Ch 12. Up to now

one-variable $\quad$ multiple outputs

$t \longrightarrow \vec{r}(t) = (x(t), y(t))$ plane curve

$\theta \longrightarrow \vec{r}(t) = (x(t), y(t), z(t))$ Space curve

<u>Now</u> input multiple variables

$x, y \xrightarrow{f}$ a single output $f(x,y)$

$x, y, z \longrightarrow f(x,y,z)$

$x_1, x_2, \ldots \longrightarrow f(x_1, x_2, \ldots)$

Multivariate Functions

<u>Ex</u> $f(x,y) = x^2 - y$

Calculate $f(2,1) = 2^2 - 1 = 3$ $\qquad f(1,2) = 1^2 - 2 = -1$

Ex $f(x,y) = \dfrac{1}{x^2+y^2+1}$

Calculate $f(2,1) = \dfrac{1}{2^2+1^2+1} = \dfrac{1}{6}$

$f(1,2) = \text{- - - -} = \dfrac{1}{6}$

same, becaus in formula x,y are symmetric

Ex windchill temp (temp, windspeed)

Two-variable fuctions $\longrightarrow$ two variables in, one quantity cones out

three quantities

$\longrightarrow$ lends itself to plots.

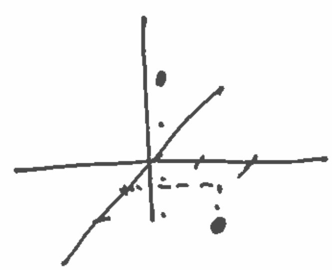
Ex $f(x,y) = x^2 - y$

$f(2,1) = 3$

$f(1,2) = -1$

View $z = f(x,y)$

$\uparrow$ output



plotted with Sage

Ex $\dfrac{1}{x^2+y^2+1}$ plot with sage

Level Curves

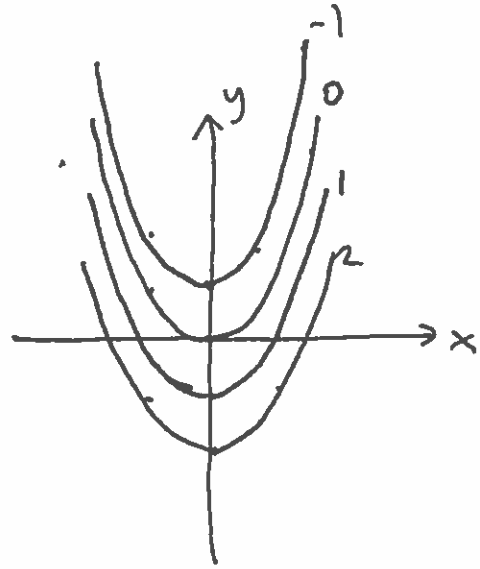
Ex $f(x,y) = x^2 - y$

Sometimes the output is 0

$0 = x^2 - y$

$\Updownarrow$

$y = x^2$



sometimes the output is 1

$1 = x^2 - y$

$y = x^2 - 1$

Level curve at level 2: $2 = x^2 - y \rightsquigarrow y = x^2 - 2$

Ex $f(x,y) = \dfrac{1}{x^2+y^2+1}$ Make a level curve plot

Level 0: $0 = \dfrac{1}{x^2+y^2+1}$ No solutions; no level 0 curve

Level 1: $1 = \dfrac{1}{x^2+y^2+1}$

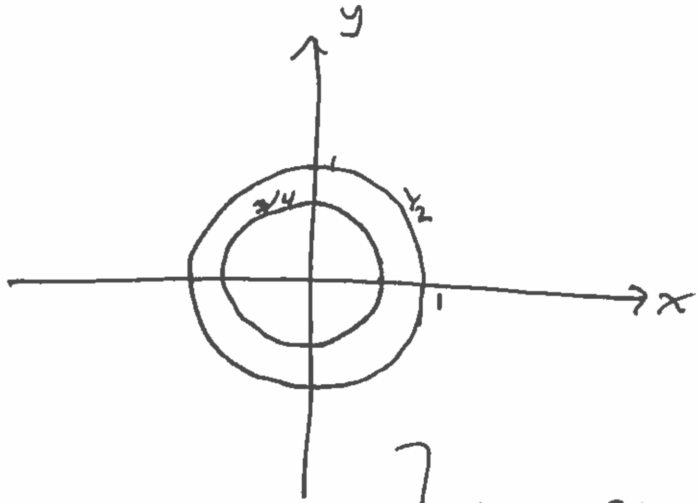
$x^2+y^2+1 = 1$

$x^2+y^2 = 0$ → only happens at $(0,0)$ (one point)

Level $\frac{1}{2}$: $\dfrac{1}{2} = \dfrac{1}{x^2+y^2+1}$

$2 = x^2+y^2+1$

$1 = x^2+y^2$ → Circle of radius 1



Level $\frac{3}{4}$:

$$\frac{3}{4} = \frac{1}{1+x^2+y^2}$$

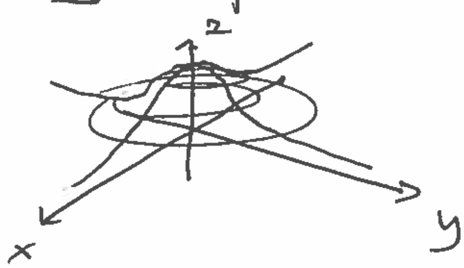
$$\frac{4}{3} = 1 + x^2+y^2$$

$$\frac{1}{3} = x^2+y^2$$

circle, radius $\frac{1}{\sqrt{3}}$

$\approx 0.57\ldots$

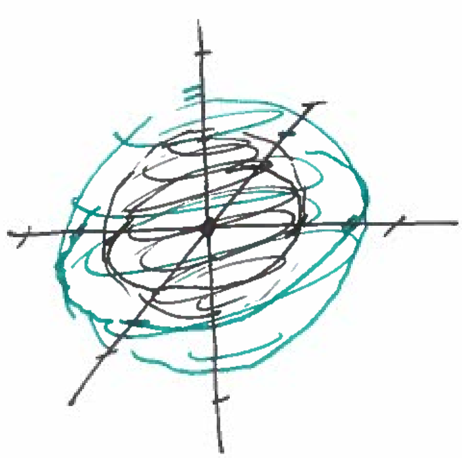
try a 3D interpretation



Functions of 3 variables

$w = f(x,y,z)$

No way to make a graph of f directly; I'd need 4D.



Consider Level Surfaces

Ex $f(x,y,z) = x^2+y^2+z^2$

Level 0: $0 = x^2+y^2+z^2$

point $(0,0,0)$

Level 1: $1 = x^2+y^2+z^2$

sphere of radius 1

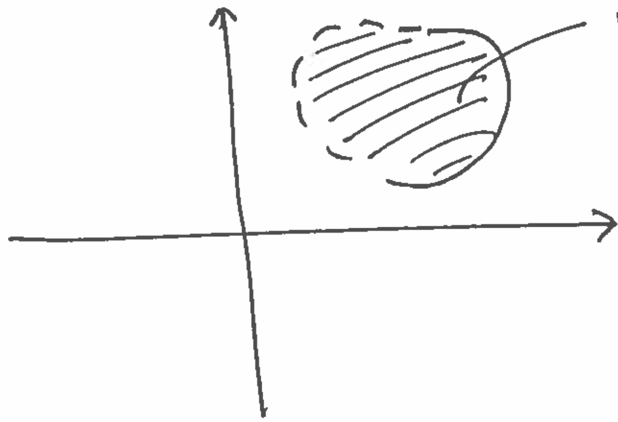
Level 2: $2 = x^2+y^2+z^2$

sphere of radius $\sqrt{2}$

Level 3: sphere of radius $\sqrt{3}$

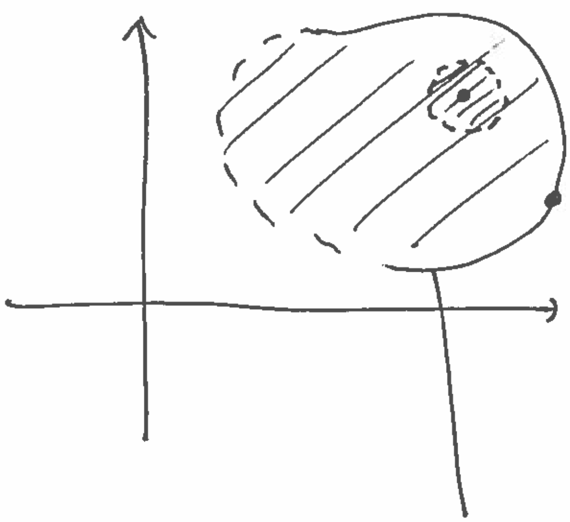
12.2 Function of multiple variables have domains.

$f(x,y)$



maybe only these (x,y) are valid input for f.

f's domain.

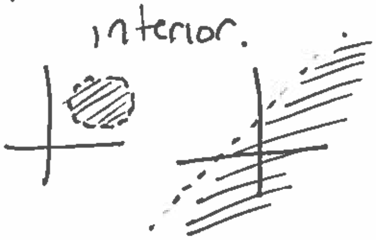


open disks

the open disc of radius ε centered at (x_0,y_0) is the collection of (x,y)

$$(x-x_0)^2+(y-y_0)^2<\varepsilon^2$$

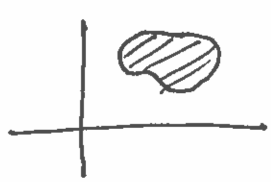
A set C is open if all point in C are interior.



with a set C, an interior point has an open disk around it that is still inside C.

A boundary point is such that every open disk around it has both points inside C and outside C

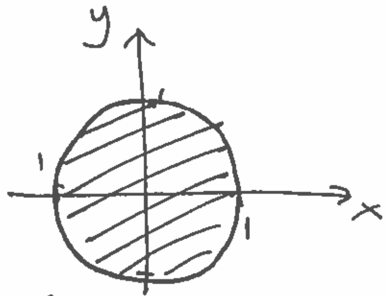
A set is closed if its complement is open



Ex $f(x,y) = \sqrt{1-x^2-y^2}$

Find / describe f's domain.

Is f's domain open, closed, or neither?



Need $1-x^2-y^2 \geq 0$ to make $\sqrt{}$ happy.

$1 \geq x^2+y^2$

$x^2+y^2 \leq 1$

($x^2+y^2 = 1$
$\Rightarrow$ circle of radius 1 centered at $(0,0)$)

Is f's domain ~~open~~, closed or neither?

✓

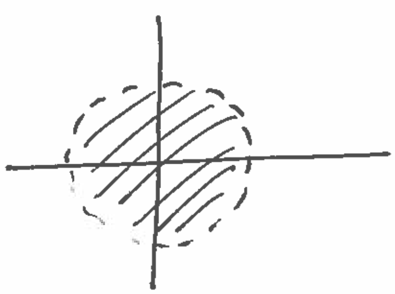
Ex $f(x,y) = \dfrac{1}{\sqrt{1-x^2-y^2}}$

Some questions...

$1-x^2-y^2 > 0$ to make $\sqrt{}$ and $\div$ happy

$1 > x^2+y^2$

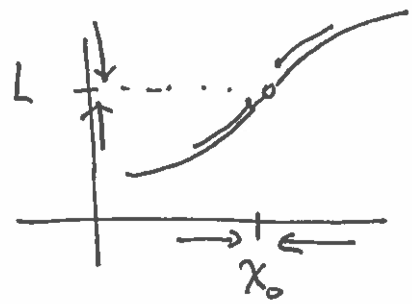
$x^2+y^2 < 1$



an <u>open</u> set

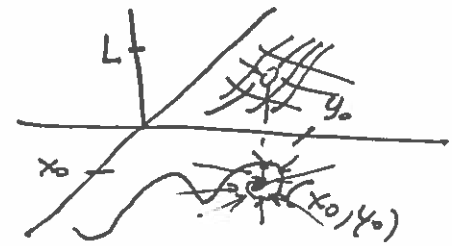
Limits...

One-variable



$$\lim_{x \to x_0} f(x) = L$$

Multivariate functions



Many ways to approach (x_0, y_0)

No formal def for $\lim_{(x,y) \to (x_0,y_0)} f(x,y)$

Ex $\lim_{(x,y)\to(1,\pi)} \left[\frac{y}{x} + \cos(x \underset{\uparrow}{\cdot} y)\right]$ times

Short version: built out of continuous functions;
$\Longrightarrow f(x,y)$ is continuous

So $= \frac{\pi}{1} + \cos(1 \cdot \pi)$
$= \pi - 1$

Ex $\lim_{(x,y)\to(0,0)} \left(\frac{3xy}{x^2+y^2}\right)$

Can't plug in $x=0, y=0$ since "$\frac{0}{0}$" is the result.

↑ Hard to analyze.. For a Hint; plot.

Picture suggest limit d.n.e.

Proving a limit does not exist

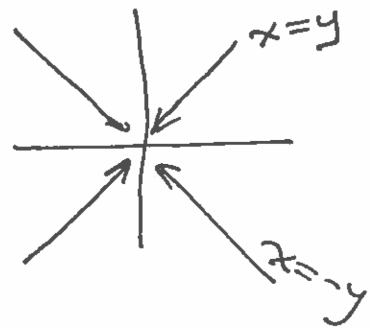
<u>Ex</u> $\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy}{x^2+y^2}$

Picture hinted that a <u>linear</u> approach to $(0,0)$ will result in different limits.

Try approaching $(0,0)$ along line $y=x$.

$$\lim_{x\to 0} \frac{3x^2}{x^2+x^2} =$$

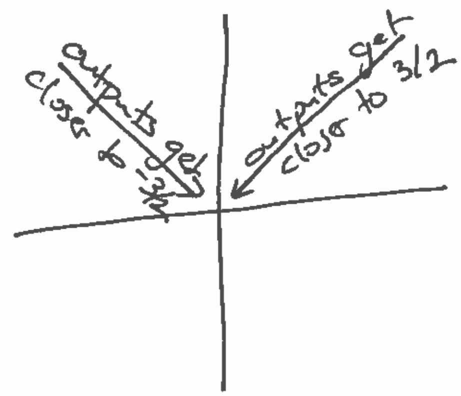
$$= \lim_{x\to 0} \frac{3x^2}{2x^2} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}$$



Now... along $y=-x$

$$\lim_{x\to 0} \frac{-3x^2}{x^2+(-x)^2}$$

$$= \lim_{x\to 0} \frac{-3x^2}{2x^2} = \frac{-3}{2}$$



Conclude:

$\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy}{x^2-y^2}$ d.n.e.

Lines: $y = m\cdot x$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy}{x^2+y^2}$ along that line

$$\lim_{x\to 0} \frac{3mx^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{3mx^2}{(1+m^2)x^2} = \frac{3m}{1+m^2}$$

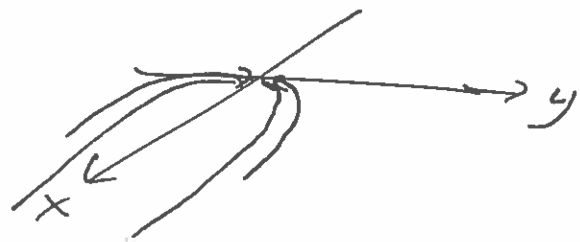
changes as m changes.

different slopes $\implies$ different limits

<u>Ex</u> $$\lim_{(x,y)\to(0,0)} \left(\frac{xy - y^2}{y^2 + x} \right)$$

Harder... try other approaches to reveal the limit does not exist.

<u>Try path $x = y^2$</u>



$$\lim_{y\to 0} \left(\frac{y^2 \cdot y - y^2}{y^2 + y^2} \right)$$

$$= \lim_{y\to 0} \left(\frac{y^3 - y^2}{2y^2} \right) = \lim_{y\to 0} \left(\frac{y-1}{2} \right) = \frac{-1}{2}$$

different!

<u>Try path $x = 2y^2$</u>

$$\lim_{y\to 0} \left(\frac{2y^2 y - y^2}{y^2 + 2y^2} \right) = \lim_{y\to 0} \left(\frac{2y^3 - y^2}{3y^2} \right)$$

$$= \lim_{y\to 0} \left(\frac{2y - 1}{3} \right) = \frac{-1}{3}$$

Ex $\lim_{(x,y)\to(0,0)} \left(\frac{x^3 y}{x^2+2y^2}\right)$

this limit <u>does</u> exist ...

proving that it does is super hard ... we won't do it in this class

Final Notes: • "$f(x,y)$ is <u>continuous</u>" at (x_0,y_0) means

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

• All we did applies to 3-variable and n-variable functions.

12.3 Partial Derivatives

$$f(x,y) = x^2+2y^2$$

"take a derivative?"

consider x the variable, only

consider y is the variable

$f_x(x,y) = 2x$

$\frac{\partial f}{\partial x}(x,y)$

$\frac{\partial}{\partial x} f(x,y)$

the partial derivatives of f

$f_y(x,y) = 4y$

$\frac{\partial f}{\partial y}(x,y)$

$\frac{\partial}{\partial y} f(x,y)$

Ex $f(x,y) = x^2y + 2x + y^3$

Find partial derivatives...

$f_x(x,y) = 2xy + 2$

$f_y(x,y) = x^2 + 3y^2$

Calculate: $f_x(2,1) = 2(2)(1) + 2$
$= 6$

$f_y(2,1) = 2^2 + 3(1)^2$
$= 7$

Ex $f(x,y) = \cos(xy^2) + \sin(x)$

$f_x(x,y) = -\sin(xy^2) \cdot y^2 + \cos(x)$

$f_y(x,y) = -\sin(xy^2) \cdot 2xy$

Calculate $f_x(2,1) = -\sin(2 \cdot 1^2) \cdot 1^2 + \cos(2)$
$= -\sin(2) + \cos(2)$
$\approx -1.33...$

$f_y(2,1) = -\sin(2 \cdot 1^2) \cdot 2 \cdot 2 \cdot 1$
$= -\sin(2) \cdot 4$
$\approx -3.6...$

Ex $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$

$$f_y(x,y) = e^{x^2y^3}\sqrt{x^2+1} \cdot 3x^2y^2$$

$$f_x(x,y) = e^{x^2y^3}\cdot 2x y^3\sqrt{x^2+1}$$
$$+ e^{x^2y^3}\cdot \frac{1}{\cancel{2}\sqrt{x^2+1}}\,\cancel{2}x$$
$$= e^{x^2y^3}\cdot x\left[2y^3\sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}\right]$$

$$f_x(2,1) = e^{2^2\cdot 1^3}\cdot 2\left[2\cdot 1^3\sqrt{2^2+1} + \frac{1}{\sqrt{2^2+1}}\right]$$
$$= e^4\cdot 2\left[2\sqrt{5} + \frac{1}{\sqrt{5}}\right] \approx 537$$

$$f_y(2,1) = e^{2^2\cdot 1^3}\sqrt{2^2+1}\cdot 3(2)^2(1)^2$$
$$= e^4\sqrt{5}\cdot 12 \approx 1465$$