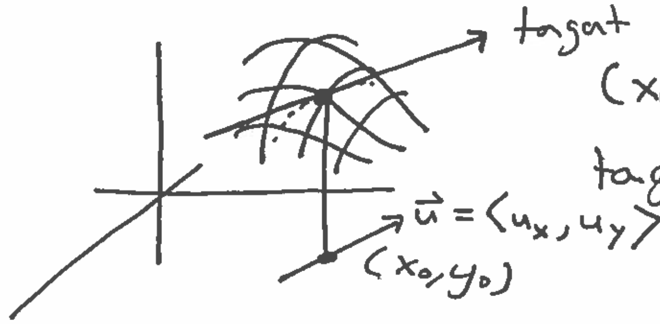
12.7 Tangent Lines, Normal Lines, Tangent Planes

$z = f(x,y)$



tangent line at $(x_0, y_0, f(x_0,y_0))$
tangent with direction $\vec{u}$.

$$\underset{\substack{\uparrow \\ f(x_0,y_0)}}{(x_0, y_0, z_0)} + t \underbrace{\langle a, b, c \rangle}_{\text{direction vector?}}$$ for a line in general.

$$\langle u_x, u_y, D_{\vec{u}} f(x_0,y_0) \rangle$$

<u>Ex</u> $f(x,y) = \sin(x) \cdot \cos(y)$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$, find <u>tangent line</u> to the surface using direction $\vec{u} = \langle 1, 0 \rangle$

∇f

direction vector $= \left\langle 1, 0, D_{\vec{u}} f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \right\rangle$

$$\nabla f = \langle \cos(x)\cos(y), -\sin(x)\sin(y) \rangle$$

$$\nabla f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \langle 0, -1 \rangle$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} \qquad D_{\vec{u}} f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \langle 0, -1 \rangle \cdot \langle 1, 0 \rangle = 0$$

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right) + t \langle 1, 0, 0 \rangle$$

Now in direction $\vec{u} = \langle 0, 1 \rangle$

$\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ ... direction vector $= \langle 0, 1, D_{\vec{u}} f(\frac{\pi}{2}, \frac{\pi}{2}) \rangle$

$\nabla f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \cdot \langle 0, 1 \rangle$

$\langle 0, -1 \rangle \cdot \langle 0, 1 \rangle = -1$

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right) + t\langle 0, 1, -1 \rangle$$

Now in direction $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ direction vector $\langle 1/\sqrt{2}, 1/\sqrt{2}, D_{\vec{u}} f(\frac{\pi}{2}, \frac{\pi}{2}) \rangle$

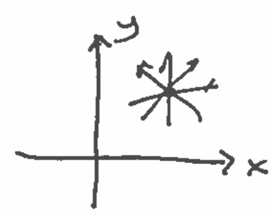
$\nabla f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\langle 0, -1 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{-1}{\sqrt{2}}$

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right) + t\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$$

<u>Ex</u> $f(x,y) = 4xy - x^4 - y^4$ at $(1,1)$. So $(1,1,2)$.

Pass through $(1,1)$ with $\vec{u} = \langle \cos\theta, \sin\theta \rangle$



direction vector $\langle \cos\theta, \sin\theta, D_{\vec{u}} f \rangle$

$\nabla f = \langle 4y - 4x^3, 4x - 4y^3 \rangle$

$\nabla f(1,1) = \langle 0, 0 \rangle \implies D_{\vec{u}} f(1,1) = 0$

line: $(1,1,2) + t\langle \cos\theta, \sin\theta, 0\rangle$

$$\left(1 + t\cdot\cos\underset{u}{\cancel{\theta}},\ 1 + t\sin\underset{u}{\cancel{\theta}},\ 2\right)$$

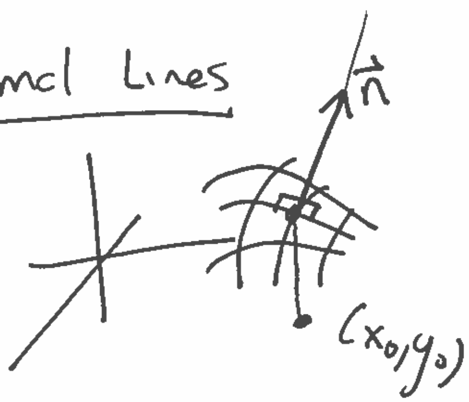
At $(0,1)$ ... so at $(0,1,-1)$. $\vec{u}$

$$\begin{aligned} D_{\vec{u}}f &= \nabla f(0,1)\cdot\langle\cos\theta, \sin\theta\rangle \\ &= \langle 4,-4\rangle\cdot\langle\cos\theta, \sin\theta\rangle \\ &= 4\cos\theta - 4\sin\theta \end{aligned}$$

$$(0,1,-1) + t\langle\cos\theta, \sin\theta, 4\cos\theta - 4\sin\theta\rangle$$

$$\left(t\cos\theta,\ 1 + t\sin\theta,\ -1 + t(4\cos\theta - 4\sin\theta)\right)$$

Normal Lines



Point

$$(x_0, y_0, f(x_0,y_0)) + t\cdot\langle \text{direction vector}\rangle$$

tangent lines had direction vector

$$\langle u_x, u_y, \underbrace{u_x\cdot f_x + u_y\cdot f_y}_{D_{\vec{u}}f}\rangle$$

Well $\langle -f_x, -f_y, 1\rangle$ is always $\perp$ to tangent lines.

$\underline{Ex}$ $f(x,y) = \sin(x) \cdot \cos(y)$

Find normal lie at $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$.

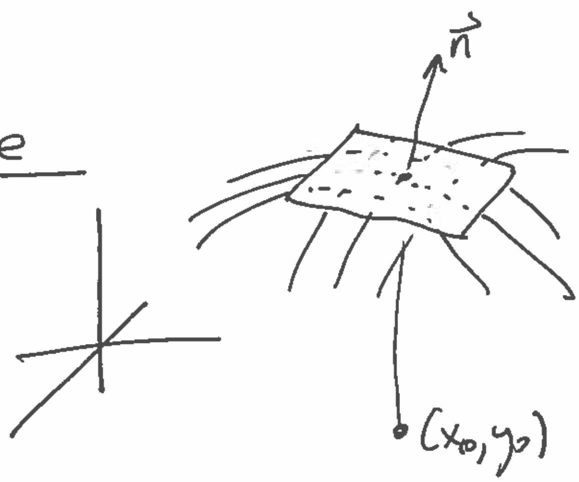
$$(\tfrac{\pi}{2}, \tfrac{\pi}{2}, 0) + t \langle -f_x, -f_y, 1 \rangle$$

$$f_x = \cos(x)\cos(y) \qquad @\,(\tfrac{\pi}{2}, \tfrac{\pi}{2}) \qquad f_x = 0$$

$$f_y = -\sin(x) \cdot \sin(y) \qquad\qquad f_y = -1$$

$\underline{\text{normal lie}}$ $$(\tfrac{\pi}{2}, \tfrac{\pi}{2}, 0) + t\langle 0, 1, 1 \rangle$$

$\underline{\text{Tagent Plae}}$



tagent plane's equation

Need $\vec{n} = \langle -f_x, -f_y, 1 \rangle$

and point...

$\underline{Ex}$ $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$, $\vec{n} = \langle 0, 1, 1 \rangle$

$\longrightarrow$ Plane $$0(x - \tfrac{\pi}{2}) + 1(y - \tfrac{\pi}{2}) + 1 \cdot (z - 0) = 0$$

$$y - \tfrac{\pi}{2} + z = 0$$

We just examined graphs of $z = \underbrace{f(x,y)}_{\text{a two-var function}}$

& identified

tangent lines,
normal line,
tangent plane
at one point.

a 3D surface

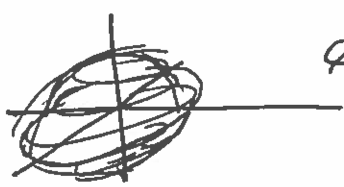
Now consider $w = f(x,y,z)$

f's level surfaces are 3D surfaces..

<u>Ex</u> $w = \underbrace{\frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4}}_{f(x,y,z)}$

consider level surface at level 1.

$$\frac{x^2}{12} + \frac{y^2}{6} + \frac{z^2}{4} = 1$$



ellipsoid.

Not a graph of f.

It's a picture of all (x,y,z) that you could feed f and get 1 for output.

$(1,2,1)$ is on this.

Tangent Plane here?

At any point, ∇f is orthogonal to the level surface there.

∇f is the normal vector!

$$\nabla f = \left\langle \frac{x}{6}, \frac{y}{3}, \frac{z}{2} \right\rangle$$

$$\nabla f(1,2,1) = \left\langle \frac{1}{6}, \frac{2}{3}, \frac{1}{2} \right\rangle$$

At $(1,2,1)$, this <u>is</u> $\perp$ to the surface...

So tangent plane: $\frac{1}{6}(x-1) + \frac{2}{3}(y-2) + \frac{1}{2}(z-1) = 0$

12.8 Extreme Values

Given $f(x,y)$, what (x_0, y_0) makes output as large or as small as possible?

<u>Ex</u> $f(x,y) = x^2 + y^2 - xy - x - 2$

Find all <u>critical points</u>: $\nabla f = \vec{0}$ $\left(\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\right)$

or

∇f dne $\left(\frac{\partial f}{\partial x} \text{ dne or } \frac{\partial f}{\partial y} \text{ dne}\right)$

$$\nabla f = \langle 2x - y - 1, 2y - x \rangle$$

want $\nabla f = \vec{0}$

$$\begin{cases} 2x - y - 1 = 0 \\ \cancel{2y - x = 0} \\ 4y - 2x = 0 \end{cases} \longrightarrow \begin{cases} 2x - y - 1 = 0 \\ \text{add eq: } 3y - 1 = 0 \end{cases}$$

$$\Downarrow$$

$$y = \frac{1}{3}$$

$$2x - \frac{1}{3} - 1 = 0 \Longleftarrow$$

$$2x = \frac{4}{3}$$

$$x = 2/3$$

$\Longleftarrow$ only one c.p at $\left(\frac{2}{3}, \frac{1}{3}\right)$.

At $\left(\frac{2}{3}, \frac{1}{3}\right)$:

$\nabla f = \vec{0}$? ? ?

Look at f's formula

$$f(x,y) = x^2 + y^2 - xy - x - 2$$

$$f_x = 2x - y - 1 \qquad f_y = 2y - x$$

$$f_{xx} = 2 \qquad f_{xy} = -1 \qquad f_{yy} = 2$$

$f_{xx} = 2 \Downarrow$ move in x-direction concave up

$f_{yy} = 2 \Downarrow$ move in y-direction concave up

Rule out $\left(\frac{2}{3}, \frac{1}{3}\right)$ being a local max...

still possible $\left(\frac{2}{3}, \frac{1}{3}\right)$ is a local min or a saddle point.

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \quad \text{Hessian...}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

determinant

$$2 \cdot 2 - (-1)(-1)$$
$$= 4 - 1 = 3$$

2nd derivative test

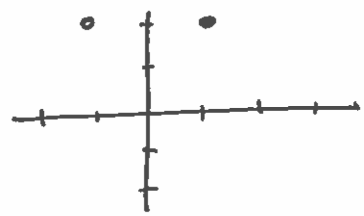
- positive? $\Rightarrow$ either have a ~~local max~~ or a local min. $\Rightarrow$ $\left(\frac{2}{3}, \frac{1}{3}\right)$ is a local min for f.
- negative? $\Rightarrow$ saddle point
- 0 $\Rightarrow$ inconclusive

Ex $f(x,y) = x^3 - 3x - y^2 + 4y$ Find local extrema & classify them.

$$\nabla f(x,y) = \langle 3x^2-3, -2y+4 \rangle$$

$$\nabla f = \vec{0} \Longrightarrow \begin{cases} 3x^2 - 3 = 0 & \Longrightarrow x=-1 \text{ or } x=1 \\ -2y+4=0 & \Longrightarrow y = 2 \end{cases}$$

two c.p.



$(-1,2)$
&
$(1,2)$

~~At $(-1,2)$~~

$$f_x = 3x^2-3 \qquad\qquad f_y = -2y+4$$

$$f_{xx} = 6x \qquad f_{xy} = 0 \qquad f_{yy} = -2$$

At $(-1,2)$

$$\begin{bmatrix} -6 & 0 \\ 0 & -2 \end{bmatrix}$$

12 pos

$\Longrightarrow$ either a local max or a local min

concave up!

our second derivs are neg.

At $(1,2)$

$$\begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}$$

-12

$\Longrightarrow$ saddle point

Ex $f(x,y) = x^2y + y^2 + xy$

Identify local extrema.

$\nabla f = \langle 2xy + y, x^2 + 2y + x \rangle$

$\nabla f = \vec{0} \implies \begin{cases} 2xy + y = 0 \\ x^2 + 2y + x = 0 \end{cases}$

$\implies \begin{cases} y(2x+1) = 0 \\ x^2 + 2y + x = 0 \end{cases} \implies y = 0 \quad \text{or} \quad x = -\frac{1}{2}$

$y = 0$:

$x^2 + x = 0$

$x(x+1) = 0$

$x = 0$ or -1

$(0,0) \qquad (-1,0)$

$x = -\frac{1}{2}$:

$\frac{1}{4} + 2y - \frac{1}{2} = 0$

$2y = \frac{1}{4}$

$y = \frac{1}{8}$

$(-\frac{1}{2}, \frac{1}{8})$

$f_{xx} = 2y \qquad f_{xy} = 2x + 1 \qquad f_{yy} = 2$

At $(0,0)$,

$H = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

$\det = -1$

saddle point

At $(-1,0)$

$H = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$

$\det = -1$

saddle pt

At $(-\frac{1}{2}, \frac{1}{8})$

$H = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 2 \end{bmatrix}$

$\det = \frac{1}{2} > 0$

local max or local min

since both 2nd derivs are positive.

$\underline{Ex}$ $f(x,y) = -\sqrt{x^2+y^2} + 2$

Find & classify critical pts.

$$\nabla f = \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

∇f is undefined ? at $(0,0)$.

$\nabla f = \vec{0} \implies x=0$ & $y=0$

$\implies \nabla f$ is undefined!

Only one c.p: $(0,0)$.

∇f is not defined here...

So 2nd deriv test can't be used

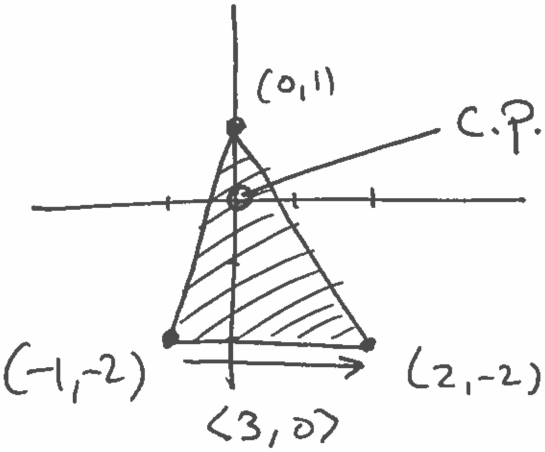
(only for c.p where $\nabla f = \vec{0}$)

$f(0,0) = 2$. $f(x,y) = 2$ minus a non-neg number.

$f(x,y) \leq 2$

So $(0,0)$ is a ~~local~~ absolute maximum

Ex $f(x,y) = x^2 - y^2 + 5$ on triangle



To find absolute extrema,

find C.P.

$\nabla f = \langle 2x, -2y \rangle$

$\nabla f = \vec{0} \Longrightarrow (0,0)$

$f_{xx} = 2 \quad f_{xy} = 0 \quad f_{yy} = -2$

$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \longrightarrow -4$

$\Longrightarrow$ saddle pt.

examine the border of domain.

Fact: closed & bounded domain

$\Downarrow$

f has an abs max & abs min.

bottom leg: $(-1,-2) + t\langle 3,0 \rangle$

$t \in [0,1]$

$(-1+3t, -2)$

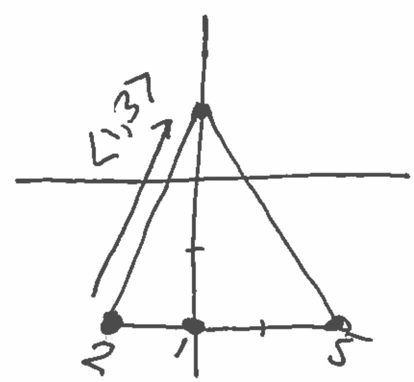
$$f = x^2 - y^2 + 5$$
$$= (-1+3t)^2 - (-2)^2 + 5$$
$$= 9t^2 - 6t + 2 \quad \text{on } [0,1]$$

$f' = 18t - 6 = 0$

$\Longrightarrow t = 1/3$

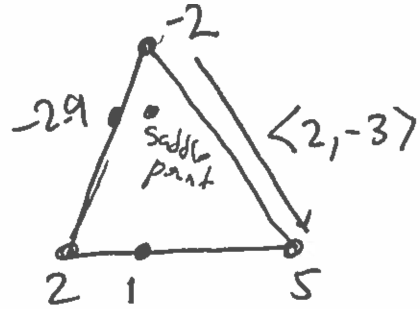
$0, \frac{1}{3}, 1$

outputs $2 \quad 1 \quad 5$



left leg: $(-1,-2)+t\langle 1,3\rangle$

$(-1+t,-2+3t)$



$$f = x^2 - y^2 + 5$$
$$= (-1+t)^2 - (-2+3t)^2 + 5$$
$$= 10t^2 - 14t + 2 \quad \text{on } [0,1]$$

$$f' = 20t - 14 = 0$$
$$\Longrightarrow t = \frac{14}{20} = 0.7$$

0, 0.7, 1

output 2, -2.9, -2

right leg: $(0,1)+t\langle 2,-3\rangle$

$(2t,\ 1-3t)$

$$f = x^2 - y^2 + 5$$
$$= (2t)^2 - (1-3t)^2 + 5$$
$$= 13t^2 + 6t + 4 \quad \text{on } [0,1]$$
$$f' = 26t + 6 = 0$$
$$t = \frac{-6}{26}$$

0, 1

outputs -2, 5

So absolute max at $(2,-2)$ with output 5

absolute min at $t=0.7$ with output -2.9.

$(-0.3, 0.1)$