11.4

position $\vec{r}(t)$

velocity $\vec{v}(t)$

acceleration $\vec{a}(t)$

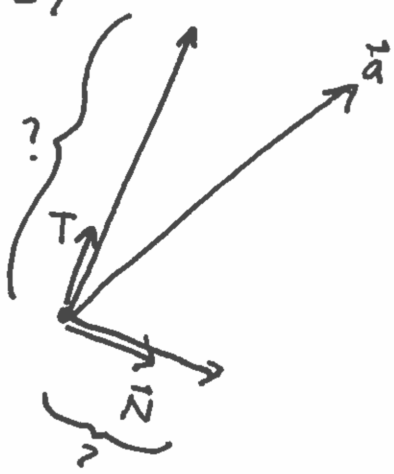
$$\vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|}$$ unit tangent vector

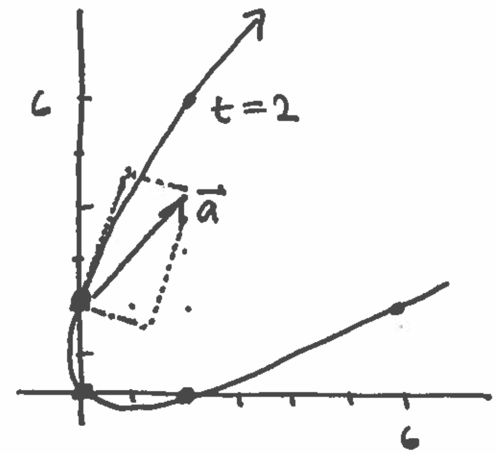
$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$ unit normal vector

Ex $\vec{r}(t) = (t^2 - t,\ t^2 + t)$

$\downarrow$

$\vec{a}(t) = \langle 2, 2 \rangle$





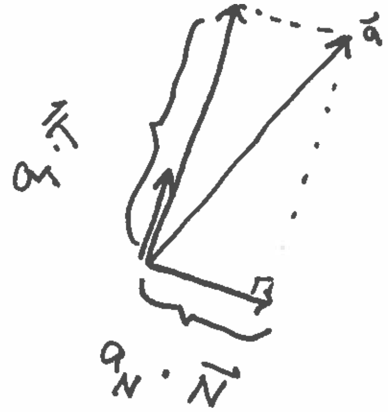
Decompose $\vec{a}(t) = \left(\text{some scalar changing over time}\right) \cdot \vec{T}(t) + \left(\text{some scalar changing over time}\right) \cdot \vec{N}(t)$

$$\vec{a}(t) = a_T(t) \cdot \vec{T}(t) + a_N(t) \cdot \vec{N}(t)$$

a_T: tangential acceleration component — how much you're pushing gas or brakes

a_N: normal acceleration component — how much sideways pull you're experiencing

Calculate:



$$\text{proj}_{\vec{T}}(\vec{a}) = a_T \cdot \vec{T}$$

$$\underbrace{\frac{\vec{a}\bullet\vec{T}}{\vec{T}\bullet\vec{T}}}_{\vec{T}\bullet\vec{T}=1} \vec{T} = a_T \cdot \vec{T}$$

$$\underbrace{(\vec{a}\bullet\vec{T})}_{\text{scalar}}\vec{T} = \underbrace{a_T}_{\text{scalar}}\vec{T}$$

$$\Longrightarrow \quad a_T = \vec{a}\bullet\vec{T}$$

$$\Longrightarrow \quad a_T(t) = \vec{a}(t)\cdot\vec{T}(t)$$

Similarly

$$a_N = \vec{a}\bullet\vec{N}$$

$$a_N(t) = \vec{a}(t)\cdot\vec{N}(t)$$

Ex $\vec{r}(t) = \langle 2\cos t, 2\sin t, 5t\rangle$

Find a_T and a_N.

$$\vec{v}(t) = \langle -2\sin t, 2\cos t, 5\rangle$$

$$\vec{a}(t) = \langle -2\cos t, -2\sin t, 0\rangle$$

$$\vec{T}(t) = \frac{\langle -2\sin t, 2\cos t, 5\rangle}{\|\langle -2\sin t, 2\cos t, 5\rangle\|} = \frac{\langle -2\sin t, 2\cos t, 5\rangle}{\sqrt{4\sin^2 t + 4\cos^2 t + 25}}$$

$$= \frac{\langle -2\sin t, 2\cos t, 5\rangle}{\sqrt{29}}$$

already has constant mag 2

$$\vec{N} = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{\frac{\langle -2\cos t, -2\sin t, 0\rangle}{\sqrt{29}}}{\left\|\frac{\langle -2\cos t, -2\sin t, 0\rangle}{\sqrt{29}}\right\|} = \langle -\cos t, -\sin t, 0\rangle$$

So $\quad a_T = \vec{a} \cdot \vec{T} = \langle -2\cos t, -2\sin t, 0\rangle \cdot \dfrac{\langle -2\sin t, 2\cos t, 5\rangle}{\sqrt{29}}$

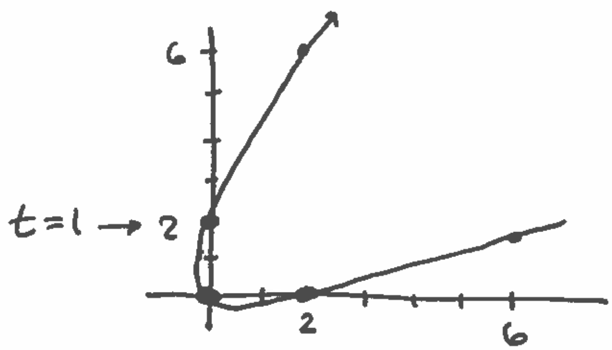
$$a_T = 0$$ (No speeding up or slowing down.)

$$a_N = \vec{a} \cdot \vec{N} = \langle -2\cos t, -2\sin t, 0\rangle \cdot \langle -\cos t, -\sin t, 0\rangle$$

$$= 2\cos^2 t + 2\sin^2 t + 0$$

$$a_N = 2$$

<u>Ex</u> $\quad \vec{r}(t) = \langle t^2 - t, t^2 + t\rangle \qquad$ Find a_T, a_N at $t=1$.



$$\vec{v}(t) = \langle 2t-1, 2t+1\rangle$$

$$\vec{a}(t) = \langle 2, 2\rangle$$

$$\vec{T} = \frac{\langle 2t-1, 2t+1\rangle}{\|\langle 2t-1, 2t+1\rangle\|} = \frac{\langle 2t-1, 2t+1\rangle}{\sqrt{8t^2+2}}$$

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|}$$

$$\vec{T}' = \frac{\sqrt{8t^2+2}\,\langle 2,2\rangle - \langle 2t-1, 2t+1\rangle \cdot \frac{1\cdot 16t}{2\sqrt{8t^2+2}}}{8t^2+2}$$

$$\vec{T}' = \frac{(8t^2+2)\langle 2,2\rangle - 8t\langle 2t-1, 2t+1\rangle}{(8t^2+2)\sqrt{8t^2+2}} \cdot \frac{1\cdot 16t}{2\sqrt{8t^2+2}}$$

$$\vec{T}' = \frac{\langle 8t+4, 4-8t\rangle}{(8t^2+2)\sqrt{8t^2+2}} \quad \underset{\text{to}}{\overset{\text{parallel}}{\sim}} \quad \langle 8t+4, 4-8t\rangle$$

$$\vec{N} = \frac{\langle 8t+4, 4-8t\rangle}{\|\langle 8t+4, 4-8t\rangle\|} = \frac{\langle 8t+4, 4-8t\rangle}{\sqrt{128t^2+32}}$$

$$a_T(t) = \vec{a}(t)\cdot\vec{T} = \langle 2,2\rangle \cdot \frac{\langle 2t-1,\ 2t+1\rangle}{\sqrt{8t^2+2}}$$

$$= \frac{4t-2 \quad + \quad 4t+2}{\sqrt{8t^2+2}} = \frac{8t}{\sqrt{8t^2+2}}$$

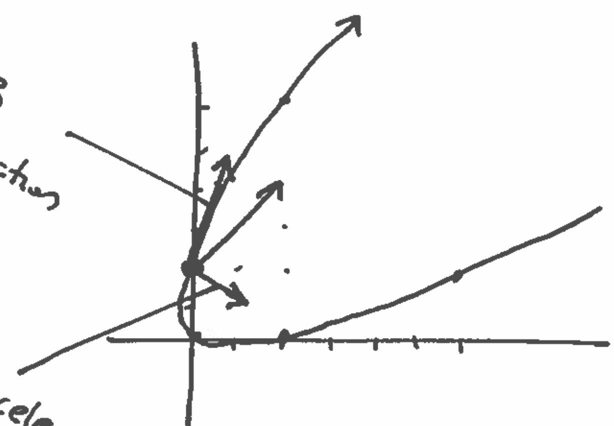
As $t\to\infty$ approaches $\sqrt{8}$

$$a_N(t) = \vec{a}(t)\cdot\vec{N} = \langle 2,2\rangle \cdot \frac{\langle 8t+4,\ 4-8t\rangle}{\sqrt{128t^2+32}}$$

$$= \frac{16t+8 \ +8-16t}{\sqrt{128t^2+32}} = \frac{16}{\sqrt{128t^2+32}}$$

As $t\to\infty$, approaches 0

2.53 acceleration units forward



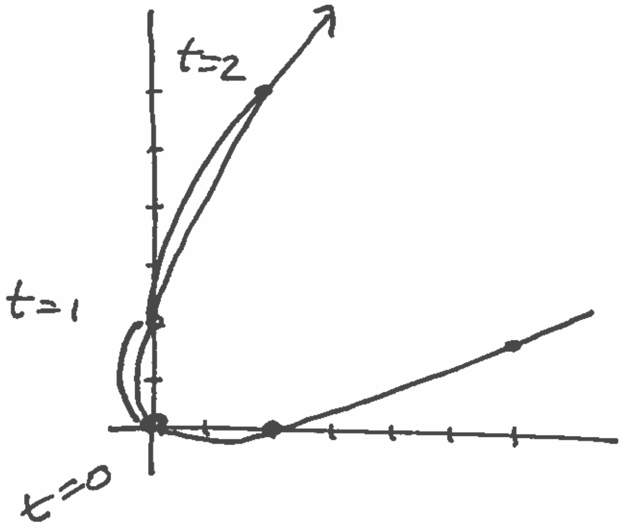
1.26 acceleration units to the right!

$$a_T(1) = \frac{8(1)}{\sqrt{8+2}} = \frac{8}{\sqrt{10}} \approx 2.53$$

$$a_N(1) = \frac{16}{\sqrt{128+32}} = \frac{16}{\sqrt{160}} \approx 1.26$$

11.5 Arclength parametrization & curvature.

$\vec{r}(t) = \langle t^2 - 1, t^2 + 1 \rangle$

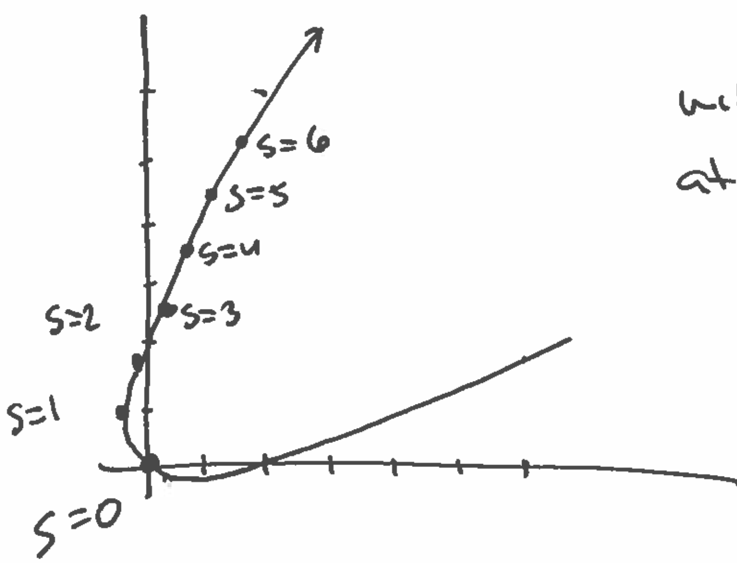


arclength on $[0,1]$

arc length on $[1,2]$

Not equal!

Attempt to <u>reparametrize</u> the same curve (shape/path) ~~at~~ with a different parameter s, so that...

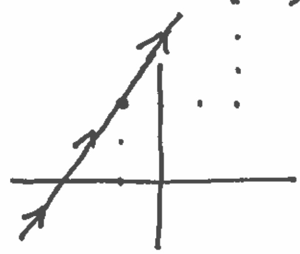


with s, we travel at constant speed $1 \dfrac{\text{unit of arc length}}{\text{unit of time } s}$

Looking for $\vec{r}(s) = \langle f(s), g(s) \rangle$ that would still make this parabola

Ex $\quad \vec{r}(t) = (3t-1,\ 4t+2)$

$= (-1,2) + t\cdot\langle 3,4\rangle$



Paramatrize this curve with respect to arc length.

$$s = \int_0^t \|\vec{v}(u)\|\,du$$

$\vec{v}(t) = \langle 3,4\rangle$

$\|\vec{v}(t)\| = \sqrt{9+16} = 5$

$$s = \int_0^t 5\,du$$

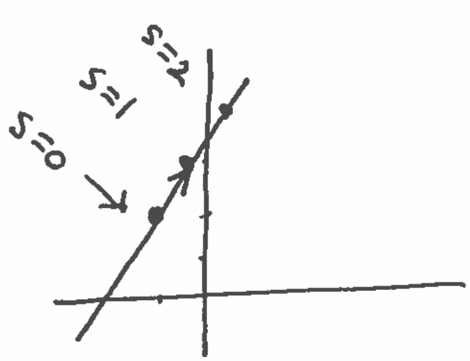
$$s = [5u]_0^t = 5t$$

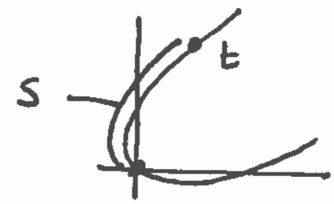
$s = 5t$

$t = \frac{1}{5}s$

$\vec{r}(t) = (3t-1,\ 4t+2)$

$\vec{r}(s) = (\frac{3}{5}s - 1,\ \frac{4}{5}s + 2)$





$$\underbrace{s = \int_0^t \|\vec{v}(t)\|\,dt}_{\text{if we can integrate this}}$$

$$\underbrace{s = \text{function of } (t)}_{}$$

if we can isolate t,

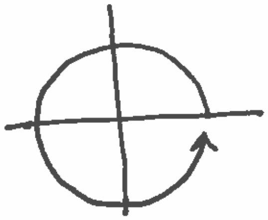
$t = \text{function of } s$

$\vec{r}(t) = \langle t^2 - t,\ t^2 + t\rangle$

$\vec{r}(s) = \langle \sim s \sim,\ \sim s \sim\rangle$

Ex $\quad \vec{r}(t) = \langle \cos(t^3), \sin(t^3) \rangle$

Parametrize by arc length.



$$s = \int_0^t \|\vec{v}(u)\| \, du$$

$$\vec{v}(t) = \langle -3t^2 \sin(t^3), 3t^2 \cos(t^3) \rangle$$

$$\|\vec{v}(t)\| = \sqrt{9t^4 \sin^2(t^3) + 9t^4 \cos^2(t^3)}$$

$$= \sqrt{9t^4 \left[\sin^2(\ ) + \cos^2(\ )\right]} = 3t^2$$

$$s = \int_0^t 3u^2 \, du = \left[u^3\right]_0^t = t^3$$

$$s = t^3$$

$$\Updownarrow$$

$$t = \sqrt[3]{s}$$

So $\langle \cos(t^3), \sin(t^3) \rangle$

$$\langle \cos((\sqrt[3]{s})^3), \sin((\sqrt[3]{s})^3) \rangle$$

$$\vec{r}(s) = \langle \cos(s), \sin(s) \rangle$$

<u>Ex</u> $\vec{r}(t) = \langle t^2 - t,\ t^2 + t\rangle$ Parametrize by arc length

$$s = \int_0^t \|\vec{v}(u)\|\, du$$

$$\vec{v}(t) = \langle 2t-1,\ 2t+1\rangle$$

$$\|\vec{v}(t)\| = \sqrt{8t^2+2}$$

$$s = \int_0^t \sqrt{8u^2+2}\ du$$

trig substitution using tables of integrals

$$s = \sqrt{8}\left(\ln\left|\underbrace{t+\sqrt{t^2+\tfrac{1}{4}}}_{\text{auto }+}\right| - \ln\left(\tfrac{1}{2}\right)\right)$$

Now try to isolate t...

$$\underbrace{\frac{s}{\sqrt{8}} + \ln\left(\tfrac{1}{2}\right)} = \ln\left(t + \sqrt{t^2+\tfrac{1}{4}}\right)$$

$$e^{[\quad]} = t + \sqrt{t^2+\tfrac{1}{4}}$$

$$e^{[\quad]} - t = \sqrt{t^2+\tfrac{1}{4}}$$

$$\left(e^{[\quad]} - t\right)^2 = t^2 + \tfrac{1}{4}$$

$$e^{2[\quad]} - 2\cdot e^{[\quad]}\cdot t + \cancel{t^2} = \cancel{t^2} + \tfrac{1}{4}$$

$$-2e^{[\quad]}t = \tfrac{1}{4} - e^{2[\quad]}$$

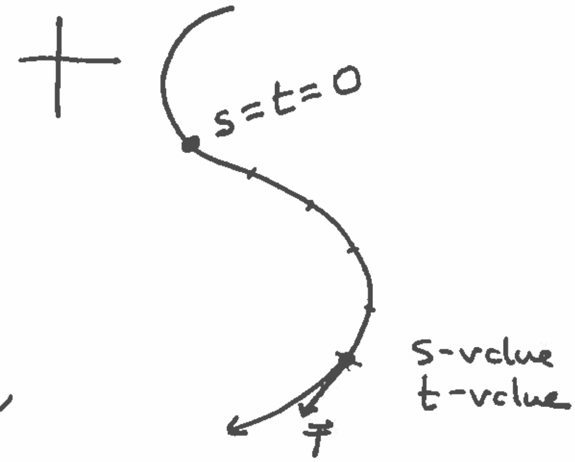
$$t = \frac{\tfrac{1}{4} - e^{2[\quad]}}{-2e^{[\quad]}}$$

So $\vec{r}(t) = \langle t^2 - t,\ t^2 + t\rangle$

$$t = \frac{\tfrac{1}{4} - e^{2\left(\frac{s}{\sqrt{8}} + \ln\left(\frac{1}{2}\right)\right)}}{-2e^{\left(s/\sqrt{8} + \ln\left(\frac{1}{2}\right)\right)}}$$

$$\vec{r}(s) = \langle \underset{s}{\sim\sim},\ \underset{s}{\sim\sim}\rangle$$

Facts: $\vec{r}\,'(s) = \vec{T}(t)$



!!!

take $\vec{r}(t)$,
reparametrize
by arc length,
$\vec{r}(s)$ has new
formulas,
differentiate those

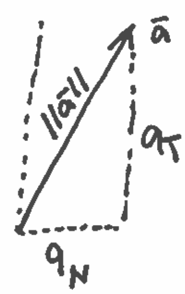
Not!!!
$\vec{r}(t)$'s formula,
differentiated,
then sub in s.

$$\Longrightarrow \|\vec{r}\,'(s)\| = 1$$

useful in formulas
useful as a check
if you
parametrized by
arc length

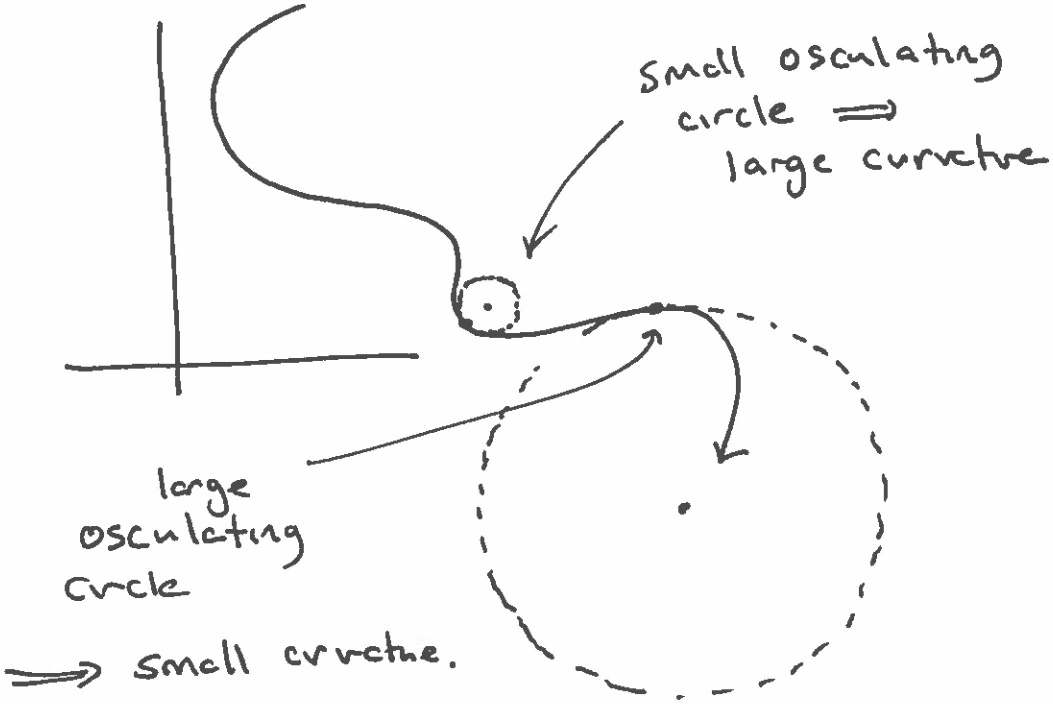
- $a_T = \vec{a}\cdot\vec{T} = \frac{d}{dt}\left(\|\vec{v}(t)\|\right)$

- $a_N = \vec{a}\cdot\vec{N} = \sqrt{\|\vec{a}\|^2 - a_T^2} = \overbrace{\frac{\|\vec{a}\times\vec{v}\|}{\|\vec{v}\|}}^{\text{for } \mathbb{R}^3\text{-situations}} = \|\vec{v}\|\cdot\|\vec{T}\,'\|$



Curvature

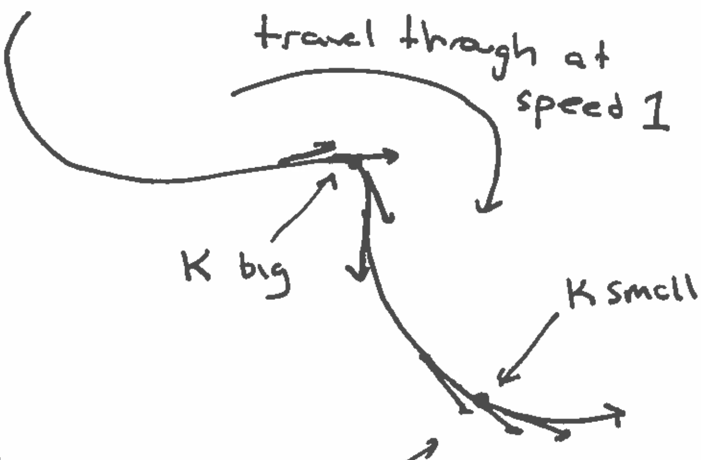
At any point, "how curvy" is the curve



Def

$$K(t) = \left\| \frac{d\vec{T}}{ds} \right\| = \| \vec{T}'(s) \|$$

curvature at time t



In a short time, $\vec{T}$ changed a lot. $\|\vec{T}'\|$ is <u>large</u>

Here, in same amount of time, $\vec{T}$ changes less, $\|\vec{T}'(s)\|$ is not so large.

Ex $\vec{r}(t) = \langle 4t+3, -2t+1 \rangle$ (line ⟹ expect $K=0$)

$\vec{v}(t) = \langle 4, -2 \rangle$

arc length?

$$s = \int_0^t \|\vec{v}(u)\|\, du$$

$$s = \int_0^t \sqrt{16+4}\, du$$

$$s = \sqrt{20} \cdot t$$

$$t = s/\sqrt{20}$$

$$\vec{r}(s) = \left\langle \frac{4}{\sqrt{20}}s + 3, \frac{-2}{\sqrt{20}}s + 1 \right\rangle$$

K needs $\vec{T}'$. Well, $\vec{T} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{\langle \frac{4}{\sqrt{20}}, \frac{-2}{\sqrt{20}} \rangle}{1}$

So $\vec{T} = \langle \frac{4}{\sqrt{20}}, \frac{-2}{\sqrt{20}} \rangle$

And $\vec{T}' = \langle 0, 0 \rangle$.

$$\underbrace{K = \|\vec{T}'(s)\|}_{\text{Nice in theory.}} = \|\langle 0,0 \rangle\| = 0$$

Other formulas for K, easier to calculate with, very hard to understand why.

$$\rightarrow \quad K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \underbrace{\frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}}_{\text{3D only}} = \frac{\vec{a}(t) \cdot \vec{N}(t)}{\|\vec{v}(t)\|^2}$$

In 2D

$\rightarrow$ If $\vec{r}(t) = (x(t), y(t))$

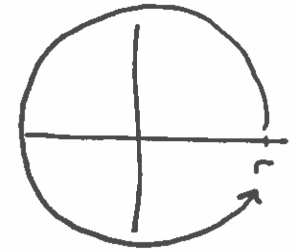
$$K(t) = \frac{|x'(t) \cdot y''(t) - x''(t) \cdot y'(t)|}{\left(x'(t)^2 + y'(t)^2 \right)^{3/2}}$$

If $\vec{r}(t) = (t, f(t))$
$(y = f(x))$

$$K(\underset{x}{t}) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}}$$

<u>Ex</u> $\vec{r}(t) = (r \cdot \cos(t), r\sin(t))$

Find $K(t)$.



$$K(t) = \frac{|x' \cdot y'' - x'' \cdot y'|}{\left((x')^2 + (y')^2\right)^{3/2}}$$

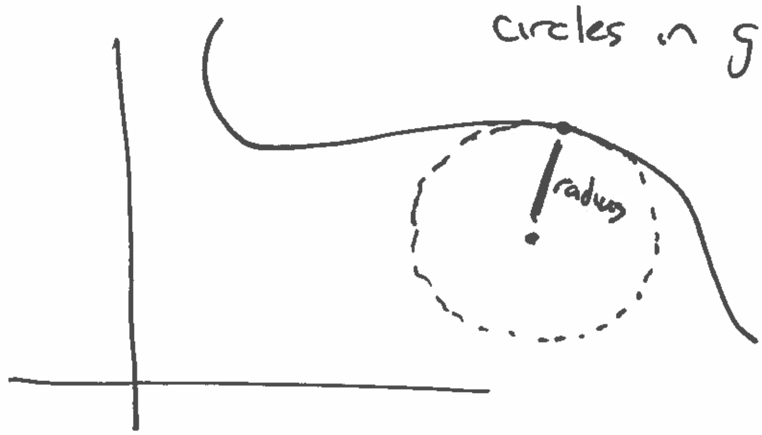
expect K to:
- constant
- large $r \longleftrightarrow$ small K
  large K $\longleftrightarrow$ small r

$$= \frac{|r \cdot -\sin(t) \cdot r \cdot -\sin(t) - r \cdot -\cos(t) \cdot r \cdot \cos(t)|}{\left((r \cdot -\sin(t))^2 + (r\cos t)^2\right)^{3/2}}$$

$$= \frac{|r^2\sin^2(t) + r^2\cos^2(t)|}{\left(r^2\sin^2(t) + r^2\cos^2(t)\right)^{3/2}} = \frac{r^2}{(r^2)^{3/2}} = \frac{r^2}{r^3}$$

$$\boxed{K = \frac{1}{r}}$$

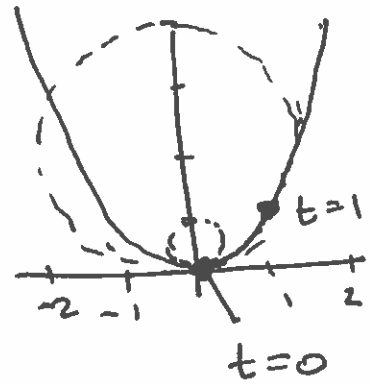
true for osculating circles in general



K is reciprocal of radius!

Consider $y = x^2$

$\vec{r}(t) = (t, t^2)$



Find $K(t) = K(x)$

Find osculatory circles at $t=0$, $t=1$

$$K(x) = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}} = \frac{2}{\left(1 + (2x)^2\right)^{3/2}}$$

$$K = \frac{2}{(1+4x^2)^{3/2}}$$

Give equations for these circles...

$(x-x_0)^2 + (y-y_0)^2 = r^2$

(x_0, y_0) center — radius

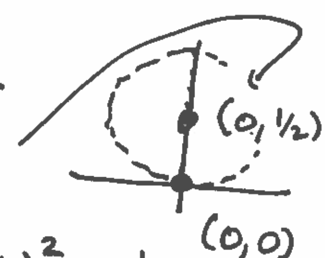
Osculatory circle at $t=0$.

radius?

$K = \frac{2}{(1+4\cdot 0)^{3/2}} = 2$

$r = \frac{1}{2}$

center

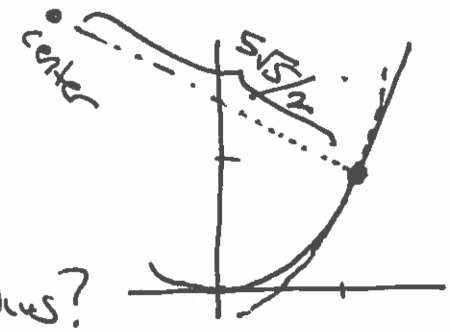


$$(x)^2 + (y - \tfrac{1}{2})^2 = \frac{1}{4}$$

When $t = x = 1$, where are we on the curve?

(1,1).

(Not center of circle!)



Calculate osculating circle's radius?

$$K = \frac{2}{(1+4x^2)^{3/2}} = \frac{2}{(1+4)^{3/2}} = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}}$$

radius is $\frac{5\sqrt{5}}{2}$

Need to know $\vec{N}$

$\vec{r} = (t, t^2)$

$\vec{v} = (1, 2t)$

$$\vec{T} = \frac{\langle 1, 2t\rangle}{\sqrt{1+4t^2}}$$

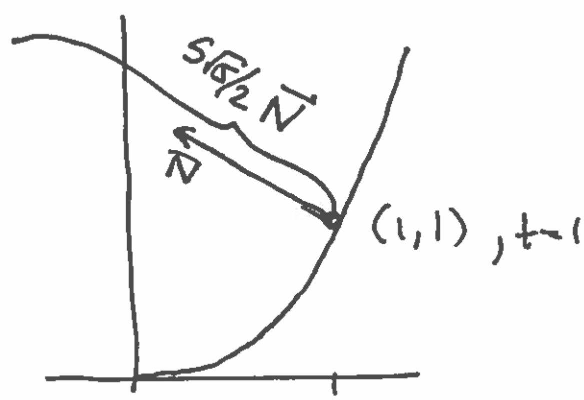
$$\vec{T}' = \frac{\sqrt{1+4t^2}\,\langle 0,2\rangle - \langle 1,2t\rangle \cdot \frac{8t}{2\sqrt{1+4t^2}}}{1+4t^2}$$

$$\vec{T}' = \frac{(1+4t^2)\langle 0,2\rangle - 4t\langle 1,2t\rangle}{(1+4t^2)\sqrt{1+4t^2}}$$

$$= \frac{\langle -4t, 2\rangle}{(1+4t^2)\sqrt{1+4t^2}} \underset{\text{to}}{\overset{\text{parallel}}{\sim}} \langle -4t, 2\rangle$$

$$\text{So } \vec{N} = \frac{\langle -4t, 2\rangle}{\sqrt{16t^2+4}}$$

$$\vec{N} = \frac{\langle -2t, 1\rangle}{\sqrt{4t^2+1}}$$



$$\frac{5\sqrt{5}}{2}\vec{N} = \frac{5\sqrt{5}}{2\sqrt{4t^2+1}}\langle -2t, 1\rangle \xrightarrow{t=1} = \frac{5\cancel{\sqrt{5}}}{2\cancel{\sqrt{5}}}\langle -2, 1\rangle = \langle -5, 5/2\rangle$$

Center: $\cancel{\ }\ (1,1) + \underbrace{\langle -5, 5/2 \rangle}_{\text{radius} \cdot \vec{N}}$

$$(-4, 7/2) \qquad \text{radius} \quad \frac{5\sqrt{5}}{2}$$

So $$(x+4)^2 + (y - 7/2)^2 = \frac{125}{4}$$