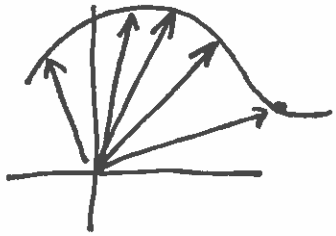
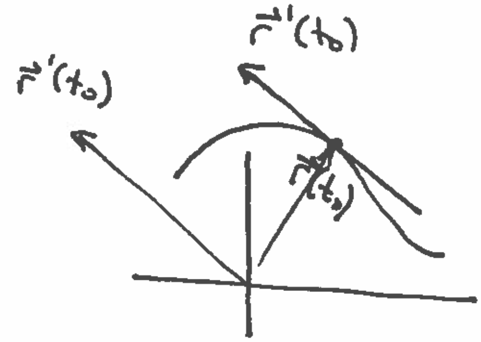
11.1, 11.2 recap



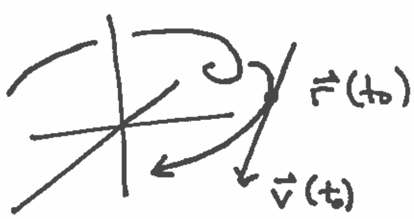
$\vec{r}(t) = \langle r_1(t), r_2(t), \dots \rangle$

$\vec{r}\,'(t) = \langle r_1'(t), r_2'(t), \dots \rangle$



11.3 Calculus of Motion

If $\vec{r}(t)$ describes a position over time in 3D



$\vec{r}\,'(t) =$ velocity at time t

speed and a direction
(magnitude)

$\vec{v}(t) := \vec{r}\,'(t)$

Maybe $\vec{r}(t) = (r_1(t), r_2(t), r_3(t))$

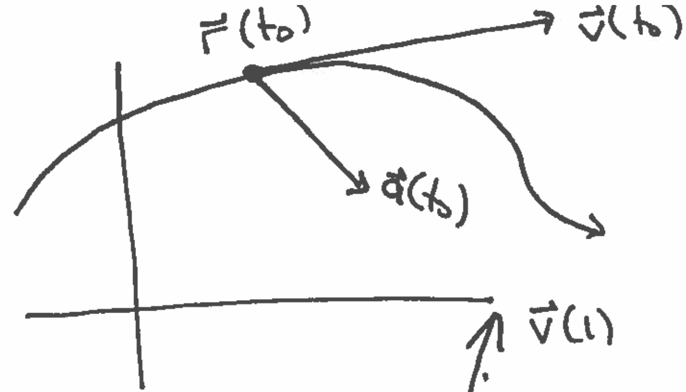
$\vec{v}(t) = \langle r_1'(t), r_2'(t), r_3'(t) \rangle$

And speed: $\underbrace{\text{speed}(t)}_{\substack{\text{one-variable in,} \\ \text{one variable out}}} = s(t) = \|\vec{v}(t)\|$

Also have

$$\vec{r}''(t)$$
$$\vec{v}'(t)$$
$$\vec{a}(t)$$

vector rate of change in velocity, acceleration

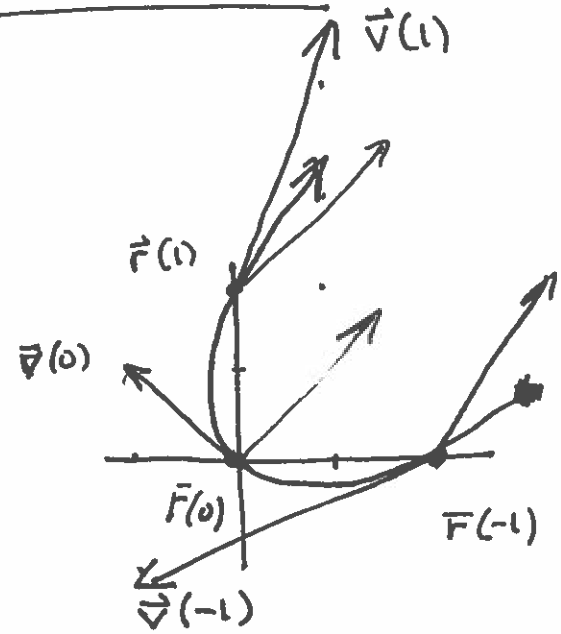


<u>Ex</u> $\vec{r}(t) = (t^2-t,\ t^2+t)$

$$\vec{v}(t) = \langle 2t-1,\ 2t+1 \rangle$$

$$\vec{a}(t) = \langle 2, 2 \rangle$$

constant!



$$\vec{v}(-1) = \langle -3, -1 \rangle$$
$$\vec{v}(0) = \langle -1, 1 \rangle$$
$$\vec{v}(1) = \langle 1, 3 \rangle$$

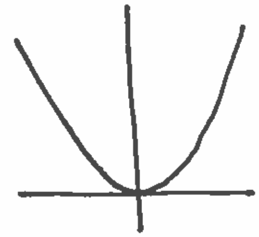
How fast? speed(t)

$s(t)$

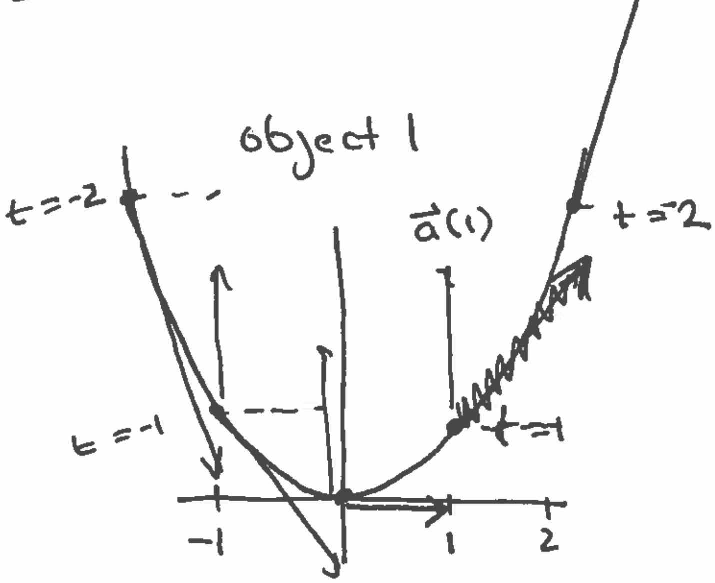
$$\|\vec{v}(t)\| = \sqrt{(2t-1)^2+(2t+1)^2}$$
$$= \sqrt{8t^2+2}$$

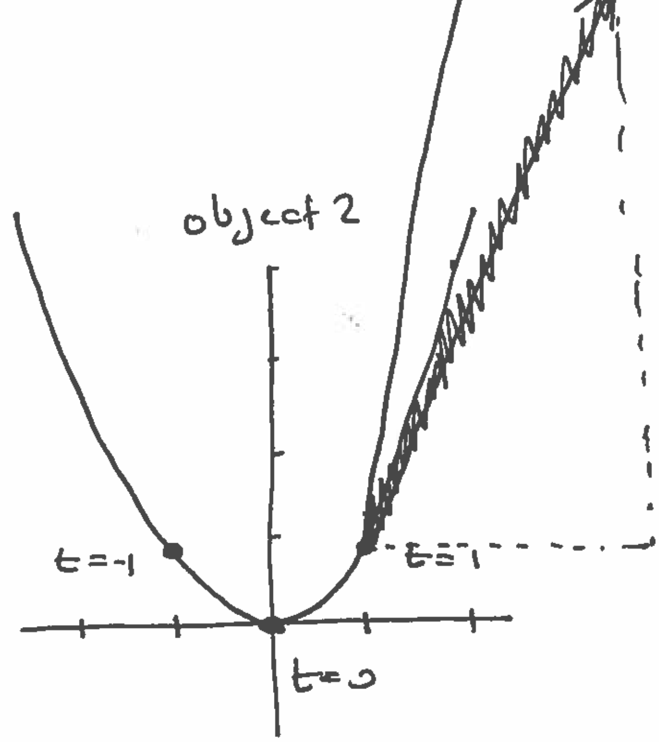
$$s(-1) = \sqrt{10}$$
$$s(0) = \sqrt{2}$$
$$s(1) = \sqrt{10}$$

$\underline{\text{Ex}}$ $\vec{r}_1(t) = \langle t, t^2 \rangle$
$\vec{r}_2(t) = \langle t^3, t^6 \rangle$ $\Rightarrow$ both have $y = x^2$



$\vec{v}_1(t) = \vec{r}_1\,'(t) = \langle 1, 2t \rangle$
$\vec{v}_2(t) = \langle 3t^2, 6t^5 \rangle$





same paths
can be traced with
wildly different velocities.

$\vec{a}_1(t) = \langle 0, 2 \rangle$

$\vec{a}_2(t) = \langle 6t, 30t^4 \rangle$
$\vec{a}_2(1) = \langle 6, 30 \rangle$

Ex

twirl ball on end of 2ft string. Speed up to 2 revs per second.
$\Longrightarrow$ period is $\frac{1}{2}$ sec.

Can we model the position at time t?

$\langle ?\, ,\, ?\, ,\, 5 \rangle$

$$\vec{r}(t) \quad \langle \underbrace{2\cos(4\pi t),\ 2\sin(4\pi t)}_{\text{parameterize a circle.}},\ 5 \rangle$$

$\cos(c \cdot t)$
has period
$\frac{2\pi}{c}$

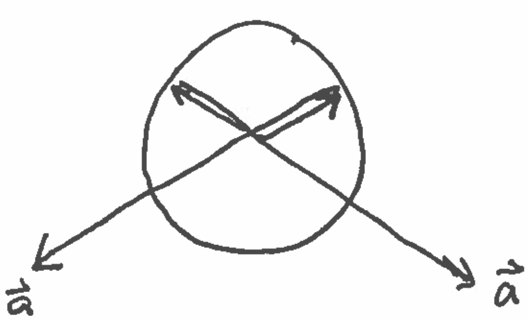
want $\frac{2\pi}{c} = \frac{1}{2}$

$4\pi = c$

Well, $\vec{v}(t) = \langle -8\pi\sin(4\pi t),\ 8\pi\cos(4\pi t),\ 0 \rangle$

$$\text{And } \vec{a}(t) = \langle \underbrace{-32\pi^2\cos(4\pi t),\ -32\pi^2\sin(4\pi t)}_{\text{compare } x \text{ \& } y \text{ here to } x \text{ \& } y \text{ in } \vec{r}(t).},\ 0 \rangle$$ make sense!

Acceleration (in this example) is opposite in direction to position.



speed? $\|\vec{v}(t)\| = \sqrt{64\pi^2\sin^2(\ ) + 64\pi^2\cos^2(\ )}$
$= \sqrt{64\pi^2}$
$= 8\pi$

Fact: constant magnitude vvf $\vec{r}$, then $\vec{r}$ & $\vec{r}\,'$ were orthogonal...

Here: $\vec{v}$ is constant magnitude (8π)

So $\vec{v}$ and $\vec{v}\,'$ are orthogonal

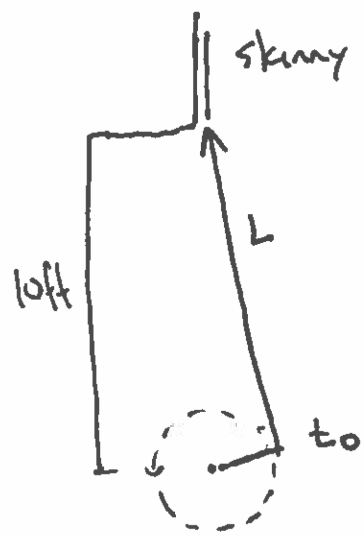
So $\vec{v}$ and $\vec{a}$ are orthogonal

So $\vec{v} \cdot \vec{a}$ should be 0...

Look! It works out!

Top-down

skinny flag pole...



when to release, to hit pole?

Calculate the line that the rock would travel through after release at some t_0.

$$\vec{r}(t_0) = \left(2\cos(4\pi t_0),\ 2\sin(4\pi t_0),\ 5\right)$$

$$\vec{v}(t_0) = \left\langle -8\pi\sin(4\pi t_0),\ 8\pi\cos(4\pi t_0),\ 0\right\rangle$$

$$L(t_0) = \left(2\cos(4\pi t_0),\ 2\sin(4\pi t_0),\ 5\right) + t\left\langle -8\pi\sin(4\pi t_0),\ 8\pi\cos(4\pi t_0),\ 0\right\rangle$$

The issue is does this pass through $(0, 10, 5)$?

$$\begin{array}{ll} x\text{-position} & 2\cos(4\pi t_0) - 8\pi t \cdot \sin(4\pi t_0) = 0 \\ y\text{ position} & 2\sin(4\pi t_0) + 8\pi t \cdot \cos(4\pi t_0) = 10 \end{array} \Bigg\} \begin{array}{c}\text{2 equations}\\ \text{in}\\ \text{2 unknowns}\end{array}$$

Want to solve for t_0.

$$2\cos(4\pi t_0) = 8\pi t \sin(4\pi t_0)$$

$$\cancel{4\cos^2(4\pi t_0) = 64\pi^2 t^2 \sin^2(4\pi t_0)}$$

$$\cancel{4\cos^2(4\pi t_0) = 64\pi^2 t^2 (1 - \cos^2(4\pi t_0))}$$

$$t = \frac{2\cos(4\pi t_0)}{8\pi \sin(4\pi t_0)} = \frac{1}{4\pi} \cdot \cot(4\pi t_0)$$

$$\Longrightarrow 2\sin(4\pi t_0) + 8\pi \frac{1}{4\pi} \underbrace{\cot(4\pi t_0)}_{\text{denominator?}} \cdot \cos(4\pi t_0) = 10$$

multiply by $\sin(4\pi t_0)$

$$\underbrace{2\sin^2(4\pi t_0) + 2\cos^2(4\pi t_0)} = 10\sin(4\pi t_0)$$

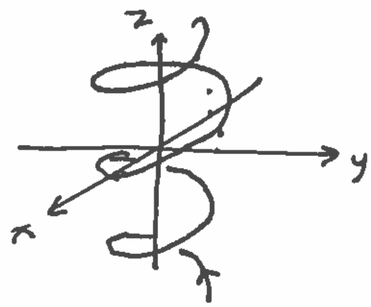
$$2 = 10\sin(4\pi t_0)$$

$$1/5 = \sin(4\pi t_0)$$

$$0.016 \approx \frac{\arcsin(1/5)}{4\pi} = t_0$$

Release the rock 0.016 seconds into a revolution.

Ex Spiral Helix $\quad \vec{r}(t) = (\cos t, \sin t, t)$

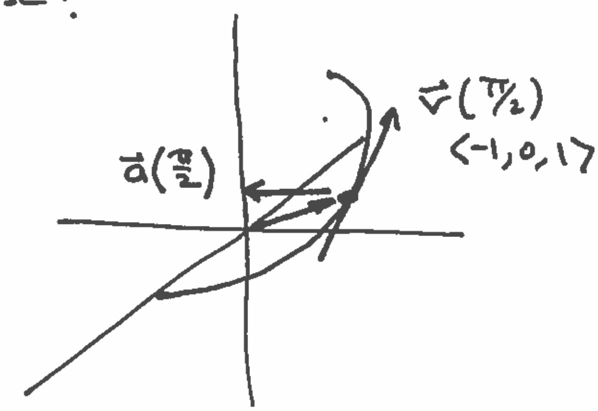


Analyze.

$$\vec{v}(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{a}(t) = \langle \underbrace{-\cos t, -\sin t}_{\text{opposite of } x,y \text{ from position}}, 0 \rangle$$

make sense?



$$\text{Speed}(t) = \sqrt{\sin^2(t) + \cos^2(t) + 1^2}$$
$$= \sqrt{2}, \text{ constant.}$$

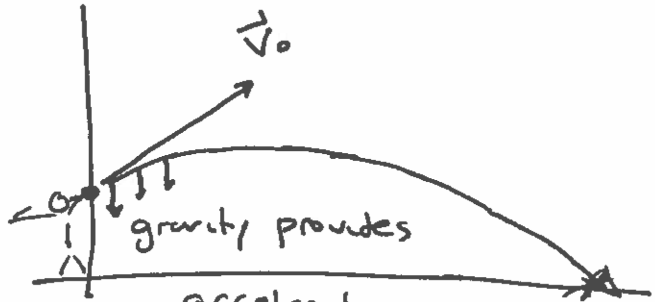
So yeah $\vec{v}$ has const mag.

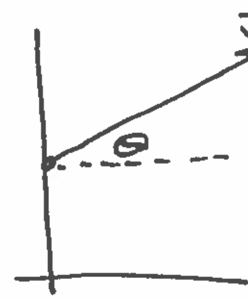
$\Longrightarrow$ $\vec{v}$ & $\vec{v}'$ are $\perp$

$\vec{v}$ & $\vec{a}$ are $\perp$

we ca see that!

Projectiles





angle of elevation.

Ex Throw a ball 80mph at an angle of 30°.
How far away will it be when it lands?
(You are 6ft tall.)

gravity will affect y ---

First principle: $\vec{a} = \langle 0, -32 \rangle$

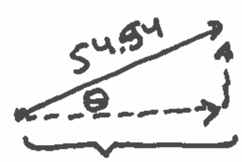
$-32\ ft/s^2$

$$\Longrightarrow \quad \vec{v} = \int_0^t \vec{a}(t)\, dt \quad \left(\text{not quite true}\right)$$

$$\cancel{\vec{v} = \int_0^t \langle 0, -32 \rangle\, dt = \Big[\langle 0, -32t \rangle\Big]_0^t}$$

? $\quad 80 \frac{mi}{h} = 54.54\ldots \frac{ft}{s}$

$$\vec{v}(t) = \int_0^t \vec{a}(t)\, dt + \vec{v}(0)$$



$54.54 \sin(30°)$
$54.54 \cdot \frac{1}{2}$

$54.54 \cdot \cos(30°)$
$54.54 \cdot \sqrt{3}/2$

$$= \int_0^t \langle 0, -32 \rangle\, dt + \left\langle 54.54 \frac{\sqrt{3}}{2}, 54.54 \cdot \frac{1}{2} \right\rangle$$

$$= \Big[\langle 0, -32t \rangle\Big]_0^t \quad + \text{ —— '' ——}$$

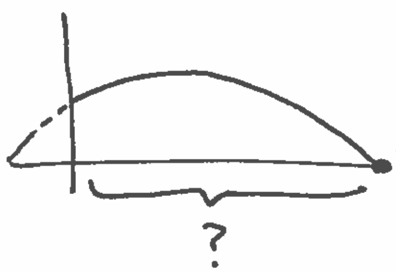
$$= \langle 0, -32t \rangle + \langle 27.27\sqrt{3}, 27.27 \rangle$$

$$\vec{v}(t) = \langle 27.27\sqrt{3}, -32t + 27.27 \rangle$$

$$\vec{r}(t) = \int_0^t \langle 27.27\sqrt{3}, -32t + 27.27 \rangle \, dt + \vec{r}(0)$$

$$= \Big[\langle 27.27\sqrt{3}\, t, -16t^2 + 27.27t \rangle \Big]_0^t + \langle 0, 6 \rangle$$

$$\vec{r}(t) = \langle 27.27\sqrt{3}\, t, -16t^2 + 27.27t + 6 \rangle$$



special thing here is $y = 0$

find t_1, special time when ball hits ground

$$-16t_1^2 + 27.27t_1 + 6 = 0$$

can be solved with QF.

$t_1 \approx 1.9$ seconds.

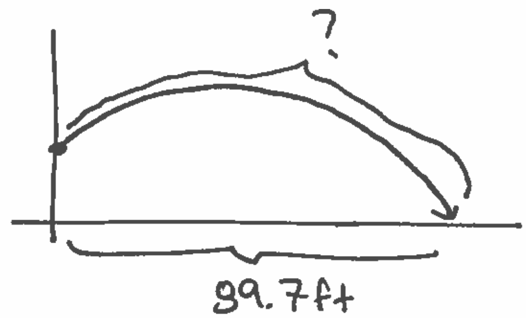
what is x at this time?

$$27.27\sqrt{3}\,(1.9)$$

$$\approx 89.7$$

we threw it 89.7 ft.

Distance Travelled



$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

(from param. curves)

~~distance~~ distance traveled for $\vec{r}$ a position function. $\longrightarrow$ $\boxed{L = \int_a^b \|\vec{v}(t)\|\, dt}$

$$\vec{v}(t) = \langle 27.27\sqrt{3}\ ,\ -32t + 27.27 \rangle$$

$$\|\vec{v}(t)\| = \sqrt{27.27^2 \cdot 3 + (-32t + 27.27)^2}$$

use calculator!

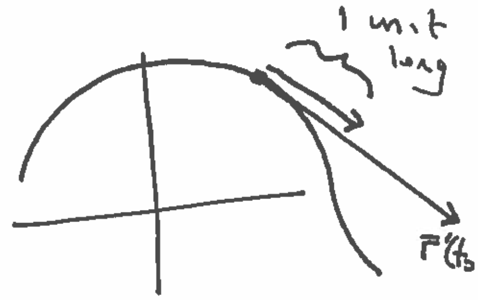
$$\int_0^{1.9} \sqrt{27.27^2(3) + (-32t + 27.27)^2}\, dt \approx$$

11.4 Unit Tangent & Normal Vectors

$\vec{r}(t)$ position

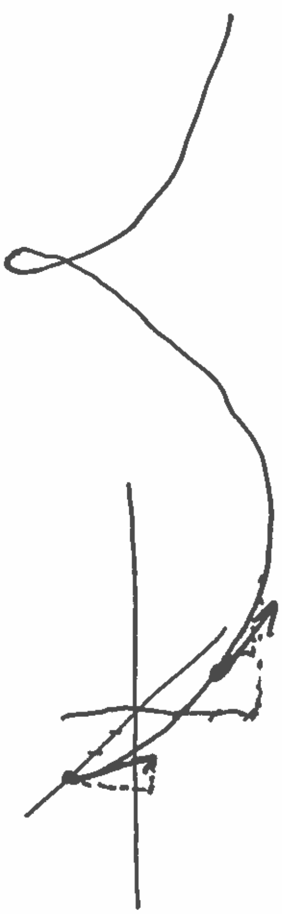
$\vec{r}\,'(t)$ derivative / velocity / tangent

unlikely this is a unit length tangent ...



$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

the unit tangent vector at time t.



Ex $\vec{r}(t) = (3\cos t,\ 3\sin t,\ 4t)$

Find $\vec{T}(t)$ ....

$\vec{r}\,'(t) = \langle -3\sin t,\ 3\cos t,\ 4 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$

So $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{\langle -3\sin t,\ 3\cos t,\ 4 \rangle}{5}$

$\vec{T}(t) = \left\langle -\frac{3}{5}\sin t,\ \frac{3}{5}\cos t,\ \frac{4}{5} \right\rangle$

$\vec{T}(0) = \langle 0,\ \frac{3}{5},\ \frac{4}{5} \rangle$ $\quad \vec{T}(1) \approx \langle -0.51,\ 0.32,\ 0.8 \rangle$

Ex $\quad \vec{r}(t) = \langle t^2 - t,\ t^2 + t \rangle$

Find $\vec{T}(t)$.

$$\vec{r}\,'(t) = \langle 2t-1,\ 2t+1 \rangle$$

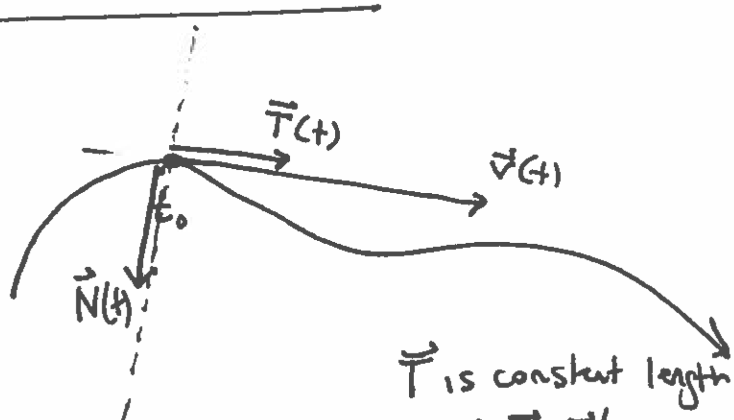
$$\|\vec{r}\,'(t)\| = \sqrt{(2t-1)^2 + (2t+1)^2}$$

$$= \sqrt{8t^2 + 2}$$

So $$\vec{T}(t) = \frac{\langle 2t-1,\ 2t+1\rangle}{\sqrt{8t^2+2}}$$

$$= \left\langle \frac{2t-1}{\sqrt{8t^2+2}},\ \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle$$

Unit Normal Vector



$\vec{T}$ is constant length $\rightarrow \vec{T}, \vec{T}\,'$ are $\perp$.

Want $\vec{N}(t)$ to be the "unit normal vector";

orthogonal to $\vec{T}$.

be in the plane of motion,

generally in direction of any acceleration

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

makes $\vec{N} \perp \vec{T}$.

here so that $\vec{N}$ is unit length

Ex $\quad \vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ 4t \rangle$

Find $\vec{N}(t)$.

tangent: $\quad \vec{v}(t) = \langle -3\sin t,\ 3\cos t,\ 4 \rangle$

$$\vec{T}(t) = \frac{\langle -3\sin t,\ 3\cos t,\ 4 \rangle}{5}$$

$$= \left\langle \frac{-3}{5}\sin t,\ \frac{3}{5}\cos t,\ \frac{4}{5} \right\rangle$$

$$\longrightarrow \quad \frac{d}{dt}\vec{T}(t) = \underbrace{\left\langle \frac{-3}{5}\cos t,\ \frac{-3}{5}\sin t,\ 0 \right\rangle}_{\text{has magnitude}}$$

$$\sqrt{} = \cdots = \frac{3}{5}$$

$$\text{So} \quad \vec{N}(t) = \frac{\frac{d}{dt}\vec{T}(t)}{3/5} = \langle -\cos t,\ -\sin t,\ 0 \rangle$$

- $\vec{N}$ is unit length ✓
- $\vec{N} \perp \vec{T}$? ✓ check $\vec{N} \cdot \vec{T} = 0$
- 

$\vec{N}$ point generally in direction of acceleration.

Ex $\vec{r}(t) = \langle t^2 - t,\ t^2 + t \rangle$

Find $\vec{N}(t)$.

Already found $\vec{T}(t) = \left\langle \dfrac{2t-1}{\sqrt{8t^2+2}},\ \dfrac{2t+1}{\sqrt{8t^2+2}} \right\rangle$

Need $$\vec{T}'(t) = \left\langle \frac{\sqrt{8t^2+2}\,(2) - (2t-1)\cdot\frac{1}{2}(8t^2+2)^{-1/2}(16t)}{8t^2+2},\right.$$

$$\left. \frac{\sqrt{8t^2+2}\,(2) - (2t+1)\,\frac{1}{2}(8t^2+2)^{-1/2}(16t}{8t^2+2} \right\rangle$$

$$\vec{T}'(t) = \left\langle \frac{(8t^2+2)(2) - 8(2t-1)t}{(8t^2+2)^{3/2}},\ \frac{(8t^2+2)(2) - 8(2t+1)t}{(8t^2+2)^{3/2}} \right\rangle$$

$$= \left\langle \frac{16t^2+4-16t^2+8t}{\sim},\ \frac{16t^2+4-16t^2-8t}{\sim} \right\rangle$$

$$= \left\langle \frac{4+8t}{(8t^2+2)^{3/2}},\ \frac{4-8t}{(8t^2+2)^{3/2}} \right\rangle$$

Just want some vector parallel to this, but unit length

parallel to $\approx$ $\langle 4+8t,\ 4-8t \rangle$

So $$\vec{N}(t) = \frac{\langle 4+8t,\ 4-8t \rangle}{\sqrt{(4+8t)^2 + (4-8t)^2}} = \left\langle \frac{4+8t}{\sqrt{32+128t^2}},\ \frac{4-8t}{\sqrt{32+128t^2}} \right\rangle$$

To verify this $\vec{N}$... $\vec{T}(t) \cdot \vec{N}(t) \stackrel{?}{=} 0$

to avoid too much work, let's just check $\vec{T}(1) \cdot \vec{N}(1)$. $\checkmark$

$$\left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle \cdot \left\langle \frac{12}{\sqrt{160}}, \frac{-4}{\sqrt{160}} \right\rangle = 0$$

Also... $\vec{N}$ is unit length?

$$N(1) = \left\langle \frac{12}{\sqrt{160}}, \frac{-4}{\sqrt{160}} \right\rangle$$

has mag $\sqrt{\frac{144}{160} + \frac{16}{160}} = 1 \checkmark$