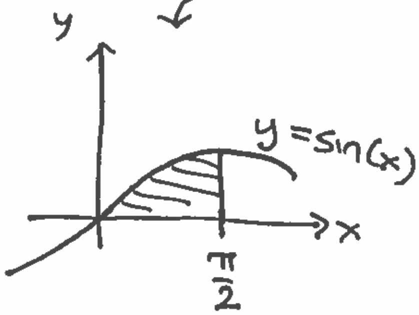
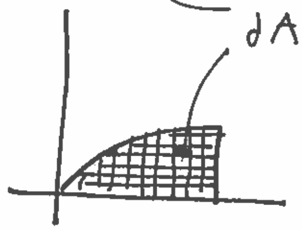
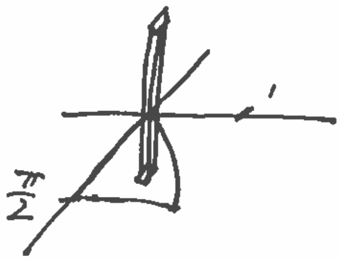
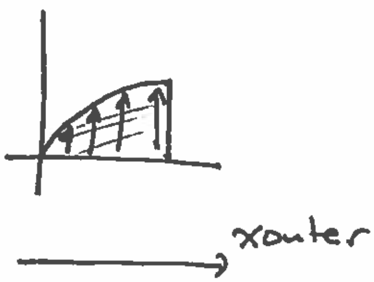
3.2 Ex $\iint_R \cos(x)\cdot y \; dA$

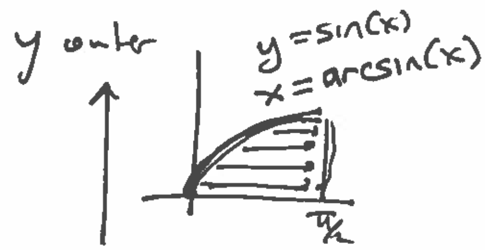




volume is $\cos(x)\cdot y \cdot dA$







$$\int_{x=0}^{x=\pi/2} \int_{y=\text{bottom}}^{y=\text{top}} \cos(x)\cdot y \; dy\, dx$$

$$\int_{x=0}^{x=\pi/2} \int_{y=0}^{y=\sin(x)} \underbrace{\cos(x)\, y \; dy}\, dx$$

$$\int_{x=0}^{x=\frac{\pi}{2}} \left[\cos(x)\cdot \tfrac{1}{2} y^2\right]_{y=0}^{y=\sin(x)} dx$$

$$\int_{x=0}^{x=\pi/2} \left[\tfrac{1}{2} \underbrace{\sin^2(x)}_{u=\sin(x)} \underbrace{\cos(x)\right] dx}_{du=\cos(x)\,dx}$$

$$\int_{u=0}^{u=1} \tfrac{1}{2} u^2 \, du = \left[\tfrac{1}{6} u^3\right]_0^1 = \tfrac{1}{6}$$

$$\int_{y=0}^{y=1} \int_{x=\text{left}}^{x=\text{right}} \cos(x) y \; dx\, dy$$

$$\int_{y=0}^{y=1} \int_{x=\arcsin(y)}^{x=\pi/2} \underbrace{\cos(x) y \; dx}\, dy$$

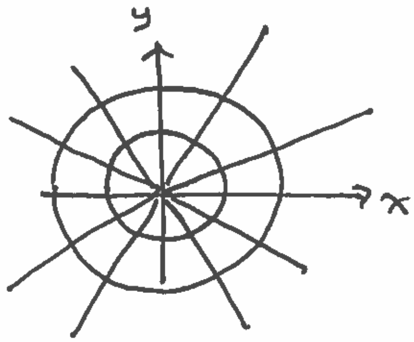
$$\int_{y=0}^{y=1} \left[\sin(x)\, y\right]_{x=\arcsin(y)}^{x=\pi/2} dy$$

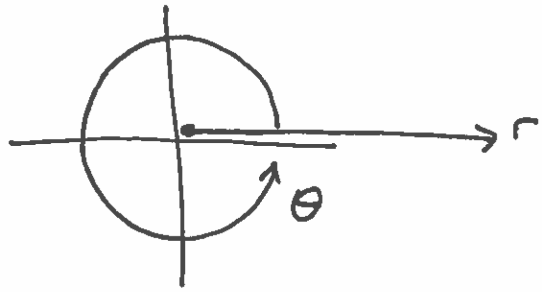
$$\int_{y=0}^{y=1} \left[y - y^2\right] dy$$

$$= \left[\tfrac{1}{2} y^2 - \tfrac{1}{3} y^3\right]_{y=0}^{y=1}$$

$$= \frac{1}{6}$$

13.3 Double Integrals using Polar coords...



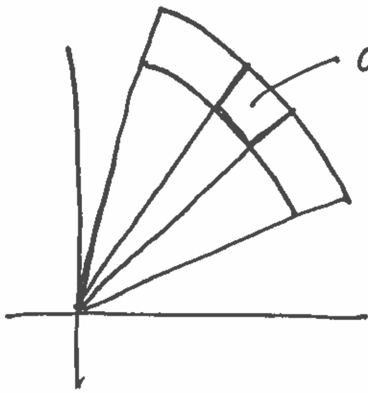


$x = r \cdot \cos\theta$

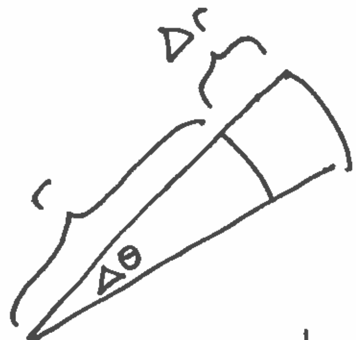
$y = r \cdot \sin\theta$

$r^2 = x^2 + y^2$

$\tan\theta = \dfrac{y}{x}$



a polar rectangle



sector's area $= \frac{1}{2}(\text{radius})^2(\text{angle})$

polar rectangle area $\Longleftarrow$ small slice $\frac{1}{2} \cdot r^2 \Delta\theta$ large slice $\frac{1}{2}(r+\Delta r)^2 \Delta\theta$

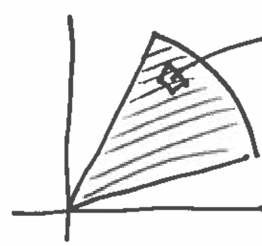
$$\frac{1}{2}\underbrace{(r+\Delta r)^2}\Delta\theta - \frac{1}{2}r^2\Delta\theta$$

$$= \left(r^2 + 2r\Delta r + \Delta r^2\right)$$

$$= r\Delta r\Delta\theta + \frac{1}{2}\Delta r^2\Delta\theta$$

for small Δr,

$$\approx r\Delta r\Delta\theta$$



$\Delta A \approx r\Delta r\Delta\theta$

limit as $\Delta r, \Delta\theta \to 0$

$dA = r \cdot dr \cdot d\theta$

$$\iint_R f \; dA = \underbrace{\iint_R f(x,y)\, dy\,dx}_{13.2} = \iint_R f(r,\theta) \cdot r\,dr\,d\theta$$

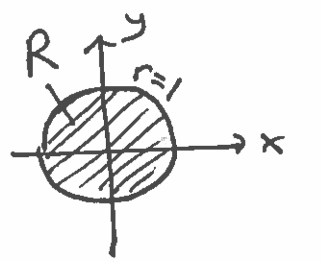
Ex $f(x,y) = 4 - x - 2y$ (graph is a plane ...

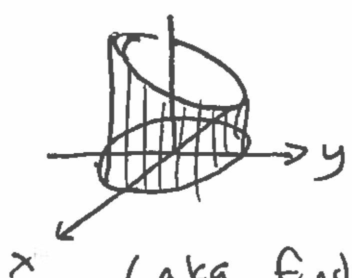
$z = 4 - x - 2y$

R the disc of radius 1 $\quad x + 2y + z = 4$

plane with $\vec{n} = \langle 1, 2, 1 \rangle$

$$\iint_R f \, dA$$

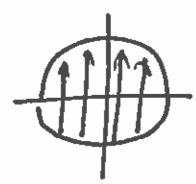


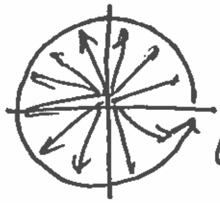


R seems very nice for polar coordinates

(aka find volume of this truncated cylinder)

$\longrightarrow$ x outer





θ outer

13.2 way

$$\int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (4 - x - 2y) \, dy \, dx$$

Let θ run from 0 to 2π

At each θ, let r run from 0 to 1

$f(x,y) = 4 - x - 2y$

$f(r,\theta) = 4 - r \cdot \cos\theta - 2r\sin\theta$

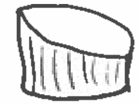
$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} f \; r \cdot dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (4 - r\cos\theta - 2r\sin\theta) \, r \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \underbrace{(4r - r^2\cos\theta - 2r^2\sin\theta)}\, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[2r^2 - \frac{1}{3} r^3 \cos\theta - \frac{2}{3} r^3 \sin\theta \right]_{r=0}^{r=1} d\theta$$

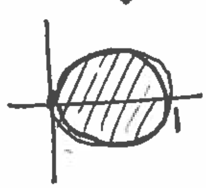
$$= \int_{\theta=0}^{\theta=2\pi} \left[2 - \frac{1}{3}\cos\theta - \frac{2}{3}\sin\theta\right] d\theta$$



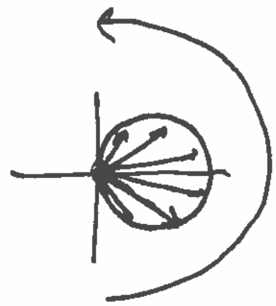
volume

$$= \left[2\theta - \frac{1}{3}\sin\theta + \frac{2}{3}\cos\theta\right]_0^{2\pi} = 2(2\pi) - 2(0) = 4\pi$$

Ex $\iint_R x \, dA$



surrounded by $r = \cos\theta$



$$\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=\cos\theta} r\cdot\cos\theta \; r\,dr\,d\theta$$

!!!

$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} \left[\frac{1}{3} r^3 \cdot \cos\theta\right]_0^{\cos\theta} d\theta$$

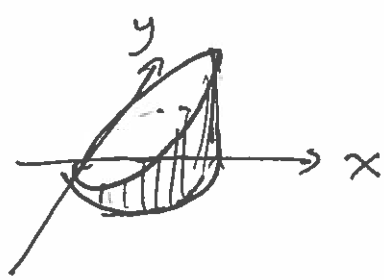
$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\pi/2} \frac{1}{3}\cos^4(\theta) \, d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \frac{1}{3}\left(\cos^2(\theta)\right)^2 d\theta = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \frac{1}{3}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \frac{1}{12}\left(1 + 2\cos(2\theta) + \cancel{\cos^2(2\theta)}\right) d\theta$$

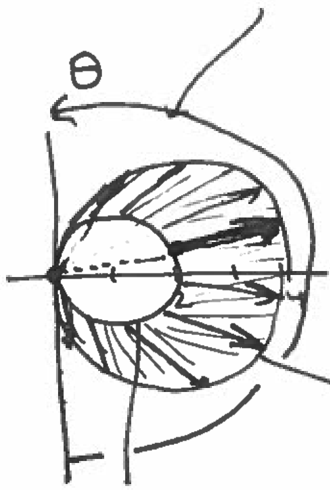
$$\frac{1+\cos(4\theta)}{2}$$

$$= \frac{1}{12}\left[\theta + \sin(2\theta) + \frac{1}{2}\theta + \frac{\sin(4\theta)}{8}\right]_{\theta=-\frac{\pi}{2}}^{\theta=\pi/2} = \frac{1}{12}\left[\frac{3}{2}\theta\right]_{\theta=-\pi/2}^{\theta=\pi/2}$$



$$= \frac{1}{12}\cdot\frac{3}{2}(\pi) = \frac{1}{8}\pi.$$

Ex $\iint_R (4-(x-2)^2-y^2)\,dA$



$$\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\int_{r=2\cos\theta}^{r=4\cos\theta}(\cancel{4}-x^2+4x-\cancel{4}-y^2)\,\underset{r}{\wedge}dr\,d\theta$$

$$=\int_{\theta=\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\int_{r=2\cos\theta}^{r=4\cos\theta}(4x\underbrace{-x^2-y^2})\,r\,dr\,d\theta$$

$r^2=x^2+y^2$

$-r^2=-x^2-y^2$

$$=\int_{\theta=\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\int_{r=2\cos\theta}^{r=4\cos\theta}\underbrace{(4r\cos\theta-r^2)r\,dr}\,d\theta$$

$$=\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\left[\frac{4}{3}r^3\cos\theta-\frac{1}{4}r^4\right]_{r=2\cos\theta}^{r=4\cos\theta}d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\left[\frac{256}{3}\cos^4(\theta)-64\cos^4(\theta)-\frac{32}{3}\cos^4(\theta)+4\cos^4(\theta)\right]d\theta$$

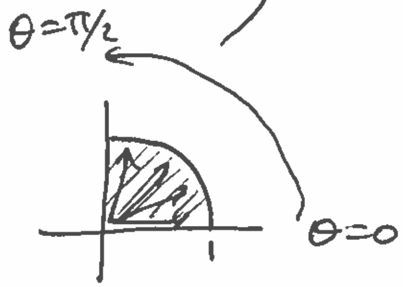
$$=\int_{-\pi/2}^{\pi/2}C\cdot\underbrace{\cos^4(\theta)}\,d\theta \quad = \;---$$

totally doable!

$(\cos^2\theta)^2$

$\left(\frac{1+\cos(2\theta)}{2}\right)^2$

Ex $\iint_R \frac{1}{x^2+y^2+1}\,dA$



$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=1} \frac{1}{r^2+1}\, r\,dr\,d\theta$$

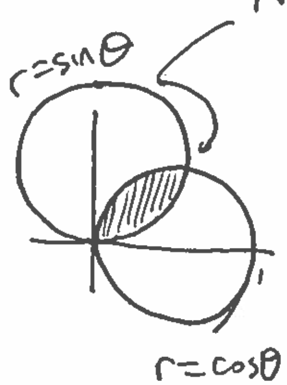
$$u = r^2+1$$
$$du = 2r\,dr$$
$$\tfrac{1}{2}du = r\,dr$$

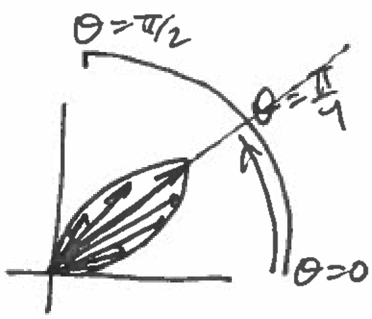
$$= \int_{\theta=0}^{\theta=\pi/2}\int_{u=1}^{u=2} \frac{1}{u}\cdot\frac{1}{2}\,du\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2}\left[\frac{1}{2}\ln|u|\right]_1^2 d\theta = \int_{\theta=0}^{\theta=\pi/2}\frac{1}{2}\ln(2)\,d\theta$$

$$= \frac{1}{2}\ln(2)\cdot\frac{\pi}{2} = \frac{\pi}{4}\ln(2).$$

Ex $\iint_R xy\,dA$





$$\int_{\theta=0}^{\theta=\frac{\pi}{4}}\int_{r=0}^{r=\sin\theta} xy\,r\,dr\,d\theta$$

$$+\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=\cos\theta} xy\,r\,dr\,d\theta$$

$r\cos\theta$ $\quad$ $r\sin\theta$

= totally doable!

$$\pi = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dA = \int_{x=-\infty}^{x=\infty} \int_{y=-\infty}^{y=\infty} e^{-x^2} \cdot e^{-y^2}\, dy\, dx$$

$$= \int_{x=-\infty}^{x=\infty} e^{-x^2} \left[\int_{y=-\infty}^{\infty} e^{-y^2}\, dy \right] dx$$

$$= \left[\int_{y=-\infty}^{\infty} e^{-y^2}\, dy \right] \cdot \left[\int_{x=-\infty}^{x=\infty} e^{-x^2}\, dx \right]$$

$$\pi = \left[\int_{x=-\infty}^{x=\infty} e^{-x^2}\, dx \right]^2$$

$$\Longrightarrow \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

(despite no antiderivative)

13.6 Volume of 3D objects

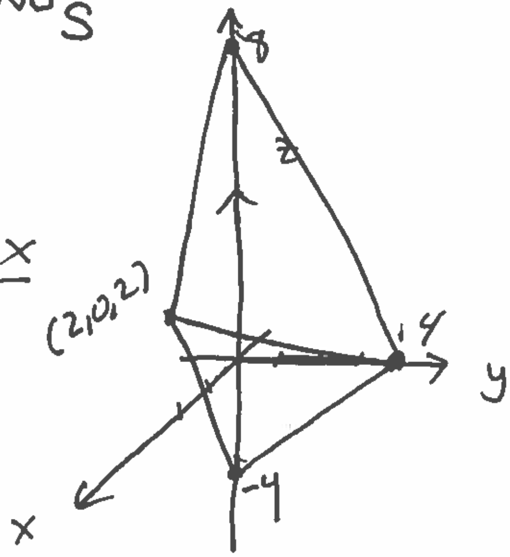
$$\iiint_S 1 \cdot dV = \text{Volume of } S$$

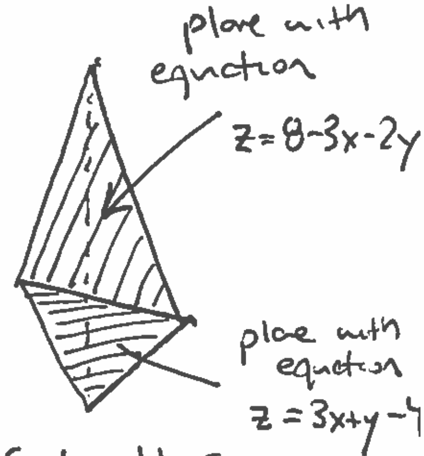
13.1

$$\iint_R 1 \cdot dA$$

= area of R

Ex





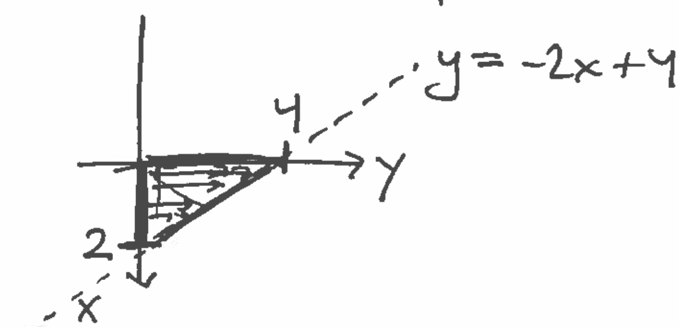
S be this solid (interior)

Find its volume

Find S's volume.

$$\iiint_S 1 \cdot dV$$

Observation: top-down



$$\int_{x=0}^{x=2} \int_{y=0}^{y=-2x+4} \int_{z=3x+y-4}^{z=8-3x-2y} 1 \, dz\,dy\,dx$$

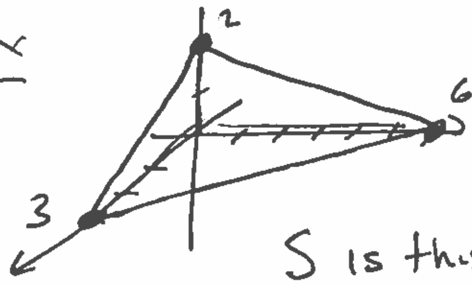
$$= \int_{x=0}^{2} \int_{y=0}^{-2x+4} \left[z \right]_{3x+y-4}^{8-3x-2y} dy\,dx = \underbrace{\int_{x=0}^{x=2} \int_{y=0}^{-2x+4} (12-6x-3y)\,dy\,dx}_{\text{Back in 13.2-style}}$$

$$= \int_{x=0}^{2} \left[12y - 6xy - \frac{3}{2}y^2 \right]_0^{-2x+4} dx$$

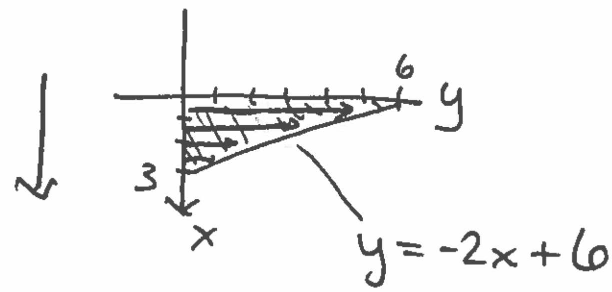
$$= \int_{x=0}^{2} \left[12(-2x+4) - 6x(-2x+4) - \frac{3}{2}(-2x+4)^2 \right] dx$$

$$= \ \cdots \ = 16 \longrightarrow$$ is the volume of that ~ tetrahedron.

<u>Ex</u>



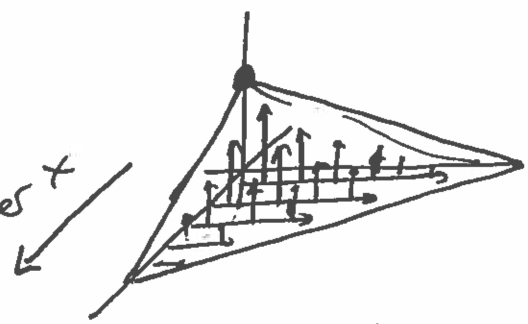
S is this solid.



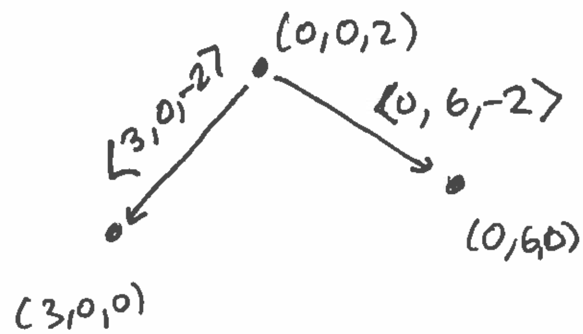
Find S's volume.

$$\int_{x=0}^{x=3} \int_{y=0}^{y=-2x+6} \int_{z=0}^{z=-\frac{2}{3}x-\frac{1}{3}y+2} 1 \cdot dz\,dy\,dx$$

outer x



Need to know equation for that plane



$$12(x) + 6y + 18(z-2) = 0$$

$$18z \ \cancel{-36} = -12x - 6y + 36$$

$$z = \frac{-2}{3}x - \frac{1}{3}y + 2$$

$$\vec{n} = \langle 3,0,-2\rangle \times \langle 0,6,-2\rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -2 \\ 0 & 6 & -2 \end{vmatrix}$$

$$\vec{n} = [\text{scribbled out}]$$

$$= \langle 12, 6, 18\rangle$$

$$\int_{x=0}^{x=3} \int_{y=0}^{y=-2x+6} \Big[z \Big]_0^{-\frac{2}{3}x - \frac{1}{3}y + 2} \, dy \, dx$$

$$\int_{x=0}^{3} \int_{y=0}^{-2x+6} \left(-\frac{2}{3}x - \frac{1}{3}y + 2 \right) dy \, dx$$

$$\int_{x=0}^{3} \left[-\frac{2}{3}xy - \frac{1}{6}y^2 + 2y \right]_0^{-2x+6} dx$$

$$\int_{x=0}^{3} \left(-\frac{2}{3}x(-2x+6) - \frac{1}{6}(-2x+6)^2 + 2(-2x+6) \right) dx$$

$$\int_0^3 \left(\frac{4}{3}x^2 - 4x - \frac{2}{3}x^2 + 6x - 6 - 4x + 12 \right) dx$$

$$\int_0^3 \left(\frac{2}{3}x^2 - 2x + 6 \right) dx = \left[\frac{2}{9}x^3 - x^2 + 6x \right]_0^3$$

$$= \frac{2}{9}(27) - 9 + 18$$

$$= 6 + 9 = 15$$

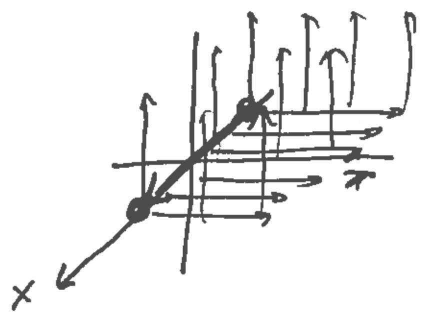
Ex $$\int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_0^{\pi} \cos(x)\sin(y)\sin(z)\, dz\, dy\, dx$$

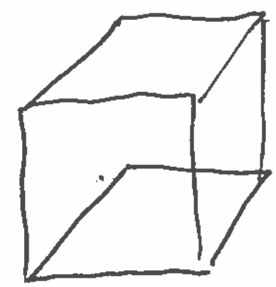
Understand what this is.

x range $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

y range 0 to π

z range 0 to π





defines a rectangula prism...

$S =$ cube: $\pi \times \pi \times \pi$

We've been asked to integrate $\cos(x)\cdot\sin(y)\sin(z)$ over this cube.

$$= \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \Big[-\cos(x)\sin(y)\underbrace{\overset{-1}{\cos(z)}}_{1}\Big]_0^{\pi}\, dy\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \Big[\cancel{\cos(x)\sin(y)} + 2\cos(x)\sin(y)\Big]\, dy\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \Big[-2\cos(x)\cos(y)\Big]_0^{\pi}\, dx$$

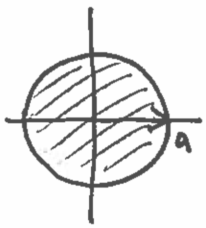
$$= \int_{-\pi/2}^{\pi/2} 4\cos(x)\, dx = \Big[4\sin(x)\Big]_{-\pi/2}^{\pi/2} = 8$$

(Avg value of $\cos(x)\sin(y)\sin(z)$ over this cube: $\frac{8}{\pi^3}$

e^{-x^2} has no <u>elementary</u> antiderivative

$\int e^{-x^2}\,dx$ can't be done by elementary functions.

$$\iint_R e^{-x^2-y^2}\,dA = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a} \underbrace{e^{-r^2}\,\underbrace{r\cdot dr}}\,d\theta$$



$$u = -r^2$$
$$du = -2\underbrace{r\,dr}$$
$$-\tfrac{1}{2}\,du = \underbrace{r\,dr}$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{u=0}^{u=-a^2} e^u\left(-\tfrac{1}{2}\,du\right)d\theta$$

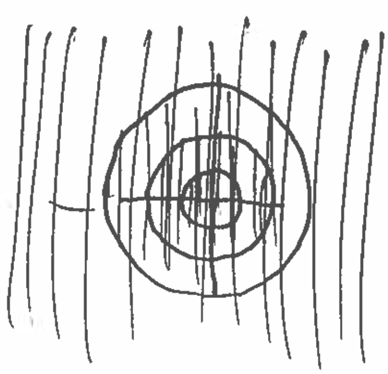
$$= \int_{\theta}^{2\pi}\left[-\tfrac{1}{2}e^u\right]_{u=0}^{-a^2} d\theta \; \cancel{= \int_0^{2\pi} -\tfrac{1}{2}e^{-a^2}}\; d\theta$$

$$= \int_0^{2\pi}\left[-\tfrac{1}{2}e^{-a^2} + \tfrac{1}{2}\right] d\theta = \left[-\tfrac{1}{2}e^{-a^2}+\tfrac{1}{2}\right]\theta\Big|_0^{2\pi} \; \cancel{-\tfrac{1}{2}e^{-a^2}[2}$$

$$= \left[\tfrac{1}{2} - \tfrac{1}{2}e^{-a^2}\right]\cdot 2\pi = \pi\left(1-e^{-a^2}\right)$$

Suppose I want $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA = \lim_{a\to\infty} \pi\left(1-e^{-a^2}\right)$

$$= \pi.$$



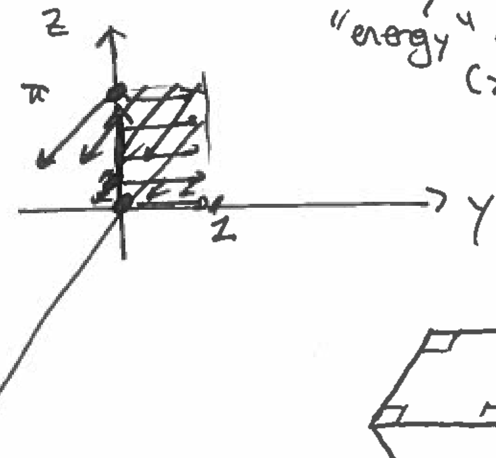
$\lim_{a\to\infty}$

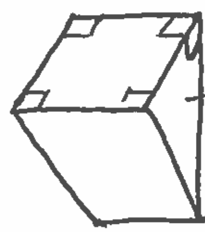
$\mathbb{R}^2$

Ex Interpret $$\int_0^{\pi}\int_0^{1}\int_0^{z} \underbrace{\sin(yz)}_{\text{density of "energy" at } (x,y,z)}\, dx\, dy\, dz$$

outer most do first

Visualize the 3D Solid





visit each (x,y,z) inside this solid, calculate $\sin(yz)$, integrate it all.

Integral gives total "energy"

$$= \int_0^{\pi}\int_0^{1} \Big[x\cdot\sin(yz)\Big]_{x=0}^{x=z}\, dy\, dz$$

$$= \int_0^{\pi}\int_0^{1} z\sin(yz)\, dy\, dz$$

$$= \int_0^{\pi} \Big[-\cos(yz)\Big]_{y=0}^{y=1} dz = \int_0^{\pi}\big(-\cos(z)+1\big)\, dz$$

$$= \Big[-\sin(z)+z\Big]_0^{\pi} = \pi$$