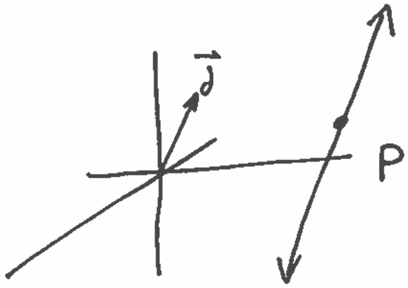
10.5 Lines in $\mathbb{R}^3$



vector Equation for a line

$$L(t) = P + \underbrace{t}_{\text{run through } \mathbb{R}} \cdot \vec{d}$$

Ex

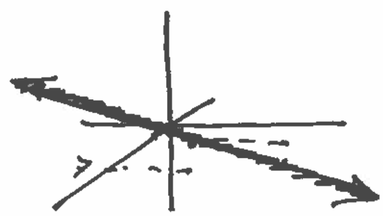
Find a vector equation for line through $\underbrace{(7,5,3)}_{\text{choose this for } P}$ and $(2,8,-2)$.

Take $\vec{d} = \overrightarrow{PQ}$

$\vec{d} = \langle 2-7, 8-5, -2-3 \rangle$

$\vec{d} = \langle -5, 3, -5 \rangle$

$$\Longrightarrow L(t) = (7,5,3) + t\langle -5, 3, -5 \rangle$$



Ex Find a vector equation for the line passing through $(1,8,-3)$, perpendicular to

$\ell_2: (0,0,1) + t\langle 2,1,0 \rangle$ $\qquad \vec{d}_2 = \langle 2,1,0 \rangle$

$\ell_3: (1,8,-1) + t\langle 3,2,1 \rangle$ $\qquad \vec{d}_3 = \langle 3,2,1 \rangle$

$\vec{d}_1 = \vec{d}_2 \times \vec{d}_3$ will be $\perp$.

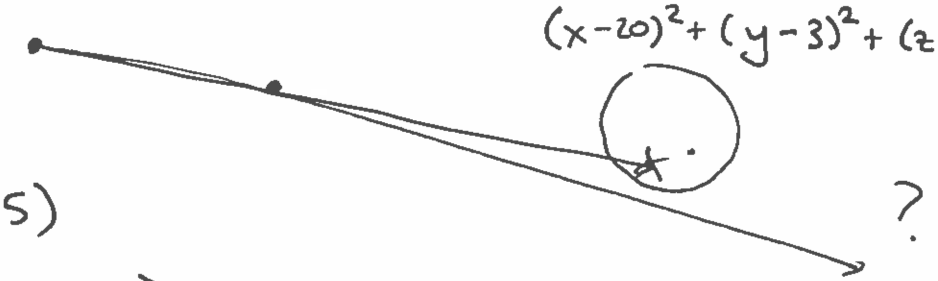
$$= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \hat{\imath} + 4\hat{k} - 2\hat{\jmath} - 3\hat{k} = \langle 1,-2,1 \rangle$$

$$L(t) = (1,8,-3) + t\langle 1,-2,1 \rangle$$

Ex Line through $(1,3,5)$ and $(10,2,8)$.

Does it hit the sphere centered at $(20,3,10)$ with radius 4 ?

$$(x-20)^2+(y-3)^2+(z-10)^2=16$$



$P=(1,3,5)$

$\vec{d}=\langle 9,-1,3\rangle$

$L(t)=(1,3,5)+t\langle 9,-1,3\rangle$

$$\begin{aligned} x&=1+9t\\ y&=3-t\\ z&=5+3t \end{aligned} \implies (1+9t-20)^2+(3-t-3)^2+(5+3t-10)^2 \overset{?}{=} 16$$

$$(9t-19)^2+t^2+(3t\ \cancel{}\ -5)^2=16$$

$$\begin{aligned} &81t^2-342t+361\\ &+t^2\\ &+9^2-30t+25=16 \end{aligned}$$

$$91t^2-372t+\cancel{386}\ 370=0$$

Discriminant: $\cancel{}\ (-372)^2-4(91)(370)$

$=$ positive!

$\implies$ 2 solutions

$\implies$ 2 times where we have intersection.

$L(t) = (1,3,-4) + t\langle 2,5,9\rangle$

vector equation for this line

$x = 1+2t$
$y = 3+5t$
$z = -4+9t$

parametric equations for the line

(scalar equations for the line)

attempt to eliminate t.

$\frac{x-1}{2} = t$

$\frac{y-3}{5} = t$

$\frac{z+4}{9} = t$

$$\frac{x-1}{2} = \frac{y-3}{5} = \frac{z+4}{9}$$

symmetric equations for the line

Ex
$x = 5-3t$
$y = -1+t$
$z = t$

$\xrightarrow[\text{equation}]{\text{Give vector}}$ $L(t) = (5,-1,0) + t\langle -3,1,1\rangle$

let $t=0$

Coefficients of t

Symmetric (isolate t)

$$\frac{x-5}{-3} = y+1 = z$$

Two lines: could be same

could be parallel
$\Big\}$ make a plane
could intersect

could be skew

Ex

Determine if $\begin{aligned} x &= 1+3t \\ y &= 2-t \\ z &= t \end{aligned}$ & $\begin{aligned} x &= -2+4t \\ y &= 3+t \\ z &= 5+2t \end{aligned}$

$\vec{d}_1 = \langle 3,-1,1 \rangle$ $\qquad \vec{d}_2 = \langle 4,1,2 \rangle$

are they parallel?

No! $c\langle 3,-1,1 \rangle \neq \langle 4,1,2 \rangle$

Do they cross?

Rewrite: $\begin{aligned} x &= -2+4s \\ y &= 3+s \\ z &= 5+2s \end{aligned} \implies \begin{aligned} 1+3t &= -2+4s \quad (\text{same } x) \\ 2-t &= 3+s \quad (\text{same } y) \\ t &= 5+2s \quad (\text{same } z) \end{aligned}$

Add

$2 = 8+3s$

$-6 = 3s$

$-2 = s \implies t = 5+2(-2)$

$t = 1$

First Eq: $1+3(1) \overset{?}{=} -2+4(-2)$

$4 = -10$

Not same!

<u>We have skew lines.</u>

Ex

$$x = -0.7 + 1.6t \qquad x = 2.8 - 2.9t$$
$$y = 4.2 + 2.72t \qquad y = 10.15 - 4.93t$$
$$z = 2.3 - 3.36t \qquad z = -5.05 + 6.09t$$

same, parallel, cross once, skew?

$\langle 1.6, 2.72, -3.36 \rangle$ $\qquad$ $\langle -2.9, -4.93, 6.09 \rangle$

parallel?

$$c \cdot 1.6 = -2.9$$
$$c = \frac{-2.9}{1.6}$$
$$\underline{c = -1.8125}$$

$$(-1.8125)(2.72) = -4.93$$
$$(-1.8125)(-3.36) = 6.09$$

Yes! But same or parallel?

l_1: $t=0$

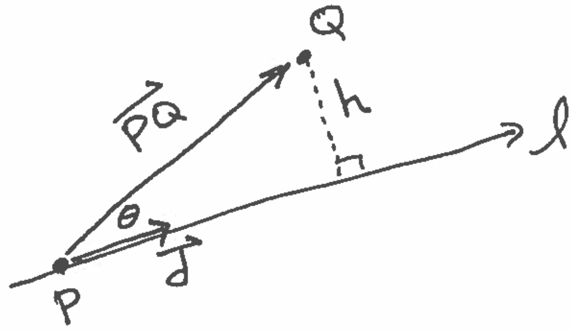
$$(-0.7, 4.2, 2.3) \Longrightarrow \begin{cases} -0.7 = 2.8 - 2.9t \\ 4.2 = 10.15 - 4.93t \\ 2.3 = -5.05 + 6.09t \end{cases}$$

$$\Longrightarrow \begin{cases} t = \frac{3.5}{2.9} \\ t = \frac{-5.95}{-4.93} \\ t = \frac{7.35}{6.09} \end{cases}$$

same ✓

So lines are the same!

<u>Distances</u>



distance from Q to ℓ is length of shortest segment connecting. where dashed line is $\perp$ to L

$$\sin\theta = \frac{h}{\|\vec{PQ}\|}$$

$$\Longrightarrow h = \|\vec{PQ}\|\cdot\sin\theta$$

$$\Longrightarrow h\cdot\|\vec{d}\| = \underbrace{\|\vec{d}\|\cdot\|\vec{PQ}\|\cdot\sin\theta}$$

$$\Longrightarrow h\cdot\|\vec{d}\| = \|\vec{d}\times\vec{PQ}\|$$

$$\Longrightarrow h = \frac{\|\vec{d}\times\vec{PQ}\|}{\|\vec{d}\|}$$

$\vec{d}$

<u>Ex</u> Line

$(1,1,2)+t\langle 1,0,3\rangle$

what is the distance to $(5,-2,1)$?

P $\qquad$ Q

$$\vec{PQ} = \langle 4,-3,-1\rangle$$

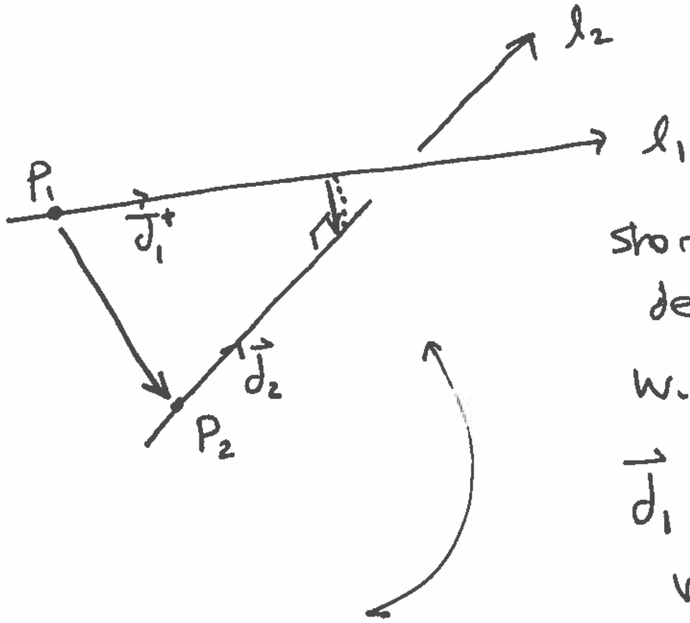
$$\vec{d}\times\vec{PQ} = \text{determinant}\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & 3\\ 4 & -3 & -1\end{bmatrix} = 12\hat{j}-3\hat{k}+9\hat{i}+\hat{j}$$

$$= \langle 9,13,-3\rangle$$

So $\text{distance} = \dfrac{\|\langle 9,13,-3\rangle\|}{\|\langle 1,0,3\rangle\|}$

$$= \frac{\sqrt{81+169+9}}{\sqrt{1+0+9}} = \frac{\sqrt{259}}{\sqrt{10}}$$

Distance between two lines



shortest line segment defined distance...

Will be $\perp$ to both lines...

$\vec{d}_1 \times \vec{d}_2$ to get a generic vector $\perp$ to both lines...

$\text{proj}_{(\vec{d}_1 \times \vec{d}_2)}(\overrightarrow{P_1P_2})$ represents this shortest path.

$$h = \left\| \text{proj}_{(\vec{d}_1 \times \vec{d}_2)}(\overrightarrow{P_1P_2}) \right\|$$

$$= \left\| \frac{\overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2)}{(\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_1 \times \vec{d}_2)} (\vec{d}_1 \times \vec{d}_2) \right\|$$

$$= \left| \frac{\overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2)}{(\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_1 \times \vec{d}_2)} \right| \left\| \vec{d}_1 \times \vec{d}_2 \right\| = \frac{\left| \overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2) \right|}{\left\| \vec{d}_1 \times \vec{d}_2 \right\|}$$

Ex

$x = 1 + 2t$	$x = 3 - t$	Find distance between...
$y = 3 - t$	$y = 1 + t$	
$z = 4$	$z = 1 + 2t$	

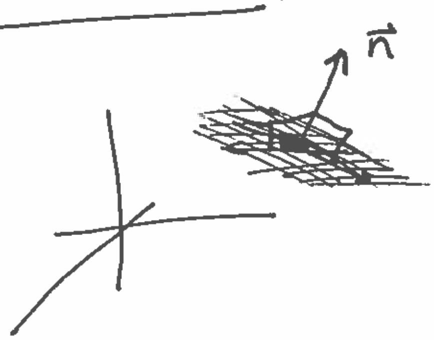
$P_1 = (1, 3, 4)$ $\quad P_2 = (3, 1, 1) \Rightarrow \overrightarrow{P_1P_2} = \langle 2, -2, -3 \rangle$

$\vec{d}_1 = \langle 2, -1, 0 \rangle$ $\quad \vec{d}_2 = \langle -1, 1, 2 \rangle \Rightarrow \vec{d}_1 \times \vec{d}_2 = \langle -2, -4, 1 \rangle$

$\Rightarrow \overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2) = -4 + 8 - 3 = 1$

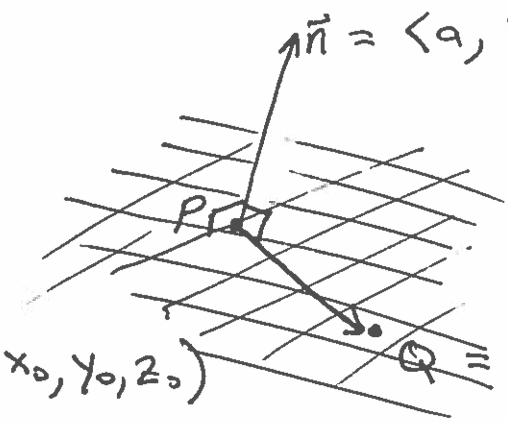
So distance $= \dfrac{1}{\|\langle -2,-4,1\rangle\|} = \dfrac{1}{\sqrt{4+16+1}} = \dfrac{1}{\sqrt{21}}$

10.6 Planes



To describe a plane, need one point...
and ~~two directions~~...
its normal vector
(a vector $\perp$ to any vector parallel to the plane.)

Lead to an equation...



$P = (x_0, y_0, z_0)$ $\quad Q = (x, y, z)$ lying on the plane

We must have that: $\overrightarrow{PQ} \cdot \vec{n} = 0$

$$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle = 0$$

Standard Eq for a plane. $\longleftarrow$ $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Ex $2(x-3) + 5y - 3(z+1) = 0$ $\longrightarrow$ Plane through $(3, 0, -1)$ with normal vector $\langle 2, 5, -3\rangle$

Ex Do $\begin{aligned} x &= -5+2t \\ y &= 1+t \\ z &= -4+2t \end{aligned}$ and $\begin{aligned} x &= 2+3t \\ y &= 1-2t \\ z &= 1+t \end{aligned}$ intersect?

If so, give equation for the plane containing them?

Try to Solve

$$\begin{cases} -5+2t = 2+3s \\ 1+t = 1-2s \\ -4+2t = 1+s \end{cases}$$

$$\begin{aligned} -2-2t &= -2+4s \\ -4+2t &= 1+s \\ \hline -6 &= -1+5s \\ -5 &= 5s \\ -1 &= s \end{aligned}$$

Last eq: $-4+2t = 1+(-1)$

$$\begin{aligned} 2t &= 4 \\ t &= 2 \end{aligned}$$

In First?

$$\begin{aligned} -5+2(2) &\overset{?}{=} 2+3(-1) \\ -5+4 &= 2-3 \\ -1 &= -1 \end{aligned}$$

✓

Now, find $\vec{n}$. $\underset{\vec{d}_1}{\langle 2,1,2\rangle} \times \underset{\vec{d}_2}{\langle 3,-2,1\rangle} = \langle 5,4,-7\rangle$

$P = (-5,1,-4)$

So $$5(x+5)+4(y-1)-7(z+4)=0$$

$$5x+4y-7z=7$$

Ex $\qquad \dfrac{x-1}{2} = \dfrac{y-1}{3} = \dfrac{z-1}{3}$

A plae passes trough

$P = (2, 1, -2)$, perpendicular to this

Write plane's eq. $\qquad \vec{d} = \langle 2, 3, 3 \rangle$

$\vec{n} = \langle 2, 3, 3 \rangle$

So: $2(x-2) + 3(y-1) + 3(z+2) = 0$

$2x + 3y + 3z = 1$

Ex How do

$x + (y-3) + (z+1) = 0 \implies \vec{n}_1 = \langle 1, 1, 1 \rangle$

and $2x + 3y - 2(z-2) = 0 \implies \vec{n}_2 = \langle 2, 3, -2 \rangle$

interact? (~~parallel~~, ~~same plane~~, line ...)

not parallel

Now, find equations for that line!

Need one pt on line... Find some point in <u>both</u> planes.

Freedom to set $x = 0$. $\implies \begin{cases} y - 3 + z + 1 = 0 \\ 3y - 2z + 4 = 0 \end{cases}$

$\begin{cases} y + z = 2 \\ 3y - 2z = -4 \end{cases}$

$\begin{cases} 2y + 2z = 4 \\ 3y - 2z = -4 \end{cases}$

$\begin{cases} 5y = 0 \\ y = 0 \end{cases}$

$\Downarrow$

$z = 2$

$(0, 0, 2)$

Now need line's direction...

We have $\vec{n}_1 = \langle 1,1,1 \rangle$
$\vec{n}_2 = \langle 2,3,-2 \rangle$

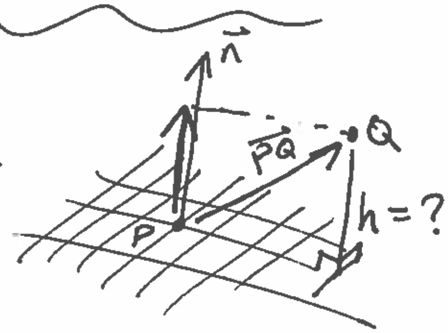
we want $\vec{d}$ lives <u>parallel</u> to both planes.

$\Longrightarrow$ $\vec{d}$ is $\perp$ to both normal vectors.

$$\vec{d} = \langle 1,1,1 \rangle \times \langle 2,3,-2 \rangle = \langle -5,4,1 \rangle$$

So $L(t) = (0,0,2) + t\langle -5,4,1 \rangle$.

<u>Distance</u>



Find distance from Q to plane...

$\uparrow$ is

$\text{proj}_{\vec{n}}(\vec{PQ})$ $\Longrightarrow$ $h = \left\| \text{proj}_{\vec{n}}(\vec{PQ}) \right\|$

$$= \left\| \frac{\vec{PQ} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \cdot \vec{n} \right\|$$

$$= \left| \frac{\vec{PQ} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \right| \cdot \|\vec{n}\| = \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}$$

<u>Ex</u> Distance from $(2,2,5)$ to $3x - 2y + 5z = 12$.

Q

$\vec{n} = \langle 3,-2,5 \rangle$

make up P: $(4,0,0)$ $\longrightarrow$ $\vec{PQ} = \langle -2,2,5 \rangle$ $\Rightarrow$ $\underbrace{\langle -2,2,5 \rangle}_{\vec{PQ}} \cdot \underbrace{\langle 3,-2,5 \rangle}_{\vec{n}}$

$= -6 - 4 + 25$
$= 15$

Also $\|\vec{n}\| = \sqrt{9+4+25} = \sqrt{38}$

So distance $= \dfrac{15}{\sqrt{38}}$.