10.3 Dot Product

$\overrightarrow{\text{vector}} \cdot \overrightarrow{\text{vector}} = $ scalar result (plain ~~scalar~~ number)

dot product for multiplying two vectors

Note

$\underbrace{\vec{u} \cdot \vec{v}} \cdot \vec{w}$ makes no sense!

dot products

$\vec{u} = \langle u_1, u_2 \rangle$
$\vec{v} = \langle v_1, v_2 \rangle$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

$\vec{u} = \langle u_1, u_2, u_3 \rangle$
$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Ex

$$\langle 1,3 \rangle \cdot \langle 5,-1 \rangle = 1 \cdot 5 + 3(-1) = 5 - 3 = 2$$

$$\langle 2,0,8 \rangle \cdot \langle 3,12,-7 \rangle = 6 + 0 - 56 = -50$$

Facts

$$\vec{0} \cdot \vec{v} = 0$$

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$$

regular number × number multiplication

dot product

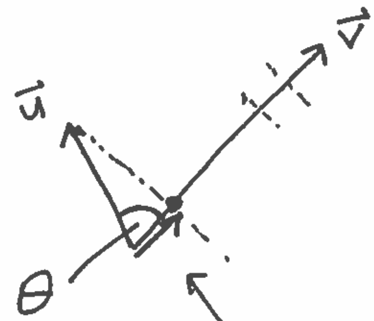
scalar-vector multiplication

dot

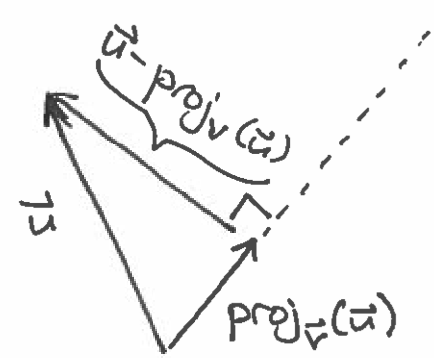
Orthogonal Projection

$\underbrace{\text{Ortho}}_{\text{straight}}\underbrace{\text{gonal}}_{\text{angle}}$

(right angle)



"project $\vec{u}$ onto $\vec{v}$, using a vector orthogonal to $\vec{v}$." $\Big\}$ $\text{proj}_{\vec{v}}(\vec{u})$



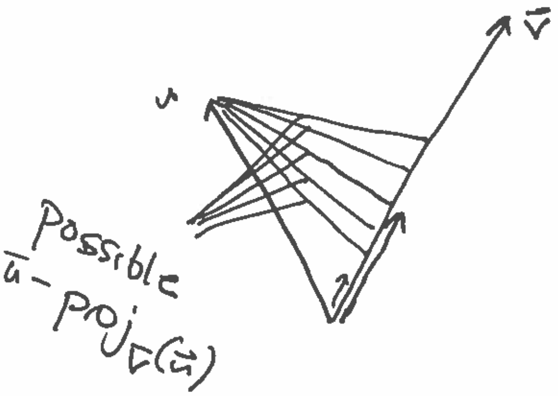
Note: $\text{proj}_{\vec{v}}(\vec{u})$ is parallel to $\vec{v}$.

$\vec{u}-\text{proj}_{\vec{v}}(\vec{u})$ is perpendicular to $\vec{v}$ (orthogonal)

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}(\vec{u})}_{\text{parallel to } \vec{v}} + \underbrace{\left(\vec{u} - \text{proj}_{\vec{v}}(\vec{u})\right)}_{\substack{\text{simplify, and} \\ \text{this is orthogonal} \\ \text{to } \vec{v}.}}$$

<u>Decomposes</u> $\vec{u}$ into a parallel component & a perpendicular component with respect to $\vec{v}$.

We know $\text{proj}_{\vec{v}}(\vec{u}) = \underset{?}{\underset{\uparrow}{c}} \cdot \vec{v}$ because its parallel to $\vec{v}$.



Choose c to minimize

$$\|\vec{u} - \text{proj}_{\vec{v}}(\vec{u})\|$$

(will give us a right angle)

Same as minimizing

$$\|\vec{u} - \text{proj}_{\vec{v}}(\vec{u})\|^2$$

$$= (\vec{u} - \text{proj}_{\vec{v}}(\vec{u})) \cdot (\vec{u} - \text{proj}_{\vec{v}}(\vec{u}))$$

$$= \vec{u} \cdot \vec{u} - 2\,\vec{u} \cdot \underbrace{\text{proj}_{\vec{v}}(\vec{u})}_{c \cdot \vec{v}} + \underbrace{\text{proj}_{\vec{v}}(\vec{u})}\cdot \underbrace{\text{proj}_{\vec{v}}(\vec{u})}$$

$$= \vec{u} \cdot \vec{u} - 2c\,\vec{u} \cdot \vec{v} + c^2\,\vec{v} \cdot \vec{v}$$

$\vec{u}, \vec{v}$ are fixed... but what c will make the expression minimal.

This is quadratic in c....

$$-\frac{B}{2A} = \frac{-(-2\vec{u} \cdot \vec{v})}{2(\vec{v} \cdot \vec{v})}$$ is where the extreme value is for c.

$$c = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

So $$\boxed{\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\,\vec{v}}$$

$\underline{\text{Ex}}$ $\vec{u} = \langle 1,2,3 \rangle$ Calculate $\text{proj}_{\vec{v}}(\vec{u})$.

$\vec{v} = \langle 4,5,6 \rangle$

$$= \frac{\langle 1,2,3\rangle \cdot \langle 4,5,6\rangle}{\langle 4,5,6\rangle \cdot \langle 4,5,6\rangle} \langle 4,5,6 \rangle$$

$$= \frac{4+10+18}{16+25+36} \langle 4,5,6 \rangle$$

$$= \frac{32}{77} \langle 4,5,6 \rangle$$

$$= \left\langle \frac{128}{77}, \frac{160}{77}, \frac{192}{77} \right\rangle$$

Also, $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$

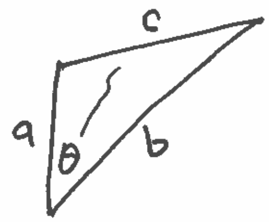
$$= \langle 1,2,3 \rangle - \left\langle \frac{128}{77}, \frac{160}{77}, \frac{192}{77} \right\rangle$$

$$= \left\langle \frac{-51}{77}, \frac{-6}{77}, \frac{39}{77} \right\rangle$$

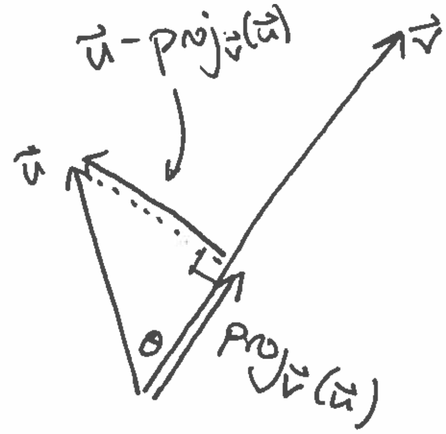
So $\underset{\langle 1,2,3 \rangle}{\vec{u}} = \underbrace{\left\langle \frac{128}{77}, \frac{160}{77}, \frac{192}{77} \right\rangle}_{\text{parallel to } \vec{v}} + \underbrace{\left\langle \frac{-51}{77}, \frac{-6}{77}, \frac{39}{77} \right\rangle}_{\substack{\text{perpendicular to } \vec{v} \\ \text{(orthogonal)}}}$

(we decomposed $\vec{u}$ with respect to $\vec{v}$.)

Law of Cosines



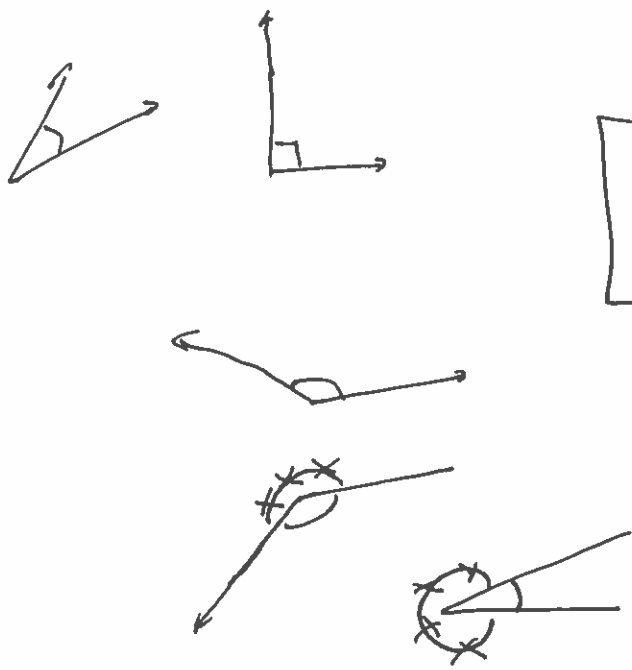
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



$$\left\|\vec{u} - \mathrm{proj}_{\vec{v}}(\vec{u})\right\|^2 = \|\vec{u}\|^2 + \|\mathrm{proj}_{\vec{v}}(\vec{u})\|^2 - 2\|\vec{u}\|\cdot\|\mathrm{proj}_{\vec{v}}(\vec{u})\|\cos\theta$$

solve for θ

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}}$$

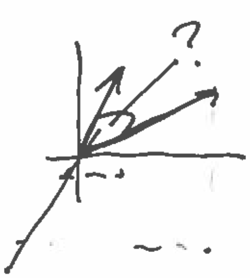


angles stuck between ~~$[0, 2\pi]$~~ $[0, \pi]$.

or

$$\boxed{\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}\right)}$$

Ex Find the angle between $\langle 1,2,3\rangle$ and $\langle 4,5,6\rangle$.



$$\cos\theta = \frac{4+10+18}{\sqrt{1+4+9}\cdot\sqrt{16+25+36}} = \frac{32}{7\sqrt{22}}$$

$$\theta = \arccos\left(\frac{32}{7\sqrt{22}}\right)$$

Implications: $\vec{u} \cdot \vec{v} = 0 \implies \cos\theta = \frac{0}{\|\vec{u}\|\|\vec{v}\|} = 0$

$\implies \theta = \underbrace{\pi/2}_{\text{right-angle}}$

Find a vector perpendicular to $\langle 1, 3 \rangle$.

imagine $\longrightarrow \langle 3, -1 \rangle$

perp to $\langle 2, 0, 8 \rangle$?

$\langle -4, 0, 1 \rangle$, $\langle -4, 15, 1 \rangle$

$\vec{u} \cdot \vec{v} = 0 \iff \vec{u}, \vec{v}$ are orthogonal

---

$\vec{u} \cdot \vec{v} > 0 \implies \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{u}\|} > 0$

$\theta < \pi/2$

So an <u>acute angle</u>.

$\vec{u} \cdot \vec{v} < 0 \implies \cos\theta < 0$

θ in $[\pi/2, \pi]$

<u>obtuse angle</u>.

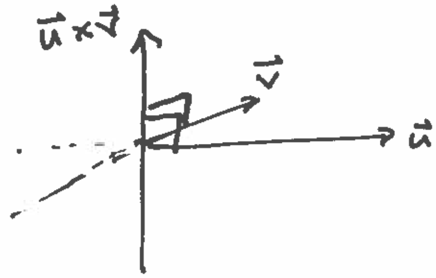
10.4 Cross Product

$\vec{u} \times \vec{v}$ = result is another vector in $\mathbb{R}^3$.

$\vec{u}, \vec{v}$ vectors in $\mathbb{R}^3$.

Cross product has many (equivalent) definitions.

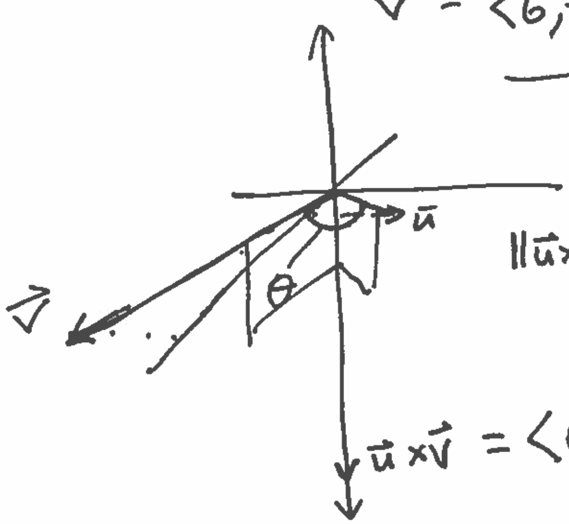
- If $\vec{u}, \vec{v}$ are parallel, then $\vec{u} \times \vec{v} = \vec{0}$.
  Otherwise, $\vec{u} \times \vec{v}$ is a vector whose magnitude is $\|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$, is perpendicular to <u>both</u> $\vec{u}$ & $\vec{v}$, and such that



$\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ respects the RHR

<u>Ex</u> If $\vec{u} = \langle 1, 2, 0 \rangle$, $\vec{v} = \langle 6, -3, 0 \rangle$ find $\vec{u} \times \vec{v}$.

both in xy-plane ... $\vec{u} \times \vec{v}$ is in z-direction



$\|\vec{u} \times \vec{v}\| = \sqrt{5} \cdot \sqrt{45} \cdot 1$
$= 15$

$\|\vec{u}\| = \sqrt{5}$, $\|\vec{v}\| = \sqrt{45}$
$\sin\theta$?
$\cos\theta = \dfrac{6-6}{mn} = 0$
$\theta = \pi/2$
$\sin\theta = 1$

$\vec{u} \times \vec{v} = \langle 0, 0, \text{neg } \# \rangle$

-15

So $\langle 1,2,0\rangle \times \langle 6,-3,0\rangle = \langle 0,0,-15\rangle$

- Matrix-based

$$\vec{u}\times\vec{v} = \text{determinant}\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

$$\left.\begin{aligned} &= u_2v_3\hat{\imath} + u_3v_1\hat{\jmath} + u_1v_2\hat{k} \\ &\quad - u_3v_2\hat{\imath} - u_1v_3\hat{\jmath} - u_2v_1\hat{k} \end{aligned}\right\}\text{ simplify}$$

Ex $\langle 2,-2,4\rangle \times \langle 1,6,-2\rangle$

$$= \text{determinant}\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 4 \\ 1 & 6 & -2 \end{pmatrix} = \overset{-2\cdot(-2)}{4\hat{\imath}} + 4\hat{\jmath} + 12\hat{k} - 24\hat{\imath} - (-4\hat{\jmath}) - (-2\hat{k})$$

$$= -20\hat{\imath} + 8\hat{\jmath} + 14\hat{k}$$

$$= \langle -20, 8, 14\rangle$$

(partial check
$\langle 2,-2,4\rangle \cdot \langle -20,8,14\rangle$ ✓
$= -40 - 16 + 56 = 0$)

- Formula to memorize:

$$\vec{u}\times\vec{v} = \langle u_2v_3 - u_3v_2,\ \underbrace{u_3v_1 - u_1v_3}_{-(u_1v_3 - u_3v_1)},\ u_1v_2 - u_2v_1\rangle$$

Ex $\langle 7,0,2\rangle \times \langle 1,3,-4\rangle = \langle 0\cdot(-4) - 2(3),\ 2\cdot 1 - 7(-4),\ 7\cdot 3 - 0\cdot 1\rangle$

$$= \langle -6, 30, 21\rangle$$

Facts:

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

$$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$

$$c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$$

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$$

$$(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$$

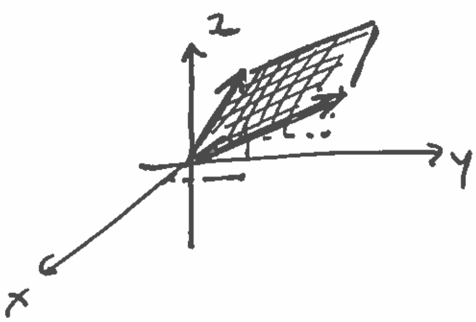
$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w} \quad \text{(Triple Scalar Product)}$$

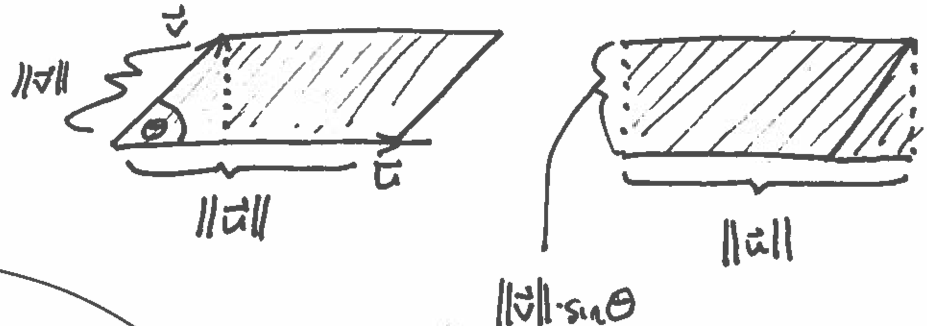
$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$$

Simple Applications

Ex Use $\langle 1,2,3 \rangle$ and $\langle -1,2,2 \rangle$ to form a parallelogram.

How much area?





$$A = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$$

$$A = \|\vec{u} \times \vec{v}\|$$

So

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 2 \end{bmatrix}$$

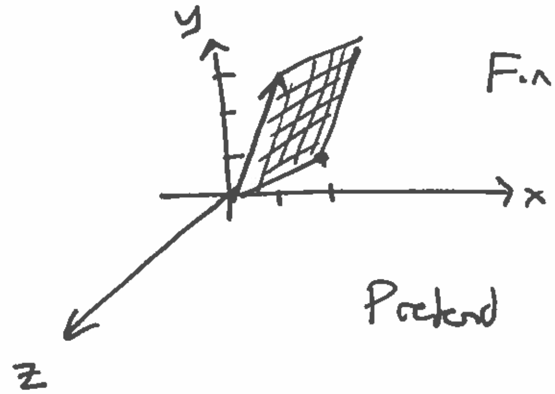
$$= \hat{\imath} \cdot 4 - 3\hat{\jmath} + 2\hat{k} - 6\hat{\imath} - 2\hat{\jmath} - (-2\hat{k}) = -2\hat{\imath} - 5\hat{\jmath} + 4\hat{k}$$

$$= \langle -2, -5, 4 \rangle$$

$$\|\langle -2,-5,4 \rangle\| = \sqrt{4+25+16} = \sqrt{45} = 3\sqrt{5}$$

In 2D...

Ex $\vec{u} = \langle 1,3 \rangle$
$\vec{v} = \langle 2,1 \rangle$



Find area...

Pretend $\langle 1,3,0 \rangle$
$\langle 2,1,0 \rangle$

$$\langle 1,3,0 \rangle \times \langle 2,1,0 \rangle$$

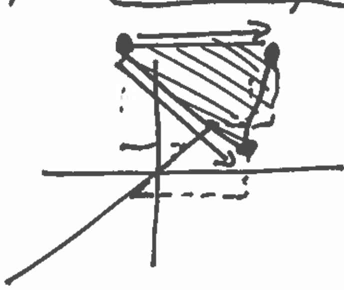
$$= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= 0\hat{i} + \hat{j}\cdot 0 + \hat{k}$$
$$-0\hat{i} - 0\hat{j} - 6\hat{k} = -5\hat{k}$$
$$= \langle 0,0,-5 \rangle$$

$\| \langle 0,0,-5 \rangle \| = 5.$ (So parallelogram has area 5.

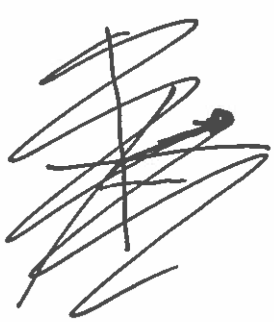
Ex In space, $\underbrace{(1,3,1), \underbrace{(-1,-2,3)}, (-2,1,2)}$,

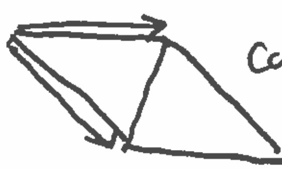
What's the area of the triangle?



$\overrightarrow{A\underline{B}} = \langle -2,-5,2 \rangle$

$\overrightarrow{C\underline{B}} = \langle 1,-3,1 \rangle$





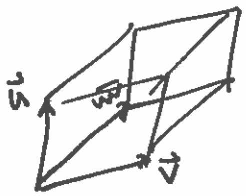
cut parallelogram area in half.

Now $\overrightarrow{AB} \times \overrightarrow{CB} = \langle 1,4,11 \rangle$

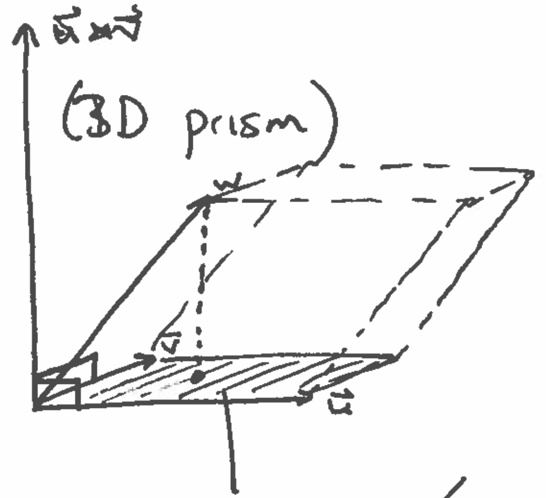
Area of $\underline{\text{par}} = \sqrt{1+16+121} = \sqrt{138}$

Area of $\Delta = \frac{1}{2}\cdot\sqrt{138}$

Volume of a parallelepiped (3D prism)



$\longrightarrow$ Lay Flat



Base Area ✓

$\|\vec{u}\times\vec{v}\|$

$$\text{Volume} = \underset{\text{Area}}{\overset{\checkmark}{\text{Base}}} \cdot \text{Height}$$

$$\text{Height} = \text{proj}_{(\vec{u}\times\vec{v})}(\vec{w})$$

$$= \|\vec{u}\times\vec{v}\| \cdot \left\|\text{proj}_{(\vec{u}\times\vec{v})}(\vec{w})\right\|$$

$$= \|\vec{u}\times\vec{v}\| \cdot \left\| \frac{(\vec{u}\times\vec{v})\cdot\vec{w}}{(\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{v})} (\vec{u}\times\vec{v}) \right\|$$

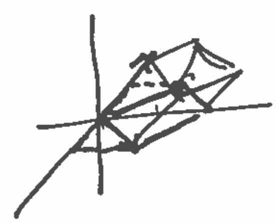
$$\text{Volume} = \cancel{\|(\vec{u}\times\vec{v})\cdot\vec{w}\|}$$

$$\boxed{V = \left|(\vec{u}\times\vec{v})\cdot\vec{w}\right|}$$

Ex

$\langle 1,1,0\rangle$
$\langle -1,1,0\rangle$
$\langle 0,1,1\rangle$

cross prod has length $\sqrt{2}\sqrt{2}\cdot 1 = 2$

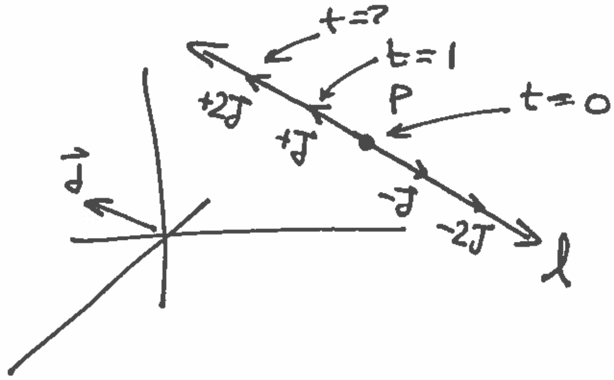


$$V = \left|\langle 0,0,2\rangle \cdot \langle 0,1,1\rangle\right|$$

$$= |0+0+2|$$

$$= 2$$

10.5 Lines in Space



Need a starting point $P = (x_0, y_0, z_0)$

And a direction to travel, $\vec{d}$.

l is given by: $\cancel{\vec{l}(t) = P + t\cdot\vec{d}}$

vector equation for a line.

$$L(t) = P + t\cdot\vec{d}$$

points (locations)

vector

$$\vec{l}(t) = \vec{p} + t\cdot\vec{d}$$

vectors

Ex Find the vector equation for a line passing through $(7, 2, -2)$ parallel to $\langle 3, -1, -3\rangle$.

$$L(t) = (7, 2, -2) + t\cdot\langle 3, -1, -3\rangle$$

Ex Find ~~the~~ a vector equation for the line passing through $(1,8,-3)$ and is perpendicular to both $\langle 2,1,0\rangle$ and $\langle 3,2,1\rangle$.

$$L(t) = \overset{\checkmark}{\vec{P}} + t\cdot\vec{d}$$

their cross-product!

$$\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{array}$$

$$\longrightarrow \hat{i} + 0\hat{j} + 4\hat{k} - 0\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\vec{d} = \langle 1,-2,1\rangle$$

$$L(t) = (1,8,-3) + t\langle 1,-2,1\rangle$$

Ex Find a vector eq. for the line perpendicular to

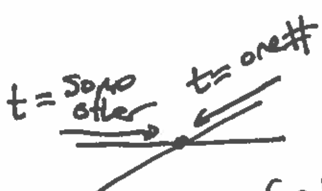
$$K(t) = (1,1,-2) + t\langle 3,-1,1\rangle$$
$$M(t) = (7,1,3) + t\langle -6,-2,-8\rangle$$

and passes through their intersection.

$$L(t) = P + t\cdot\vec{d}$$

P: ?

$\vec{d}$: $\langle 3,-1,1\rangle \times \langle -6,-2,-8\rangle$



Set $(1,1,-2) + t\langle 3,-1,1\rangle = (7,1,3) + s\langle -6,-2,-8\rangle$

$$\left.\begin{array}{l} 1+3t = 7-6s \\ 1-t = 1-2s \end{array}\right\} \Longrightarrow \text{Can solve for } s,t.$$

Use to find P