12.3 continued

Function of 2 or more variables

$$f(x_1, x_2, x_3, \ldots)$$

$$f(x, y, z)$$

$$f(x, y)$$

view y as constant, apply $\frac{\partial}{\partial x}$

$f_x(x,y)$ aka $\frac{\partial f}{\partial x}(x,y)$	$f_y(x,y)$ aka $\frac{\partial f}{\partial y}(x,y)$	Partial Derivatives

$f_{xx}(x,y)$ aka $\frac{\partial^2 f}{\partial x^2}(x,y)$	$f_{xy}(x,y)$ aka $\frac{\partial^2 f}{\partial y\, \partial x}$	$f_{yx}(x,y)$ aka $\frac{\partial^2 f}{\partial x\, \partial y}$	$f_{yy}(x,y)$ aka $\frac{\partial^2 f}{\partial y^2}(x,y)$	Second Partial Derivatives

If these are continuous, these will be equal.

Ex

Find 2nd derivatives

$$f(x,y) = x^3y^2 + 2xy^3 + \cos(x)$$

$$f_x(x,y) = 3x^2y^2 + 2y^3 - \sin(x)$$

$$f_y(x,y) = 2x^3y + 6xy^2$$

$$f_{xx} = 6xy^2 - \cos(x)$$

$$f_{xy} = 6x^2y + 6y^2$$

$$f_{yx} = 6x^2y + 6y^2$$

$$f_{yy} = 2x^3 + 12xy$$

At $(2,1)$.

$$f(2,1) = 2^3 \cdot 1^2 + 2 \cdot 2 \cdot 1^3 + \cos(2)$$
$$= 12 + \cos(2)$$

$$f_x(2,1) = 14 - \sin(2)$$

$$f_y(2,1) = 28$$

$$f_{xx}(2,1) = 12 - \cos(2)$$

$$f_{xy}(2,1) = f_{yx}(2,1) = 30$$

$$f_{yy}(2,1) = 40$$

Ex $f(x,y) = \frac{x^3}{y^2} = x^3 \cdot y^{-2}$

$f_x(x,y) = \frac{3x^2}{y^2}$

$f_y(x,y) = -2x^3 y^{-3} = \frac{-2x^3}{y^3}$

$f_{xx}(x,y) = \frac{6x}{y^2}$

$f_{xy} = \frac{-6x^2}{y^3}$

$f_{yy} = \frac{6x^3}{y^4}$

At $(1,1)$.

$f(1,1) = 1$

$f_x(1,1) = 3$

$f_y(1,1) = -2$

$f_{xx}(1,1) = 6$

$f_{xy}(1,1) = -6$

$f_{yy}(1,1) = 6$

$\underline{\text{Ex}}$ $f(x,y) = e^x \cdot \sin(x^2y)$

$f_x = e^x \cdot \sin(x^2y) + \underbrace{\underline{e^x} \cdot \underline{\cos(x^2y)}} \cdot \underbrace{\underline{2xy}}$

$f_y = e^x \cdot \cos(x^2y) \cdot x^2$

$$f_{xx} = e^x \cdot \sin(x^2y) + e^x \cdot \cos(x^2y) \cdot 2xy$$

$$+ \underset{\substack{\uparrow \\ \partial/\partial x}}{e^x} \cdot \cos(x^2y) \cdot 2xy + e^x \cdot \underbrace{\left(-\sin(x^2y) \cdot 2xy\right)}_{\partial/\partial x} \cdot 2xy$$

$$+ e^x \cdot \cos(x^2y) \cdot 2y$$

$$f_{xy} = e^x \cdot \cos(x^2y) \cdot x^2 + e^x\left(-\sin(x^2y) \cdot x^2\right) \cdot 2xy + e^x \cos(x^2y) \cdot 2x$$

$$f_{yy} = e^x \cdot x^2 \left(-\sin(x^2y)\right) \cdot x^2$$

Skip 12.4 $\longrightarrow$ 12.5 <u>Multivariate Chain Rule</u>

Recall one-variable chain rule:

if z depends on y, and y depends on x, ultimately z depends on x.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Now consider $z = f(x,y)$, where $x = g(t)$ and $y = h(t)$ . . .

$\Longrightarrow$ z ultimately depends on t...

so we can talk about $\frac{dz}{dt}$.

$z = f(x,y)$

$\langle x(t), y(t) \rangle$

Multivariate Chain Rule (Part I)

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$

well, what can I take derivative of f with respect to?

fill in to get dt's on bottom

<u>Ex</u> $f(x,y) = xy^2 + x^3y$ Find $\frac{df}{dt}$.

with

$x(t) = \frac{1}{t}$

$y(t) = e^t$

$$\frac{df}{dt} = (y^2 + 3x^2y) \cdot \left(\frac{-1}{t^2}\right) + (2xy + x^3) \cdot e^t$$

$$\frac{df}{dt} = (y^2+3x^2y)\left(\frac{-1}{t^2}\right) + (2xy + x^3)(e^t)$$

Leave alone.

$t = t_0$

$x = x(t_0)$

$y = y(t_0)$

Sub in x & y's formulas

$$\frac{df}{dt} = \left(e^{2t} + \frac{3}{t^2}e^t\right)\left(\frac{-1}{t^2}\right) + \left(\frac{2}{t}e^t + \frac{1}{t^3}\right)e^t$$

(only has "t")

<u>Ex</u> $f(x,y) = x^2y + x$

$x = \sin(t)$

$y = e^{5t}$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= (2xy+1)\cdot\cos(t) + x^2\cdot 5e^{5t}$$

OR

$$\frac{df}{dt} = \left(2\sin(t)e^{5t}+1\right)\cdot\cos(t) + \sin^2(t)\cdot 5e^{5t}$$

$\underline{\text{Ex}}$ $\frac{d}{dx}\left(x^x\right)$

251 way

$$\frac{d}{dx}\left(e^{x\cdot\ln(x)}\right)$$

use Chain Rule

& Product Rule

$$\Delta(x) = (x, x)$$

diagonal

$$f(x,y) = x^y$$

$x(x) = x$ $y(x) = x$

$$\tilde{f}(x) = (f\circ\Delta)(x) = f(\Delta(x)) = f(x,x) = x^x$$

$$\frac{d\tilde{f}}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$= y\cdot x^{y-1}\cdot 1 + x^y\cdot\ln(x)\cdot 1$$

$$= x\cdot x^{x-1} + x^x\cdot\ln(x)$$

$$= x^x + x^x\ln(x)$$

$$\frac{d}{dx}\left(x^x\right)$$

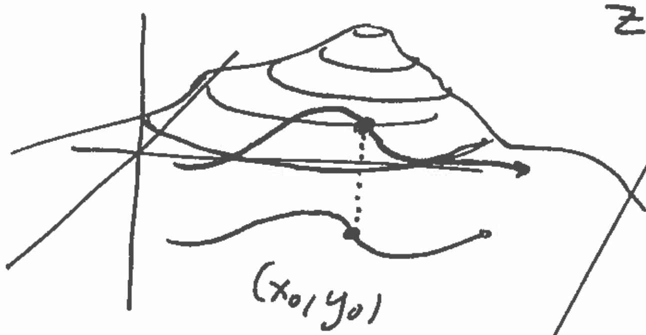
$$= x\cdot x^{x-1} + x^x\ln(x)$$

$\checkmark$

$\underline{\text{Ex}}$ $\frac{d}{dx}\left[\frac{x\cdot\ln(x)}{x^2+1}\right] = \frac{\ln(x)}{x^2+1} + \frac{x}{x^2+1}\cdot\frac{1}{x} + x\ln(x)\left(x^2+1\right)^{-2}\cdot 2x$

Not multivariate,
in our heads, treat
each "x" as a separate
variable

Ex



$z = f(x,y)$

$x = g(t)$

$y = h(t)$

real-world: no formulas ...

We can use measuring devices to measure:

$$\frac{\partial z}{\partial x} = 5 \qquad \frac{\partial z}{\partial y} = -2 \qquad \frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = 7$$

(at $t_0 \Rightarrow (x_0 y_0)$)

Martian Rover.

$Z = f(x,y)$ info from satellite survey.

Find $\frac{dz}{dt}$ at this time/location?

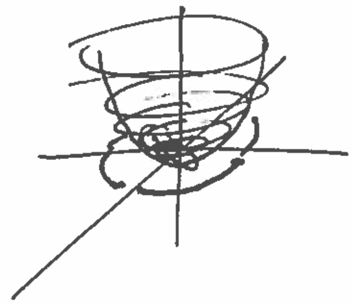
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= 5(3) + (-2)(7) = 1$$

$x = g(t)$
$y = h(t)$
are "seen" by satellite cameras.

tells us rover's elevation change per unit time.

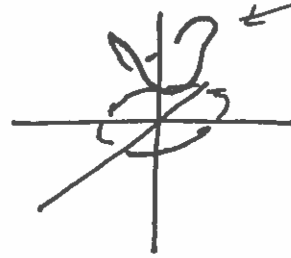
$\underline{\text{Ex}}$ $f(x,y) = x^2 + y^2 - xy$ is an elliptic paraboloid.



$x = \cos(t)$

$y = \sin(t)$

$(x, y, f(x,y))$



As we traverse this path, at what time(s) is the height maximized?

height → z, what time(s) → t

$$\Longrightarrow \frac{dz}{dt} \overset{\text{want}}{=} 0$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2x-y)(-\sin t) + (2y-x)\cos(t) = 0$$

$$0 = (2\cdot\cos t - \sin t)(-\sin t) + (2\sin t - \cos t)\cdot\cos t$$

$$0 = \cancel{-2\sin t\cos t} + \sin^2 t + \cancel{2\sin t\cos t} - \cos^2 t$$

$$\cos^2(t) = \sin^2(t) \Longrightarrow 1 = \tan^2(t)$$

$$\Longrightarrow \tan(t) = \pm 1$$

$$\Longrightarrow t = \underbrace{\frac{\pi}{4} \text{ or } \frac{5\pi}{4}}_{\tan(t)=1} \text{ or } \underbrace{\frac{-\pi}{4} \text{ or } \frac{3\pi}{4}}_{\tan(t)=-1}$$

Extremal Value Thm

$\Longrightarrow \quad z(t=\frac{\pi}{4}) = ? \qquad x=\frac{\sqrt{2}}{2},\ y=\frac{\sqrt{2}}{2}$

$$z = x^2+y^2-xy$$
$$= \frac{1}{2}+\frac{1}{2}-\frac{1}{2} = \frac{1}{2}$$

$z(t=\frac{5\pi}{4}) \qquad x=-\frac{\sqrt{2}}{2},\ y=-\frac{\sqrt{2}}{2}$

$$= \cdots = \frac{1}{2}$$

$z(t=-\pi/4) \qquad x=\frac{\sqrt{2}}{2},\ y=-\frac{\sqrt{2}}{2}$

$$= \cdots =$$
$$= \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$$

$z(t=\frac{3\pi}{4}) =$

$$= \cdots = 3/2$$

$\Longrightarrow$ Peak at $t=\frac{3\pi}{4}, \frac{-\pi}{4}$ (plus $2\pi n$)

Low points at $\frac{\pi}{4}, \frac{5\pi}{4}$.

Another situation:

$$z = f(x, y)$$

where $x(s,t)$
and $y(s,t)$

(Now x, y are <u>also</u> functions of more than 1 variable...)

$\cancel{\frac{dz}{ds}}$ $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ since z ultimately depends on s, t

<u>Ex</u> $z = x^2 y + x$

$x = s^2 + 3t$

$y = 2s - t$

Find partial derivatives of z with respect to s, t.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

Multivar. Chain Rule Part 2

$$= (2xy+1)(2s) + x^2 \cdot 2$$

$$= \left[2(s^2+3t)(2s-t)+1\right](2s) + (s^2+3t)^2 \cdot 2$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$= (2xy+1) \cdot 3 + (x^2)(-1)$$

$$= \left[2(s^2+3t)(2s-t)+1\right] \cdot 3 - (s^2+3t)^2$$

$\underline{\text{Ex}}$ $\quad w = xy + z^2$

New: w depends on 3 things!

$$x = t^2 e^s$$
$$y = t \cdot \cos(s)$$
$$z = s \cdot \sin(t)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$

$$= y \cdot t^2 e^s + x \cdot (-t\sin(s)) + 2z \cdot \sin(t)$$

optionally Sub in formulas use s,t.

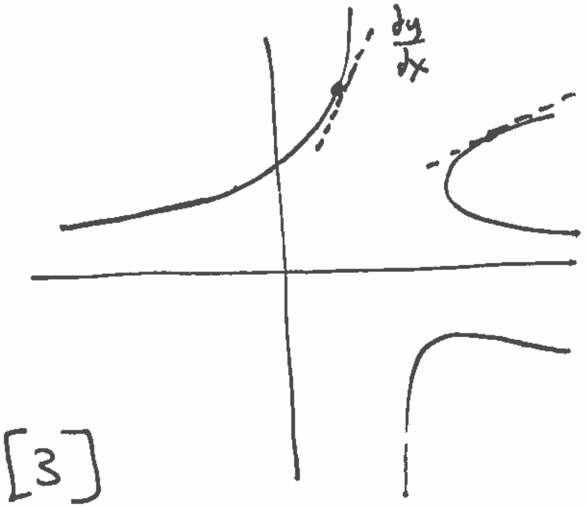
Generally, $f(x_1, x_2, \dots, x_n)$

$$x_1 = g_1(t_1, t_2, \dots, t_m)$$
$$x_2 = g_2(t_1, t_2, \dots, t_m)$$
$$\vdots$$
$$x_n = g_n(t_1, t_2, \dots, t_m)$$

$$\frac{\partial f}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j}{\partial t_i}$$

Implicit Differentiation

$$x^2y - xy^3 = 3$$



Find $\frac{dy}{dx}$

old method

$$\frac{d}{dx}\left[x^2y - xy^3\right] = \frac{d}{dx}[3]$$

$$2xy + x^2\cdot\frac{dy}{dx} - \left[y^3 + x\cdot 3y^2\cdot\frac{dy}{dx}\right] = 0$$

Here y depends on x.

Now solve for $\frac{dy}{dx} = \cdots$

New way: Create function $f(x,y) = x^2y - xy^3$

our curve is f's level curve at level 3.

Use $z = x^2y - xy^3$

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

If you are restricting motion to level curve at level 3, this is 0

$$0 = (2xy - y^3)\cdot 1 + (x^2 - 3xy^2)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-(2xy - y^3)}{x^2 - 3xy^2}$$

Some curve: $f(x,y) = \underline{\underline{\text{constant}}}$.

$$\frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)}$$

$\underline{Ex}$ $\sin(x^2y^2) + y^3 = x + y$ Find $\frac{dy}{dx}(0,0)$

$$\Rightarrow \underbrace{\sin(x^2y^2) + y^3 - x - y}_{f(x,y)} = 0$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\cos(x^2y^2)\cdot 2xy^2 - 1}{\cos(x^2y^2)\cdot 2x^2y + 3y^2 - 1}$$

$$\frac{\partial y}{\partial x}(0,0) = -\frac{-1}{-1} = -1$$