) + \sin^2(t) = 1$$

$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$$

t is gone!

Trig Identities:

$$\sin(2t) = 2 \cdot \sin(t) \cdot \cos(t)$$

$$\cos(2t) = \cos^2(t) - \sin^2(t)$$

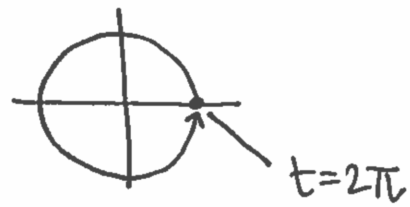
$$= 2\cos^2(t) - 1$$

$$= 1 - 2\sin^2(t)$$

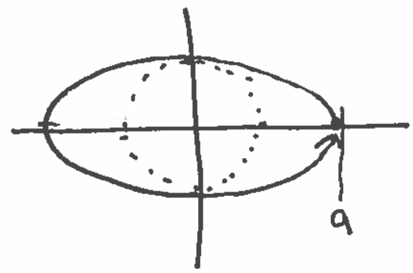
Circles & Ellipses

unit circle

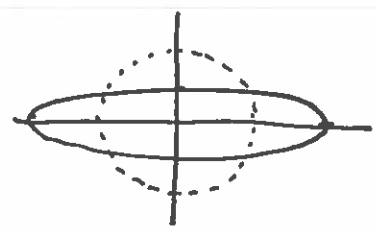
$x = \cos(t)$
$y = \sin(t)$ $\longrightarrow$



So $x = a \cdot \cos(t)$
$y = \sin(t)$ $\longrightarrow$

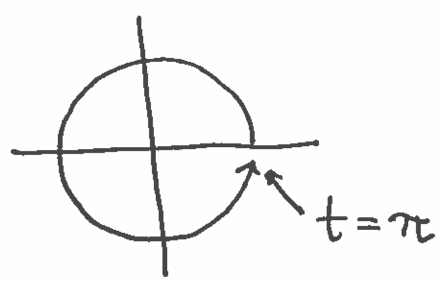


And $x = a \cdot \cos(t)$
$y = b \cdot \sin(t)$ $\longrightarrow$



Ellipse with x-radius a, y-radius b.

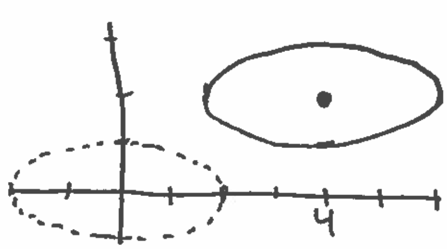
Ex $x = \cos(\boxed{2t})$
$y = \sin(\boxed{2t})$ $\xrightarrow[\text{a unit circle}]{\text{Also}}$



Shifts with parametric equations, easy!

How to shift.

Ex

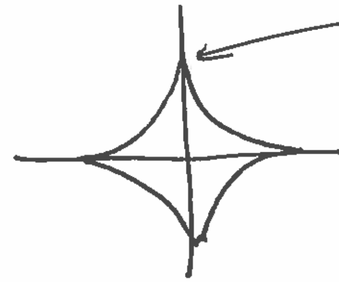


Ellipse centered at (4,2), x-radius 2, y-radius 1
Find param eq. for this.

$x = 2 \cdot \cos(t) + 4$
$y = \sin(t) + 2$

9.3

Ex $x = \cos^3(t)$
$y = \sin^3(t)$



Saw with GGB, motion comes to a stop at a cusp.

Def A parametrized curve is smooth if $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are never both 0 at the same time.

Ex Given $x = \cos^3(t)$, $y = \sin^3(t)$, find places where its not smooth.

$\frac{dx}{dt} = 3(\cos(t))^2 \cdot (-\sin(t))$ $\qquad$ $\frac{dy}{dt} = 3(\sin(t))^2 \cdot \cos(t)$

How could both $= 0$ at the same time?

$\frac{dx}{dt} = 0$ when $t = 0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad$ $\frac{dy}{dt} = 0$ when $t = 0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}$

The same! $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ are times where this curve not smooth. $\longrightarrow$ $(1,0), (0,1), (-1,0), (0,-1)$

Ex $x = 2t^3 - 3t^2$

$y = t^2 - 2t$

Is this smooth?

$\frac{dx}{dt} = 6t^2 - 6t$
$= 6t(t-1)$

is 0 when

$t = 0, \; t = 1$

$\frac{dy}{dt} = 2t - 2$
$= 2(t-1)$

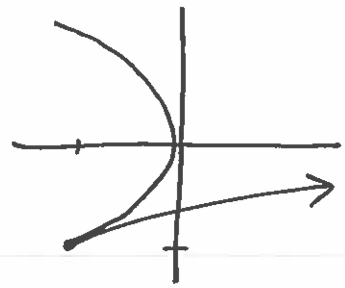
is 0 when

$t = 1$

So at $t=1$, not smooth. At $(-1,-1)$

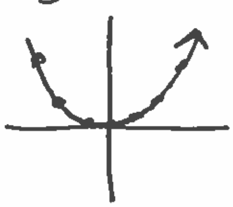
x y

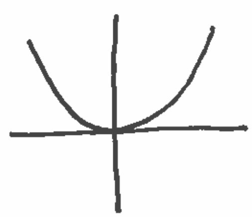
-1 -1



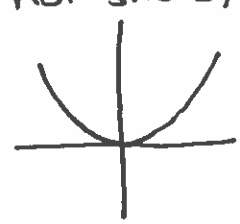
"Smooth" applies to a parametrization, not the shape it makes.

smooth





not smooth



$x = \frac{1}{2}t$

$y = \frac{1}{4}t^2$

$\frac{dx}{dt} = \frac{1}{2}$

$\frac{dy}{dt} = \frac{1}{2}t$

$x = t^3$

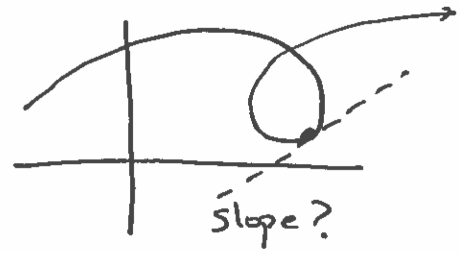
$y = t^6$

$\frac{dx}{dt} = 3t^2$

$\frac{dy}{dt} = 6t^5$

At $t=0$, both 0.

At a particular t, what is the slope, $\frac{dy}{dx}$?



$$\frac{dy}{dx} = \underbrace{\frac{dy}{dt}}_{\text{can calculate}} \cdot \frac{dt}{dx}$$ (Chain Rule

turns out:

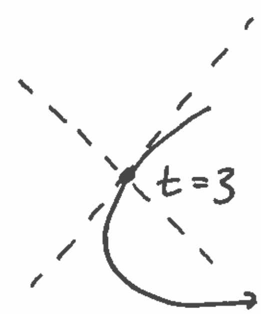
$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

<u>Ex</u> $x = 5t^2 - 6t + 4$
$y = t^2 + 6t - 1$

Find equations for tangent line & normal line, at $t = 3$.

Abstract: Picture



$\cancel{y = mx + b}$

$y = m(x - x_0) + y_0$

$\frac{dy}{dx}(t=3)$ for the tangent line; neg. reciprocal for normal!

point when $t=3$: $x_0 = 5 \cdot 3^2 - 6 \cdot 3 + 4 = 31$
$y = 3^2 + 6 \cdot 3 - 1 = 26$

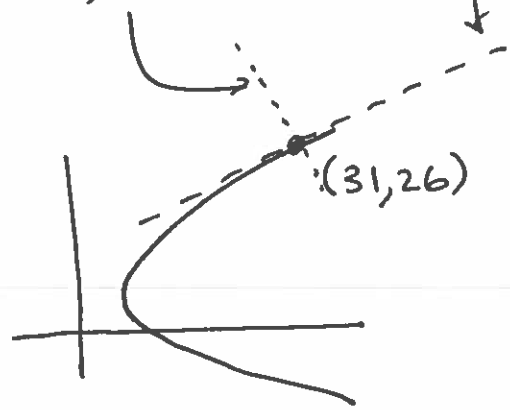
Need to know $\frac{dy}{dx}$. $= \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2t+6}{10t-6}$

So at $t=3$, $\frac{dy}{dx} = \frac{12}{24} = \frac{1}{2}$.

Tangent Line: $y = \frac{1}{2}(x-31) + 26$

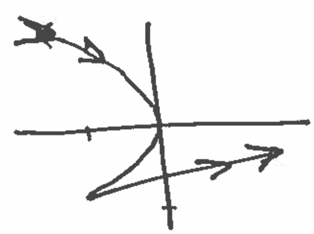
Normal Line: $y = -2(x-31) + 26$

Check with tech:



Now $\frac{dy}{dx}$ at non-smooth places...

<u>Ex</u> $x = 2t^3 - 3t^2$
$y = t^2 - 2t$ $\longrightarrow$ we found at $t=1$, $(-1,-1)$ was a not smooth location.



<u>Near</u> $t=1$, we know $\frac{dy}{dx}$.

Report this curve has slope $\frac{1}{3}$ at its cusp.

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2t-2}{6t^2-6t} = \frac{2(t-1)}{6t(t-1)} = \frac{1}{3t}$$

$$\lim_{t \to 1} \left(\frac{dy}{dx}\right) = \lim_{t \to 1} \left(\frac{1}{3t}\right) = \frac{1}{3}$$

non-smooth time