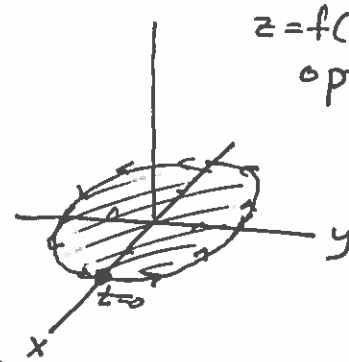
12.8 cont'd.

z = f(x,y)
optimize!



Ex $f(x,y) = 3y - 2x^2$

constrained to inside
$x^2 + y^2 = 4$

Look for critical points

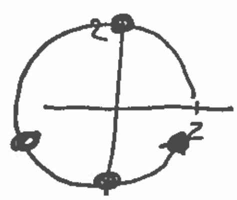
$f_x(x,y) = -4x$

$f_y(x,y) = 3 \neq 0$ no matter what (x,y) is

No critical points.

Explore boundary...

Parametrize the boundary...



$x = 2 \cdot \cos(t)$

$y = 2 \cdot \sin(t)$

$F(t) = 3(2\sin(t)) - 2(2\cos t)^2$

$6\sin(t) - 8\cos^2 t$

optimize F

$F'(t) = 6 \cdot \underbrace{\cos(t)} + 16 \underbrace{\cos(t)} \sin(t) = 0$

$\cos(t)\left[6 + 16\sin(t)\right] = 0$

$\cos(t) = 0$

$t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$(x,y) = (0,2)$

$(x,y) = (0,-2)$

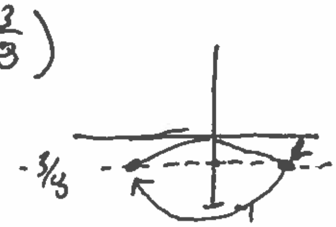
$6 + 16\sin(t) = 0$

$\sin(t) = \frac{-6}{16} = \frac{-3}{8}$

$t = \arcsin\left(\frac{-3}{8}\right)$

or

$\pi - \arcsin\left(\frac{-3}{8}\right)$



$(x,y) = (\quad, \frac{-3}{4})$

$(x,y) = (\quad, \frac{-3}{4})$

we know $x^2 + y^2 = 4$

$x^2 + \left(\frac{-3}{4}\right)^2 = 4$

$x^2 = \underset{\frac{64}{16}}{\cancel{4}} - \frac{9}{16} = \frac{55}{16} \qquad x = \pm \frac{\sqrt{55}}{4}$

$(0,2) \xrightarrow{f} 6 \longrightarrow$ largest

$(0,-2) \xrightarrow{f} -6$

$\left(\frac{\sqrt{55}}{4}, \frac{-3}{4}\right) \xrightarrow{f} 3\left(\frac{-3}{4}\right) - 2\left(\frac{\sqrt{55}}{4}\right)^2$

$\frac{-9}{4} - 2\left(\frac{55}{16}\right)$

$\frac{-9}{4} - \frac{55}{8}$

$-\frac{73}{8}$

$\left(-\frac{\sqrt{55}}{4}, \frac{-3}{4}\right) \xrightarrow{f} 3\left(\frac{-3}{4}\right) - 2\left(\frac{\sqrt{55}}{4}\right)^2$

$-\frac{73}{8}$

smallest

max value of 6 at $(0,2)$

min value of $\frac{-73}{8}$ at $\left(\pm\frac{\sqrt{55}}{4}, -\frac{3}{4}\right)$

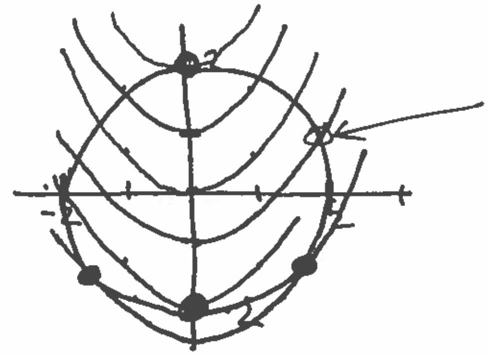
Lagrange Multipliers

$$f(x,y) = 3y - 2x^2$$

Level Curves:

$$3y - 2x^2 = c$$

$$y = \frac{2}{3}x^2 + \frac{c}{3}$$



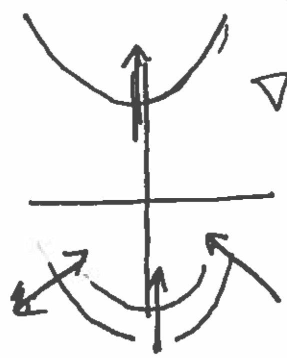
$C(x,y) = x^2 + y^2$
At level 4.

We look for where

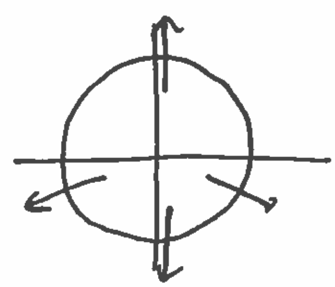
∇f is parallel to ∇C

(orthogonal to level curves)

constraint function.



∇f



∇C

$\nabla f = \langle -4x, 3 \rangle$

$\nabla C = \langle 2x, 2y \rangle$

parallel?

some scalar

$$\langle -4x, 3 \rangle = \lambda \langle 2x, 2y \rangle$$

$$\begin{cases} -4x = \lambda \cdot 2x \\ 3 = \lambda \cdot 2y \\ x^2 + y^2 = 4 \end{cases}$$

← original constraint

Eliminate λ: $\lambda = \frac{3}{2y}$ (valid as long as $y \neq 0$)

$$\begin{cases} -4x = \left(\frac{3}{2y}\right) \cdot 2x \\ x^2+y^2 = 4 \end{cases} \Longrightarrow \begin{cases} -4x = \frac{3x}{y} \\ x^2+y^2 = 4 \end{cases} \Longrightarrow -4 = \frac{3}{y}$$

(as long as $x \neq 0$)

$$\Longrightarrow \begin{cases} -4 = \frac{3}{y} \\ x^2+y^2=4 \end{cases} \Longrightarrow y = -\frac{3}{4}$$

$$\Longrightarrow x^2 + \left(-\frac{3}{4}\right)^2 = 4$$

$\vdots$

$$x = \pm \frac{\sqrt{55}}{4}$$

$\left(\frac{\sqrt{55}}{4}, \frac{-3}{4}\right)$

$\left(\frac{-\sqrt{55}}{4}, \frac{-3}{4}\right)$

What if $y=0$?

$$\begin{cases} -4x = \lambda \cdot 2x \\ 3 = 0 \\ x^2+y^2=4 \end{cases}$$

$3 \neq 0$

What if $x=0$?

$$\begin{cases} 0 = 0 \\ 3 = \lambda \cdot 2y \\ y^2 = 4 \end{cases} \Longrightarrow y = \pm 2$$

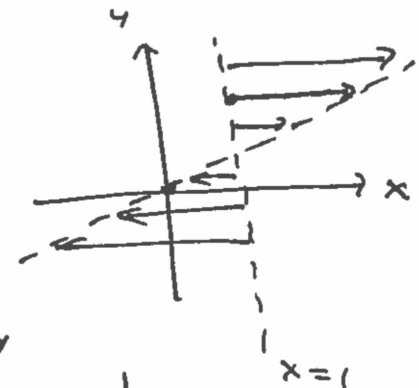
$(0,2)$ $(0,-2)$

13.1 What does $\int_1^{2y} \underbrace{2xy}_{f(x,y)}\, dx$ mean?

integral is w.r.t x.

$x = 2y$

$x = 1$



$\int_1^{2y} \cdots\, dx$

Well

$\int_1^{2y} 2xy \, dx$

x^2y is the antiderivative...
because this $\int$ was w.r.t $\underline{x}$, not y.

$$= \left[x^2 y\right]_{x=1}^{x=2y} = \left[(2y)^2 y\right] - \left[1^2 \cdot y\right]$$

$$= 4y^3 - y$$

Ex $$\int_1^x \left(5x^3y^{-3} + 6y^2\right) dy = \left[\frac{5x^3}{-2} y^{-2} + 2y^3\right]_{y=1}^{y=x}$$

$$= \frac{5x^3}{-2}(x)^{-2} + 2x^3 - \left[\frac{5x^3}{-2} + 2\right]$$

$$= -\frac{5}{2}x \; \cancel{+2x^3 + \frac{5}{2}x^3} - 2$$

$$\frac{9}{2}x^3$$

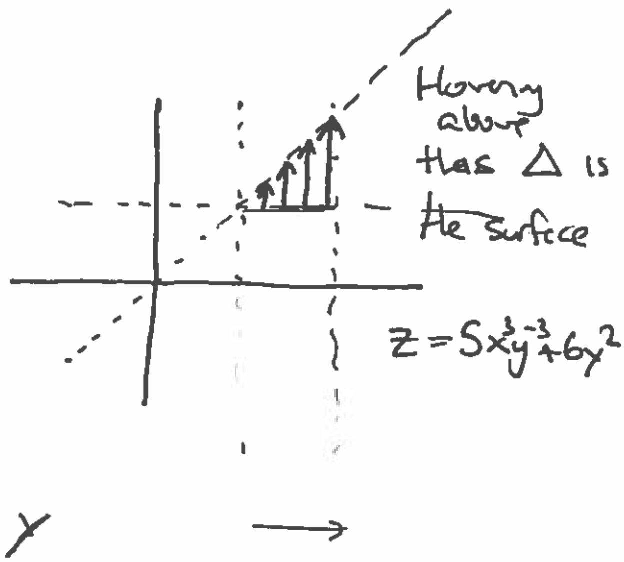
<u>Ex</u> $$\underbrace{\int_1^2 \underbrace{\int_1^x (5x^3y^{-3}+6y^2)\,dy}_{\text{first...}}\,dx}$$

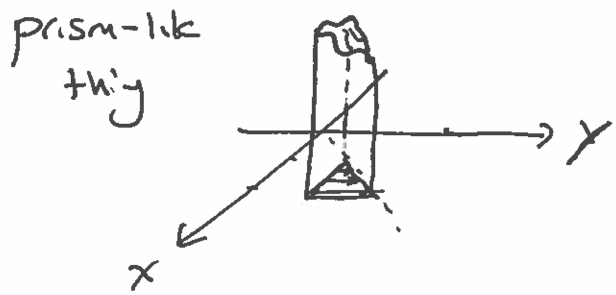
An <u>iterated</u> integral.

$$= \int_1^2 \left[-\frac{5}{2}x + \frac{9}{2}x^3 - 2\right] dx = \cdots = \frac{89}{8}$$

x went from 1 to 2 ...

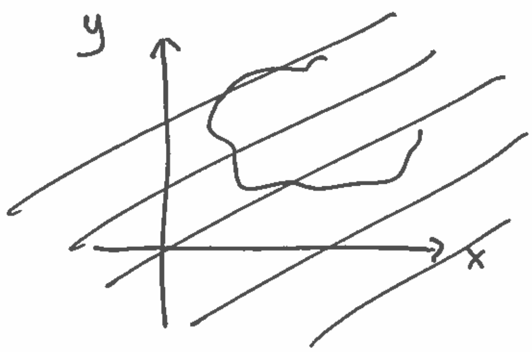
y went from $y=1$ to $y=x$

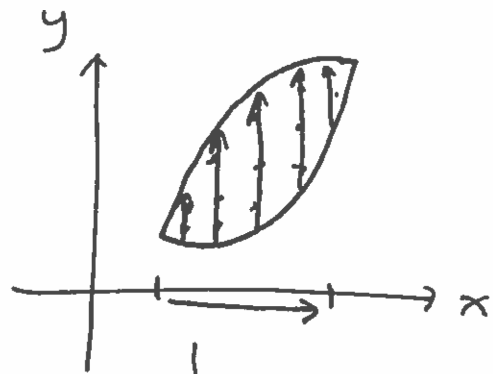




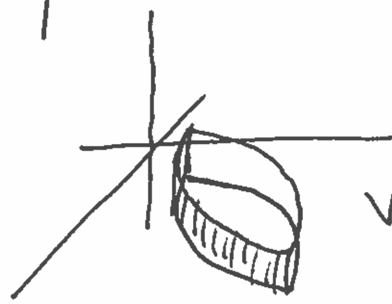
(the $\frac{89}{8}$ is its <u>volume</u>.)

Area of Planar Regions





What area is this

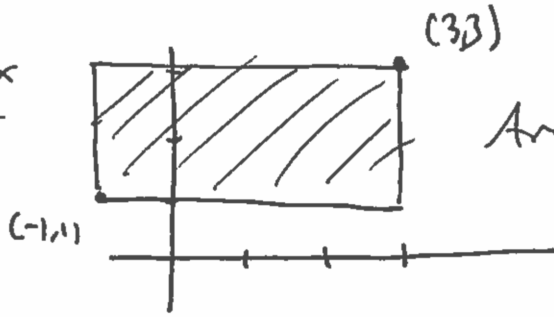


Volume = Area if height = 1.

Area of a region R

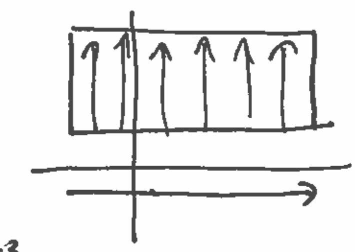
$$= \int_{?}^{?} \int_{?}^{?} 1 \cdot dy\, dx$$

Ex



Area = ?

$$\int_{x=-1}^{x=3} \int_{y=1}^{y=3} 1 \cdot dy \cdot dx$$

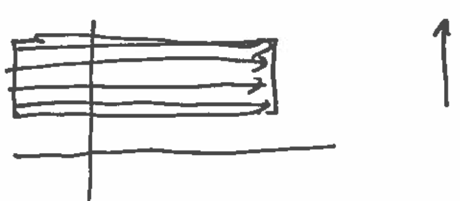


$$\int_{x=-1}^{x=3} \Big[y\Big]_{y=1}^{y=3} dx$$

$$= \int_{x=-1}^{x=3} 2\, dx = \Big[2x\Big]_{x=-1}^{x=3}$$

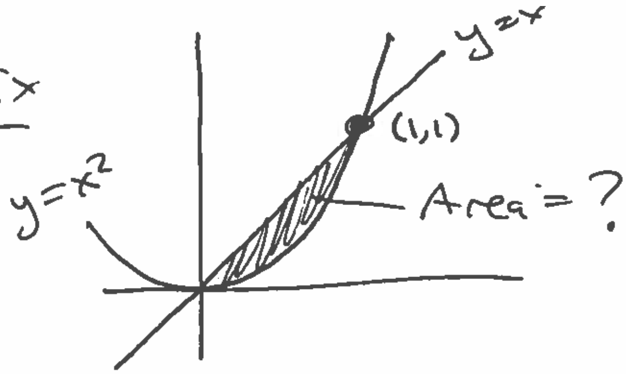
$$= 8$$

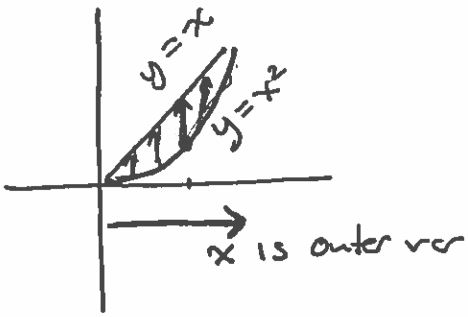
Alternatively



$$\int_{y=1}^{y=3} \int_{x=-1}^{x=3} 1 \cdot dx dy = \cdots = 8$$

Ex





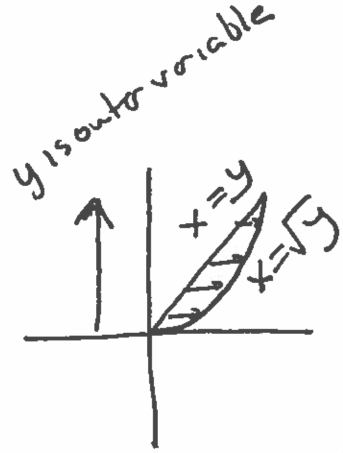
$$\int_{x=0}^{x=1} \int_{y=x^2}^{y=x} 1 \cdot dy\, dx$$

$$\int_{x=0}^{x=1} \Big[y\Big]_{y=x^2}^{y=x} dx$$

$$\int_0^1 (x - x^2)\, dx$$

$$\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$

$$= \frac{1}{6}$$



$$\int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} 1 \; dx\, dy$$

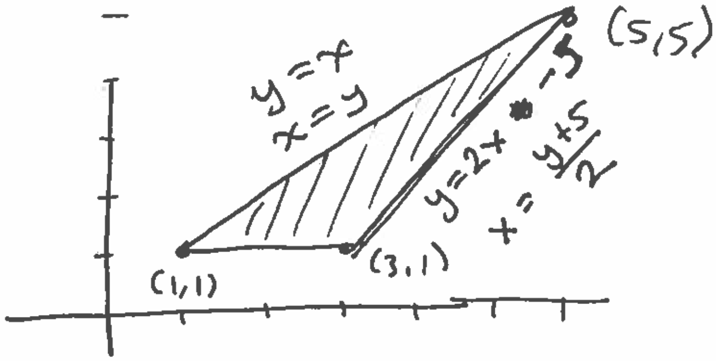
$$= \int_{y=0}^{y=1} \Big[x\Big]_{x=y}^{x=\sqrt{y}} dy$$

$$= \int_0^1 (\sqrt{y} - y)\, dy$$

$$= \left[\frac{2}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^1$$

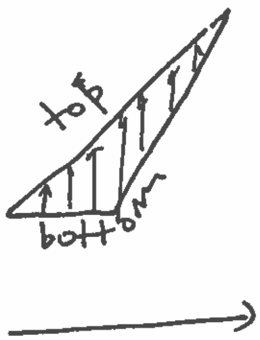
$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

Ex

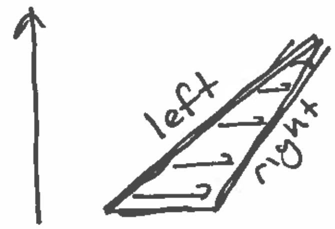


Find area
(4 by $\frac{1}{2}bh$)

x outer



y outer



$\Downarrow$ smarter choice

$$\int_{y=1}^{y=5}\int_{x=\text{left}}^{x=\text{right}} 1 \, dx \, dy$$

$$= \int_{y=1}^{y=5}\int_{x=y}^{x=\frac{y+5}{2}} 1 \, dx \, dy$$

$$= \int_{y=1}^{y=5} \Big[x \Big]_{x=y}^{x=\frac{y+5}{2}} dy$$

$$= \int_1^5 \left(\frac{y+5}{2} - y\right) dy = \int_1^5 \left(\frac{5}{2} - \frac{y}{2}\right) dy$$

$$= \left[\frac{5}{2}y - \frac{y^2}{4}\right]_1^5$$

$$= \frac{5}{2}(5) - \frac{25}{4} - \left[\frac{5}{2} - \frac{1}{4}\right] = \frac{25}{2} - \frac{25}{4} - \frac{5}{2} + \frac{1}{4}$$

$$\frac{20}{2} - \frac{24}{4}$$

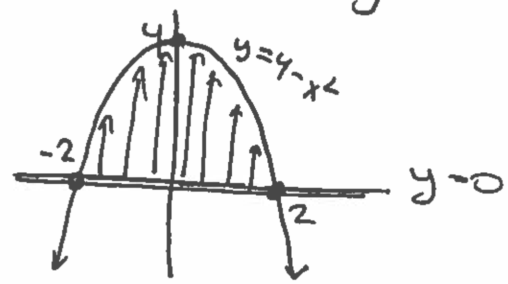
$$= 10 - 6$$

$$= 4 \checkmark$$

Identify what these integrals do...
and swap y & x (inner & outer).

$$\int_{-2}^{2}\int_{0}^{4-x^2} dy\,dx$$

x is outer...
y is inner... $\Rightarrow y=0$ to $y=4-x^2$

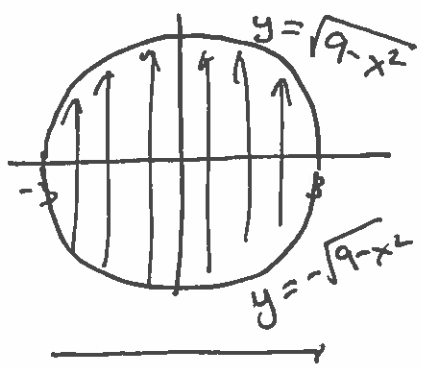


Finds the area here

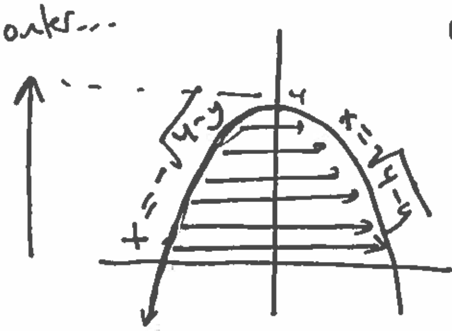
$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx$$

$y=\sqrt{9-x^2}$
$y^2=9-x^2$
$x^2+y^2=9$

$$\int_{-1}^{1}\int_{\frac{x-1}{2}}^{\frac{1-x}{2}} dy\,dx$$



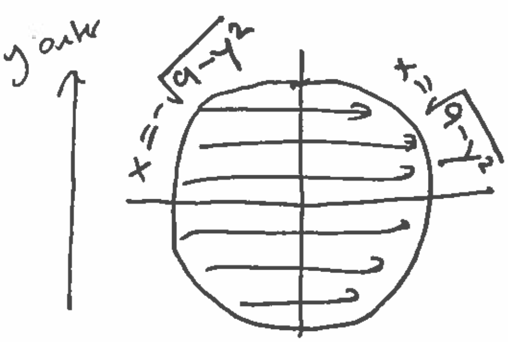
y outer...



$y=4-x^2$
$x^2=4-y$
$x=\pm\sqrt{4-y}$

$$\int_{0}^{4}\int_{x=-\sqrt{4-y}}^{x=\sqrt{4-y}} dx\,dy$$

Same as



$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} dx\,dy$$

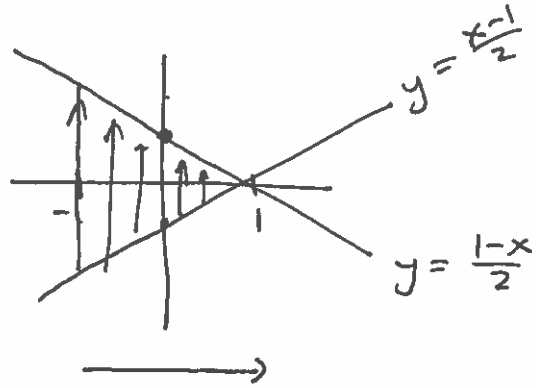
top: $y = \frac{1-x}{2}$

$$\int_{-1}^{1} \int_{\frac{x-1}{2}}^{\frac{1-x}{2}} dy\, dx$$

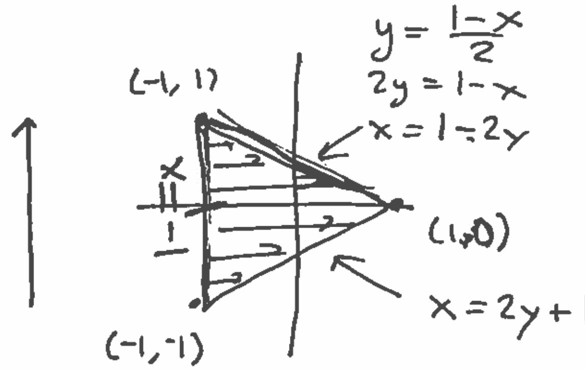
bottom

x outer

x −1 to 1



Now make y outer

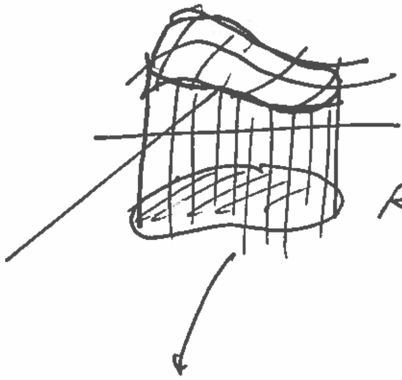


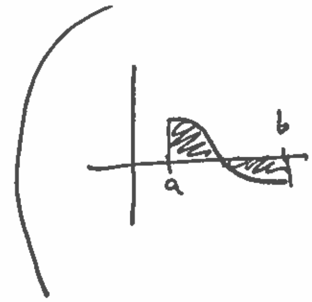
$$\int_{-1}^{0} \int_{x=\text{left}}^{x=\text{right}} dx\, dy \; + \; \int_{0}^{1} \int_{x=\text{left}}^{x=\text{right}} dx\, dy$$

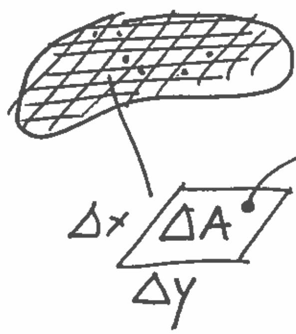
$$\int_{-1}^{0} \int_{-1}^{2y+1} dx\, dy \; + \; \int_{-1}^{0} \int_{-1}^{1-2y} dx\, dy$$

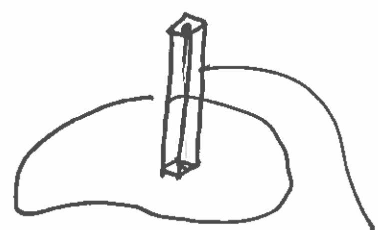
13.2

$$\int_?^? \int_?^? \underbrace{f(x,y)}_{\text{in 13.1, this was usually 1.}} \, dy \, dx$$ gives signed volume









$f\left((x,y)_i^*\right)$ is a certain height

has volume

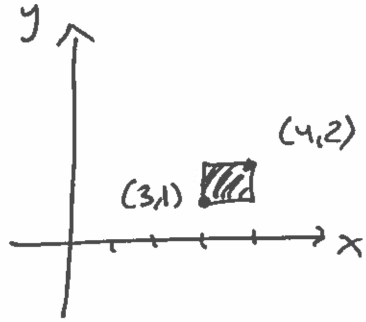
$$f\left((x,y)_i^*\right) \cdot \Delta A$$

$$\text{Volume} \approx \sum_i f\left((x,y)_i^*\right) \Delta A$$

lim as $\Delta x, \Delta y \to 0$

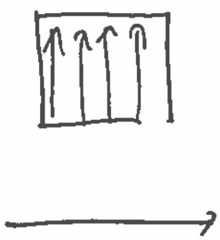
$$\underbrace{\iint_R}_{\text{double integral}} f(x,y) \, \underbrace{dA}_{dx\,dy}$$ gives the volume above R capped by $z = f(x,y)$.

Ex R is :



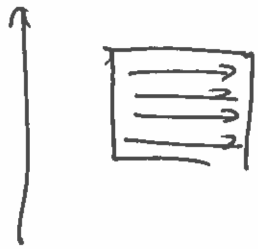
Find

$$\iint_R (xy + e^y)\, dA$$



x outer

or



(won't matter which..)

$$\int_{x=3}^{x=4} \int_{y=1}^{y=2} (xy + e^y)\, dy\, dx$$

$$= \int_{x=3}^{x=4} \left[x\frac{y^2}{2} + e^y \right]_{y=1}^{y=2} dx$$

$$= \int_3^4 \left[2x + e^2 - \left(\tfrac{1}{2}x + e\right) \right] dx$$

$$= \int_3^4 \left(\tfrac{3}{2}x + e^2 - e \right) dx$$

$$= \left[\tfrac{3}{4}x^2 + (e^2 - e)x \right]_3^4$$

$$= \cdots$$

$$= \frac{21}{4} + e^2 - e$$

$$\int_{y=1}^{y=2} \int_{x=3}^{x=4} (xy + e^y)\, \underline{\underline{dx}}\, dy$$

$$= \int_{y=1}^{y=2} \left[\frac{x^2 y}{2} + e^y \cdot x \right]_{x=3}^{x=4} dy$$

$$= \int_1^2 \left(8y + 4e^y - \left(\tfrac{9}{2}y + 3e^y\right) \right) dy$$

$$= \int_1^2 \left(\tfrac{7}{2}y + e^y \right) dy$$

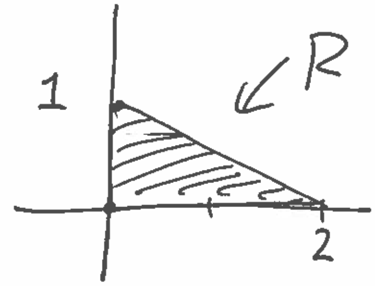
$$= \left. \tfrac{7}{4}y^2 + e^y \right]_1^2$$

$$= \frac{21}{4} + e^2 + e$$

Fubini's Theorem

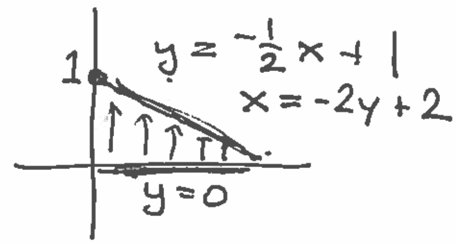
Basically doesn't matter what order...

Ex $\iint_R (3xy - x^2 - y^2 + 6)\, dA$



$$\int_{x=0}^{x=2} \int_{y=0}^{y=-\frac{1}{2}x+1} (3xy - x^2 - y^2 + 6)\, dy\, dx$$

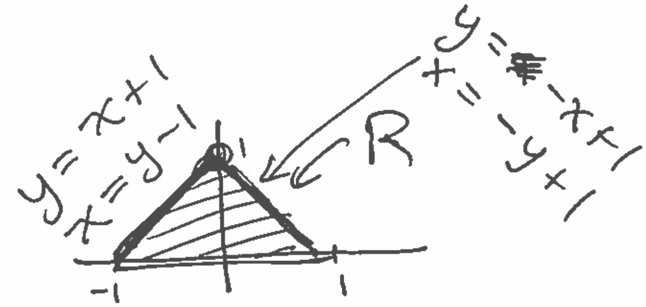
$$\int_{x=0}^{x=2} \left[\frac{3}{2} xy^2 - x^2 y - \frac{1}{3} y^3 + 6y \right]_{y=0}^{y=-\frac{1}{2}x+1} dx$$



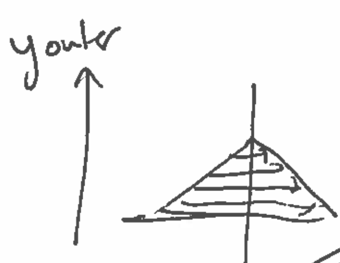
$$= \int_0^2 \left[\frac{3}{2} x \left(-\frac{1}{2}x+1\right)^2 - x^2 \left(-\frac{1}{2}x+1\right) - \frac{1}{3}\left(-\frac{1}{2}x+1\right)^3 + 6\left(-\frac{1}{2}x+1\right) \right] dx$$

$=$ one-variable calc $= \cdots = \frac{17}{3}$

Ex $\iint_R \sin(x)\cos(y)\, dA$



$$\int_{y=0}^{y=1} \int_{x=y-1}^{x=-y+1} \sin(x)\cos(y)\, dx\, dy$$



negatives

$$= \int_0^1 \left[-\cos(x)\cos(y) \right]_{x=y-1}^{x=-y+1} dy = \int_0^1 \left[-\cos(-y+1)\cos(y) + \cos(y-1)\cdot\cos(y) \right] dy$$

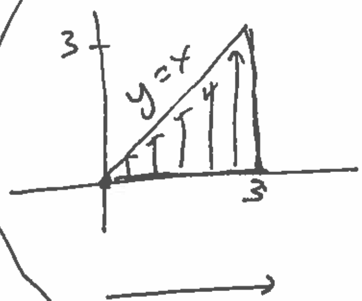
$$= \int_0^1 0\, dy = 0$$

Ex $\underbrace{\int_0^3 \int_y^3}_{\text{swap order}} \underbrace{e^{-x^2}\,dx}_{\substack{\text{famously}\\ \text{not having an}\\ \text{elementary antiderivative}}}\,dy$

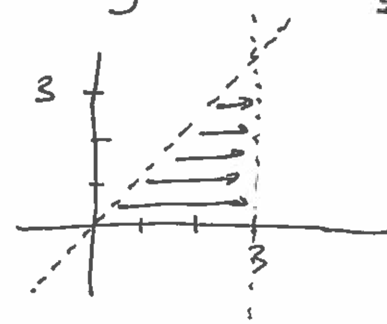
Need to understand R better....

y from 0 to 3...

x = y to x = 3



we can imagine reverse order



$$\int_{x=0}^{x=3} \int_{y=0}^{y=x} e^{-x^2}\,dy\,dx = \int_{x=0}^{x=3} \left[e^{-x^2}\cdot y\right]_{y=0}^{y=x} dx$$

$$= \int_0^3 e^{-x^2}\cdot x \; dx \qquad \begin{aligned} u &= -x^2 \\ du &= -2x\cdot dx \end{aligned} \qquad \begin{aligned} x=0 &\to u=0 \\ x=3 &\to u=-9 \end{aligned}$$

$$= \int_{u=0}^{u=-9} e^{u}\cdot \cancel{x}\left(\frac{du}{-2\cancel{x}}\right) = -2\int_{u=0}^{u=-9} e^{u}\,du$$

$$= -2\left[e^{u}\right]_0^{-9} = -2\left[e^{-9}-1\right] = 2\left[1-e^{-9}\right]$$