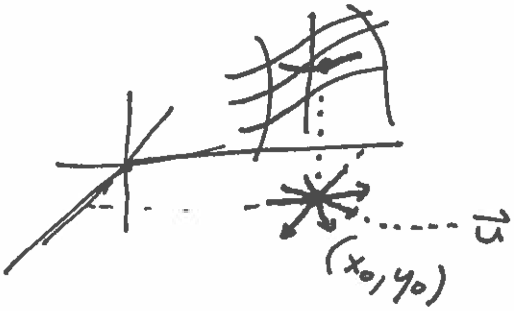
12.6 Directional Derivatives



$z = f(x,y)$

Pass through (x_0, y_0) with direction $\vec{u}$

unit length vector

At what rate would z be changing?

There's a tangent line up at $(x_0, y_0, f(x_0,y_0))$; and its slope is what I'm interested in.

$$\underbrace{\text{slope}} \quad \frac{\Delta z}{\Delta\sqrt{x^2+y^2}}$$

Definition $\quad D_{\vec{u}} f(x,y) \quad$ the directional derivative of f in direction $\vec{u}$.

Formula: $$D_{\vec{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f((x_0,y_0) + h\cdot\vec{u}) - f(x_0,y_0)}{h}$$

EX $z = 14 - x^2 - y^2$
at $(1,2)$
along $\cancel{\langle 2,-1\rangle}$ $\dfrac{\langle 2,-1\rangle}{\sqrt{5}}$

Find $D_{\vec{u}} f$

$$= \lim_{h\to 0} \frac{f\left((1,2) + h\frac{\langle 2,-1\rangle}{\sqrt{5}}\right) - f(1,2)}{h}$$

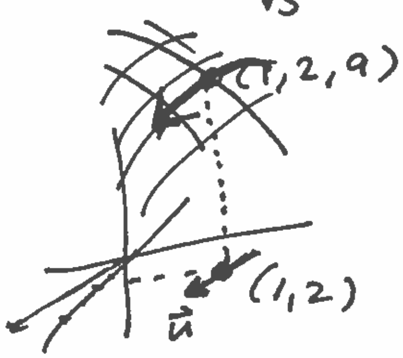
$$= \lim_{h\to 0} \frac{f\left(1+\frac{2}{\sqrt{5}}h,\ 2-\frac{1}{\sqrt{5}}h\right) - f(1,2)}{h}$$

$$= \lim_{h\to 0} \frac{14-\left(1+\frac{2}{\sqrt{5}}h\right)^2-\left(2-\frac{1}{\sqrt{5}}h\right)^2 - 9}{h}$$

$$= \lim_{h\to 0} \frac{14 - 1 - \frac{4}{\sqrt{5}}h - \frac{4}{5}h^2 - 4 + \frac{4}{\sqrt{5}}h - \frac{1}{5}h^2 - 9}{h}$$

$$= \lim_{h\to 0} \left(-\frac{4}{\sqrt{5}} - \frac{4}{5}h + \frac{4}{\sqrt{5}} - \frac{1}{5}h\right) = 0$$

$$D_{\frac{\langle 2,-1\rangle}{\sqrt{5}}} f(1,2) = 0$$



Let t run $[-1,1]$

$$\left(x(t),\ y(t),\ z(t)\right)$$

$$\left(1+\frac{2}{\sqrt{5}}t,\ 2-\frac{1}{\sqrt{5}}t,\ 0\right)$$

$$\left(1+\frac{2}{\sqrt{5}}t,\ 2-\frac{1}{\sqrt{5}}t,\ 14-\left(1+\frac{2}{\sqrt{5}}t\right)^2-\left(2-\frac{1}{\sqrt{5}}t\right)^2\right)$$

Given $f(x,y)$, define the gradient of f

$$\underbrace{\nabla f}_{\text{put partial derivatives into a vector}} = \langle f_x, f_y \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

"nabla"

"harp"

put partial derivatives into a vector

$$\left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$$

or

$$\left\langle \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \right\rangle$$

Fact $D_{\vec{u}} f = \underbrace{\underbrace{\nabla f}_{\text{vector}} \cdot \underbrace{\vec{u}}_{\text{vector}}}_{\text{scalar}}$

Ex Calculate ∇f for $f(x,y) = 14 - x^2 - y^2$

$$\nabla f = \langle -2x, -2y \rangle$$

Ex $f(x,y) = e^{x^2 + y^2/4}$

So

$$\nabla f = \langle e^{x^2+y^2/4} \cdot 2x, e^{x^2+y^2/4} \cdot y/2 \rangle$$

$$= e^{x^2+y^2/4} \langle 2x, y/2 \rangle$$

Ex Use ∇f to calculate $D_{\frac{\langle 2,-1\rangle}{\sqrt{5}}}(14-x^2-y^2)$ at $(1,2)$

(Long way, with limit ... we got 0)

$$\nabla f = \langle -2x, -2y\rangle$$

at $(1,2)$

$$\nabla f(1,2) = \langle -2,-4\rangle. \qquad \vec{u} = \frac{\langle 2,-1\rangle}{\sqrt{5}}$$

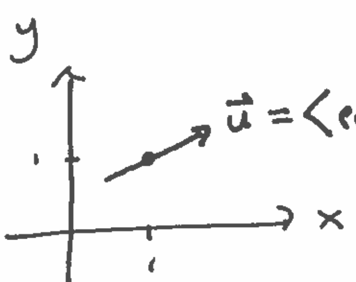
$$D_{\vec{u}} f = \langle -2,-4\rangle \cdot \frac{\langle 2,-1\rangle}{\sqrt{5}} = 0$$

Ex A hill has height $z = e^{x^2+y^2/4}$.

You are at $(1,1)$. Start walking 30° North of East.

$(1,1,\text{mu})$

~~What will~~ be your rate of change in altitude compared to your x-y speed.



$\vec{u} = \langle \cos(30^\circ), \sin(30^\circ)\rangle$

$$= \langle \sqrt{3}/2, 1/2\rangle$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$$\nabla f = e^{x^2+y^2/4}\langle 2x, y/2\rangle$$

$$\nabla f(1,1) = e^{5/4}\langle 2, 1/2\rangle$$

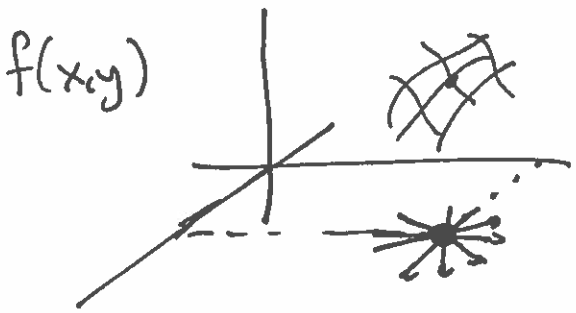
$$= e^{5/4}\langle 2, 1/2\rangle \cdot \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\rangle$$

$$= e^{5/4}\left(\sqrt{3} + \tfrac{1}{4}\right)$$

(Note: positive $\Rightarrow$ uphill)

<u>Observations</u> What direction would you travel in to make your ascent steepest?

$f(x,y)$



Maximize

$D_{\vec{u}} f$ with respect to $\vec{u}$.

$$\underbrace{\nabla f}_{\text{no control over this}} \cdot \underbrace{\vec{u}}_{\text{choose a } \vec{u} \text{ that makes the result as large as possible.}} = \|\nabla f\| \cdot \overbrace{\|\vec{u}\|}^{1} \cdot \cos\theta$$

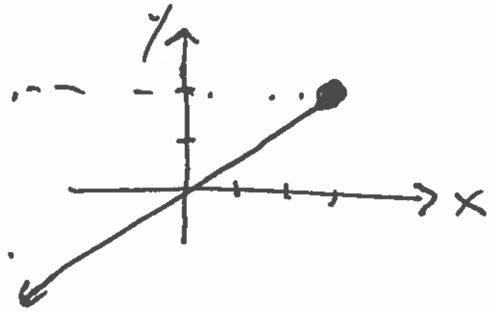
$\cos\theta$ maximum when $\theta = 0$.

So head in same direction as ∇f!

$\vec{u} = \dfrac{\nabla f}{\|\nabla f\|}$ is steepest direction to head in.

<u>Ex</u> $f(x,y) = 14 - x^2 - y^2 \implies \nabla f(x,y) = \langle -2x, -2y \rangle$

If you stand at $(3,2)$, $\nabla f(3,2) = \underbrace{\langle -6, -4 \rangle}_{\text{this is the direction to walk in to rise most quickly}}$



And likewise to drop in height most quickly head in $-\nabla f$.

What happens if you head in a direction where $D_{\vec{u}} f = 0$?

$$\nabla f \cdot \underbrace{\vec{u}}_{} = 0$$

directions that are $\perp$ to ∇f...

moving in these directions z stays constant.

With $z = f(x,y)$. So directions $\perp$ to ∇f define level curves.

Ex $f(x,y) = 14 - x^2 - y^2$

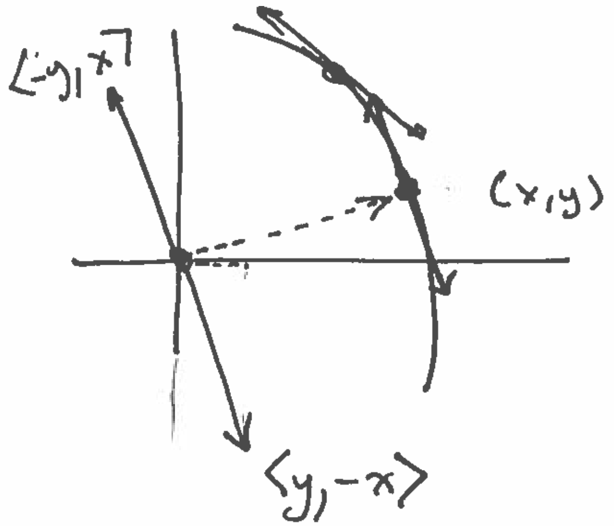
(We know level curves
$14 - x^2 - y^2 = c$
$x^2 + y^2 = 14 - c$
circles ...)

$$\Downarrow$$

$$\nabla f(x,y) = \langle -2x, -2y \rangle$$

Find vectors $\perp$ to this

$$\frac{\langle y, -x \rangle}{\sqrt{y^2 + x^2}} \quad \text{and} \quad \frac{\langle -y, x \rangle}{\sqrt{y^2 + x^2}}$$



Because (in this case)

∇f was parallel to (x,y),

Things $\perp$ to ∇f are $\perp$ to (x,y)

Directions to travel in that are $\perp$ to ∇f form a circle.

So far ∇f's <u>direction</u> gives us information

- travel in ∇f direction, and z increases fastest
- travel in $-\nabla f$ direction, z decreases fastest
- travel $\perp$ to ∇f, z doesn't changes

Magnitude of ∇f ?

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$$D_{\frac{\nabla f}{\|\nabla f\|}} f = \nabla f \cdot \frac{\nabla f}{\|\nabla f\|}$$

$$\left\| D_{\frac{\nabla f}{\|\nabla f\|}} f \right\| = \underbrace{\|\nabla f\|}\cdot 1$$

magnitude $\|\nabla f\|$ tells you <u>how</u> fast you'd increase if you headed straight up hill.

Ex $f(x,y) = 14 - x^2 - y^2$ at $(3,2)$...

$\nabla f(3,2) = \langle -6, -4 \rangle$

Head in direction $\langle -6,-4\rangle$, my ascent will be steepest possible, and it will be $\sqrt{36+16} = \sqrt{52}$ units of z / unit of xy-plane

Ex $f(x,y) = \sin(x)\cos(y)$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

what's happening here?

$\left(\frac{\pi}{3}, \frac{\pi}{3}, f\left(\frac{\pi}{3}, \frac{\pi}{3}\right)\right)$

$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\sqrt{3}}{4}\right)$

$\nabla f(x,y) = \langle \cos(x)\cos(y), \sin(x)(-\sin y) \rangle$

$\nabla f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) =$

$= \left\langle \frac{1}{4}, -\frac{3}{4} \right\rangle$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\sin\left(\frac{\pi}{3}\right) = \sqrt{3}/2$

From $(\pi/3, \pi/3)$, heading in $\left\langle \frac{1}{4}, \frac{-3}{4} \right\rangle$ direction ascends hill most quickly. (specifically, at rate $\sqrt{\frac{1}{16} + \frac{9}{16}} = \frac{\sqrt{10}}{4}$.)

Ex $f(x,y) = -x^2 + 2x - y^2 + 2y + 1$

At $(1,1)$, what is going on?

$(1,1,3)$

$\nabla f = \langle -2x + 2, -2y + 2 \rangle$

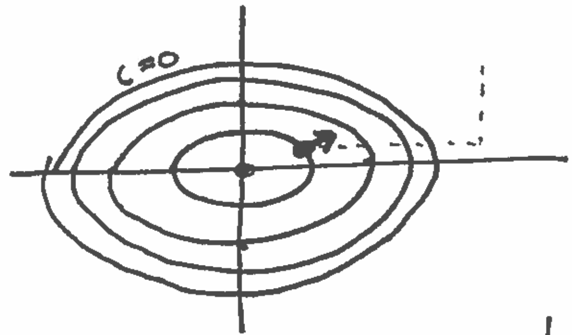
$\nabla f(1,1) = \langle 0, 0 \rangle$

∇f tells us what direction to travel in to "climb fastest".

when $\nabla f(1,1) = \langle 0,0 \rangle$ it means something special is happening at $(1,1)$.

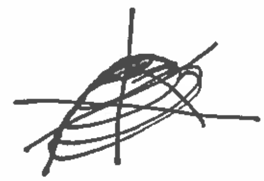
Ex Hill described by $z = 20 - x^2 - 2y^2$

Visualize with level curves



$20 - x^2 - 2y^2 = c$

$x^2 + 2y^2 = 20 - c$



At $(1, \frac{1}{4})$ . . .

$(1, \frac{1}{4}, 18\frac{7}{8})$

~~Drop~~ Place a golf ball . . . gravity takes over . . .

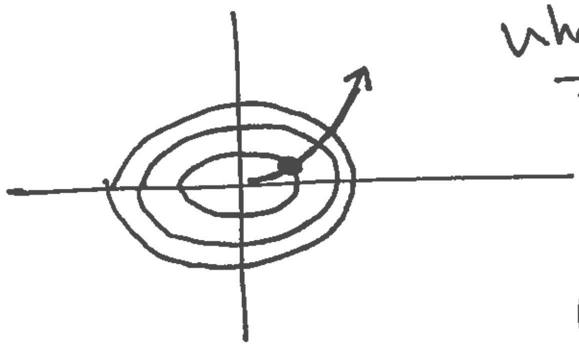
where does it go?

$-\nabla f$ is the steepest downward direction . . . so gravity pulls that way.

At $(1, \frac{1}{4})$, $\nabla f(x,y) = \langle -2x, -4y \rangle$

$\nabla f(1, \frac{1}{4}) = \langle -2, -1 \rangle$

So $-\nabla f(1, \frac{1}{4}) = \langle 2, 1 \rangle$ . . . ball start heading in this direction.



what path will the ball take?

t as a parameter

$(x(t), y(t))$ is my notation for the path.

$\underbrace{\text{direction of travel}}$ should be parallel to $\underbrace{-\nabla f}$

$$\langle x'(t), y'(t)\rangle = c \cdot \langle 2x, 4y\rangle$$

$$\begin{cases} x'(t) = c \cdot 2x(t) \\ y'(t) = c \cdot 4y(t) \end{cases}$$

System of two differential equations.

eliminate from equations

$$c = \frac{x'(t)}{2x(t)} \qquad c = \frac{y'(t)}{4y(t)}$$

$$\Rightarrow \quad \frac{x'(t)}{2x(t)} = \frac{y'(t)}{4y(t)}$$

$$\int \frac{x'(t)}{2x(t)}\,dt = \int \frac{y'(t)}{4y(t)}\,dt$$

$$\frac{1}{2}\ln|x(t)| = \frac{1}{4}\ln|y(t)| + C$$

Started at $x=1$, $y=\frac{1}{4}$

$$\frac{1}{2}\ln|1| = \frac{1}{4}\ln\left|\frac{1}{4}\right| + C$$

$$0 = \frac{1}{4}\ln\left(\frac{1}{4}\right) + C$$

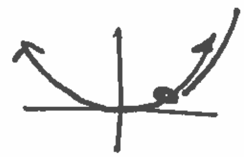
$$C = -\frac{1}{4}\ln\left(\frac{1}{4}\right)$$

$$\frac{1}{2}\ln|x(t)| = \frac{1}{4}\ln|y(t)| - \frac{1}{4}\ln\left(\frac{1}{4}\right)$$

$$2\cdot\ln|x(t)| = \ln|y(t)| - \ln\left(\frac{1}{4}\right)$$

$$\ln\left((x(t))^2\right) = \ln|4\cdot y(t)|$$

$$\Rightarrow \quad x^2 = 4y \rightsquigarrow y = \frac{1}{4}x^2$$



So the ball's path is $\left(x, \frac{1}{4}x^2, 20 - x^2 - 2\left(\frac{1}{4}x^2\right)^2\right)$.

Three Variables?

Same basic ideas.

$w = f(x,y,z)$

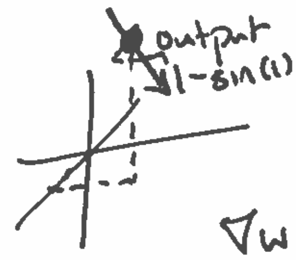
3-input vars
1-output var.

could make level curves.

<u>Ex</u> $w = x^2y - z\sin(x)$

At $(1,1,2)$.

$w(1,1,2) = 1 - 2\sin(1)$



What direction to head to make w increase quickly?

$\nabla w(1,1,2)$
$= \langle 2-2\cos 1, 1, -\sin(1)\rangle$

$$\nabla w = \langle 2xy - z\cos(x), x^2, -\sin(x)\rangle$$

12.6 Last thing for Exam 3.

Exam 1 Ch 9

Exam 2 Ch 10

Exam 3 Ch 11 12.1, 12.2, 12.3, 12.5, 12.6

Formulas to Memorize

$$K = \|\vec{T}'(s)\|$$

Formula don't have to ... will be provided

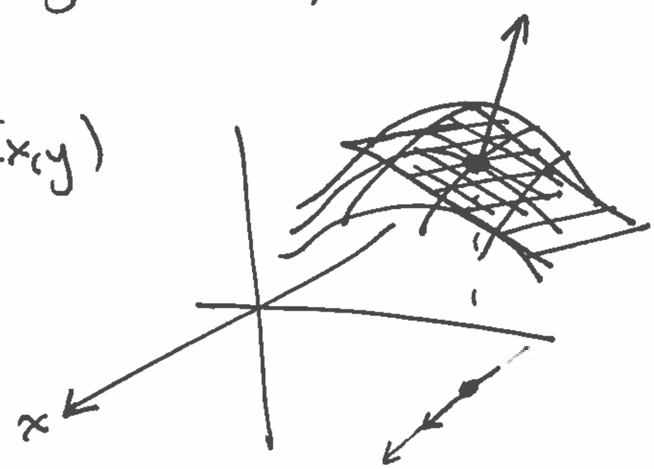
Most curvature formulas ...

Same Format: calculator & no-calculator

Make up one question

12.7 Tangent Lines, Normal Lines, & Tangent Planes.

$z = f(x,y)$



$\underline{Ex}$ $\quad z = \sin(x)\cdot\cos(y)$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$

Find the tangent line in the x-direction.

l_x will have direction vector

like $\langle 1, 0, f_x \rangle$

$\langle 1, 0, \cos(x)\cos(y) \rangle$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\quad \langle 1, 0, 0 \rangle$

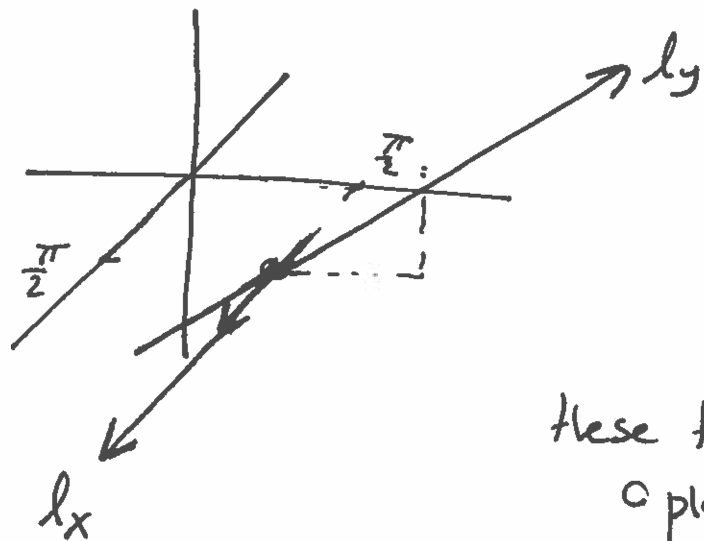
$l_x: \quad \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right) + t\cdot\langle 1, 0, 0 \rangle$

~~direction~~ direction : $\langle 0, 1, f_y \rangle$

$\langle 0, 1, \sin(x)(-\sin(y)) \rangle$

l_y : $\langle 0, 1, -1 \rangle$

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right) + t\langle 0, 1, -1 \rangle$$



these two lines make a plane.

$z = \sin(x)\cos(y)$