The Alternating Series Test

1. Use the Alternating Series Test to determine the convergence status of the following series. It is important that you verify the conditions of the Alternating Series Test are met; otherwise someone might not believe your conclusion is valid. And yes, you have to brush up your integration skills!

a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n}{n^2+1}$ Check conditions: is a_n an alternating sequence? Using the notational convention $b_n = |a_n|$, is b_n (eventually) a decreasing sequence? Does $\lim_{n\to\infty} b_n$ equal 0? You have to *explain* why b_n is decreasing and converges to 0, not merely assert that these things are true.

$b_n = \dfrac{n}{n^2+1}$

$\dfrac{d}{dn}\left(\dfrac{n}{n^2+1}\right) = \dfrac{1-n^2}{(n^2+1)^2}$

is negative for n in $[2,\infty)$

So b_n is decreasing.

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{1/n}{1+\frac{1}{n^2}}$

$= \dfrac{0}{1+0} = 0$

So by the Alternating Series Test, $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n}{n^2+1}$ converges.

b) $\displaystyle\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{n^2+1}$

$b_n = \dfrac{1}{n^2+1}$

b_n is clearly decreasing.

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\dfrac{1}{n^2+1} = 0$

So by the Alternating Series Test, $\displaystyle\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{n^2+1}$ is convergent.

c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+3}ne^{-n/8}$

$b_n = \dfrac{n}{e^{n/8}}$

$\dfrac{d}{dn}\left(\dfrac{n}{e^{n/8}}\right) = \left(1-\dfrac{n}{8}\right)e^{-n/8}$

is negative for n in $[9,\infty)$.

So b_n is eventually decreasing.

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\dfrac{n}{e^{n/8}}$

$= \lim\limits_{n\to\infty}\dfrac{1}{\frac{1}{8}e^{n/8}} = 0$

So by the A.S.T., $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+3}ne^{-n/8}$ is convergent.

d) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{(-n)^n}$

$b_n = \dfrac{n!}{n^n} \geq \dfrac{n!}{(n+1)^n} = \dfrac{(n+1)!}{(n+1)^{n+1}} = b_{n+1}$

So b_n is decreasing.

$\lim\limits_{n\to\infty} b_n = 0$ by the squeeze theorem since $0 < \dfrac{n!}{n^n} < \dfrac{1}{n}$.

So by the AST, $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{(-n)^n}$ converges.

2. Could you use the Alternating Series Test for each of the following? This question is about whether the conditions for the Alternating Series Test are met. Are the conditions met? Why or why not?

a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\arctan(n)}$ b) $\displaystyle\sum_{n=0}^{\infty}(-1)^{[2\cos(n\pi/3)]}0.9^n$ c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1+\cos(n)}{n}$

$\lim_{n\to\infty} b_n = \frac{4}{\pi} \neq 0$

Not an alternating series. Pattern is
$+ + - - - + + + - - - \cdots$

b_n is not decreasing.

3. This exploration is intended to demonstrate why the conditions for using the Alternating Series Test are important to check. Consider the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\cos^2(n\pi/2)n+\sin^2(n\pi/2)2^n}$$

a) The terms of this series are in fact nice fractions. Compute the first ten terms of the series as a way to more quantitatively understand the series.

$\frac{1}{2} - \frac{1}{2} + \frac{1}{8} - \frac{1}{4} + \frac{1}{32} - \frac{1}{6} + \frac{1}{128} - \frac{1}{8} + \frac{1}{512} - \frac{1}{10}$

b) Is the series alternating?

yes

c) Does b_n (the absolute value of the sequence terms) converge to 0?

yes. numerator always 1. denominator either n or 2^n.

d) Based on your ten terms, is b_n decreasing? You should see that it is not...

No...

e) This series in fact diverges and we will show that now. But the important things to see is that it *was* an alternating series and the terms *do* approach 0. We're not contradicting the Alternating Series Test because the b_n were not decreasing. The lesson is that checking this condition (and all conditions with all tests) really can make all the difference. Show that $\cancel{a_{2n-1}+a_{2n} = \frac{1}{2n-1} - \frac{1}{2^{2n}} > \frac{1}{2n}}$. So the sequence of partial sums for our series would have a subsequence (every other partial sum) that is larger than one half of the partial sum sequence of the Harmonic series (which we know diverges to ∞.)

$a_{2n-1}+a_{2n} = \frac{1}{2^{2n-1}} - \frac{1}{2n}$ is negative... more negative than $-\frac{1}{2n-2}$.

So this series has a subsequence of partial sums growing negative faster than a Harmonic Series.