MTH 111–112 Supplement

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Contents

Chapter 1

MTH 111 Supplement

1.1 Graph Transformations

Example 1.1.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 1.1.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x) + 5$									
$f(x + 2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x - 3)$									

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

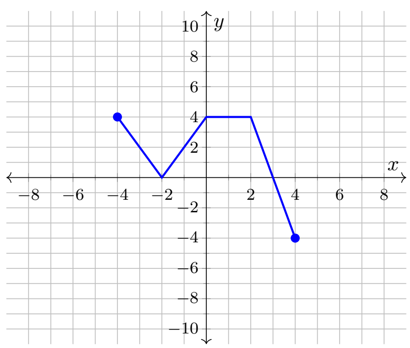
6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

7. $f(x) = \frac{1}{x}$ $\quad g(x) = \frac{2}{x} + 3$

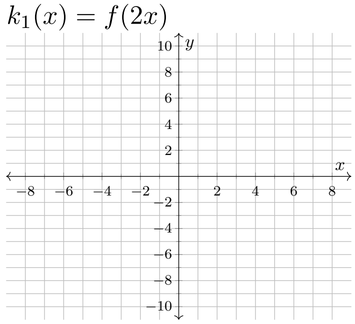
8. $f(x) = x^2$ $\quad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

9. $f(x) = \sqrt[3]{x}$ $\quad g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

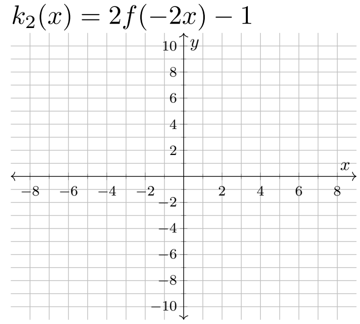
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



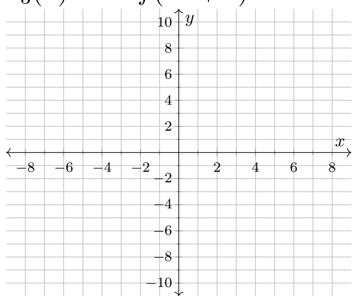
10. $k_1(x) = f(2x)$



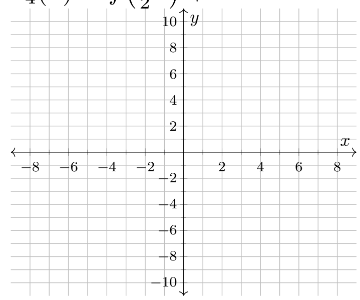
11. $k_2(x) = 2f(-2x) - 1$



12. $k_3(x) = -2f(2x+4)$



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



1.2 Inverse Functions

These exercises examine the invertibility of a function defined using a table.

Exercises

1. The table below defines the function m. Is m an invertible function? Why or why not? If your answer is yes, construct a table-of-values for m^{-1}.

x	1	2	3	4	5
$m(x)$	0	5	10	15	20

2. The table below defines the function p. Is p an invertible function? Why or why not? If your answer is yes, construct a table-of-values for p^{-1}.

x	1	2	3	4	5
$p(x)$	4	0	−2	0	2

1.3 Exponential Functions

These exercises find the formula for an exponential funciton given a pair of input-output coordinates.

Exercises

Find the Formula. In Exercises 1–6, find an algebraic rule for an exponential function f that passes through the given two points.

1. $(0, 50)$ and $(3, 400)$

2. $(0, 4)$ and $\left(4, \frac{1}{4}\right)$

3. $\left(-1, \frac{2}{3}\right)$ and $(2, 18)$

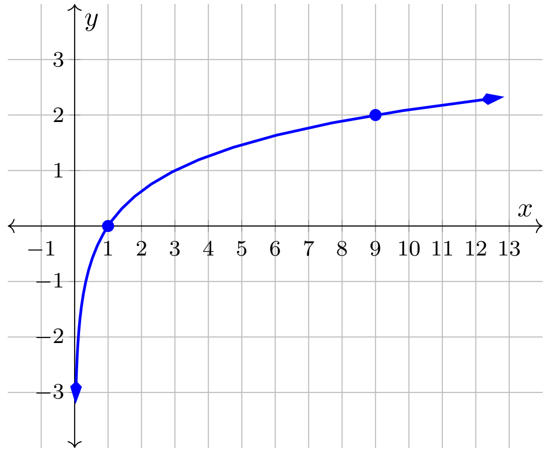
4. $\left(-2, \frac{125}{8}\right)$ and $(1, 8)$

5. $(-2, 125)$ and $\left(3, \frac{1}{25}\right)$

6. $\left(-3, \frac{27}{16}\right)$ and $\left(3, \frac{4}{27}\right)$

1.4 Logarithmic Functions

Example 1.4.1 The graph of $f(x) = \log_a(x)$ is given in the graph below. Find the value of a. Note, the points $(1, 0)$ and $(9, 2)$ are on the graph of f.



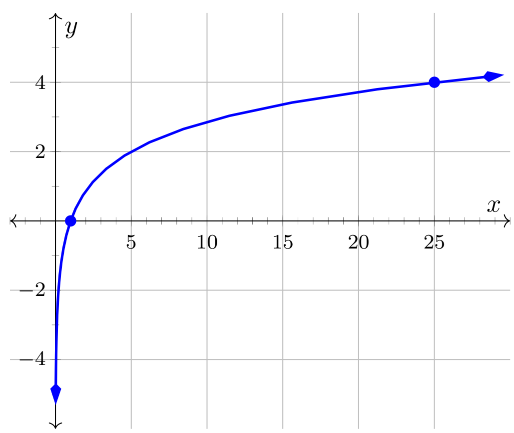
Solution. Since the function has the form $f(x) = \log_a(x)$ and $(9, 2)$ is on the graph, we know that $f(9) = 2$. Thus,

$$\begin{aligned} f(9) = 2 &\implies \log_a(9) = 2 && \text{(since } f(9) = \log_a(9)\text{)} \\ &\implies a^2 = 9 && \text{(translate to an exponential statement)} \\ &\implies a = 3 && \text{(positive square root because bases are positive)} \end{aligned}$$

Notice that we didn't attempt to use $(1, 0)$, the other obvious point on the graph of $f(x) = \log_a(x)$, to find the value of a. Why not? The point $(1, 0)$ is on the graph of all functions of the form $f(x) = \log_a(x)$, so it doesn't provide information that will help us find the paerticular function graphed here. □

Exercises

1. The graph of $f(x) = \log_a(x)$ is given below. Find the value of a. Note, the points $(1, 0)$ and $(25, 4)$ are on the graph of f.



Find the Base. In Exercises 2–3, each table represents a table-of-values for a function $f(x) = \log_a(x)$. Find the value of a.

2.

x	0.000125	0.05	1	$2\sqrt{5}$	400
$f(x)$	−3	−1	0	0.5	2

3.

x	$\frac{1}{9}$	1	3	81	243
$f(x)$	-4	0	2	8	10

Chapter 2

MTH 112 Supplement

2.1 Angles

2.1.1 Coterminal Angles

Definition 2.1.1 Two angles are **coterminal** if they have the same terminal side when in standard position. ◊

Since 360° represents a complete revolution, if we add integer-multiples of 360° to an angle measured in degrees, we'll obtain a coterminal angle. Similarly, since 2π represents a complete revolution in radians, if we add integer-multiples of 2π to an angle measured in radians, we'll obtain a coterminal angle. We can summarize this information as follows

If θ is measured in degrees, θ and $\theta + 360° \cdot k$, where $k \in \mathbb{Z}$, are coterminal.

If θ is measured in radians, θ and $\theta + 2\pi \cdot k$, where $k \in \mathbb{Z}$, are coterminal.

Example 2.1.2 The angles 45°, 405°, and −315° are coterminal as illustrated in Figure 2.1.3.

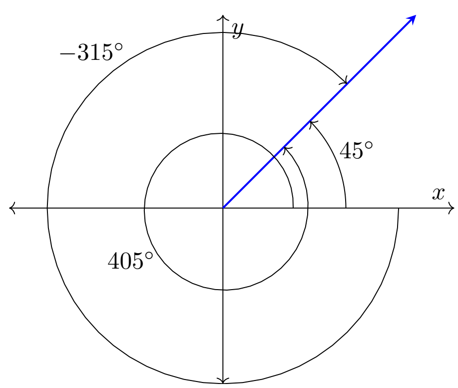


Figure 2.1.3 Coterminal angles

□

2.1.2 Reference Angles

Definition 2.1.4 The **reference angle** for an angle in standard position is the positive acute angle formed by the x-axis and the terminal side of the angle. ◊

Depending on the location of the angle's terminal side, we'll have to use a different calculation to determine the angle's reference angle.

Example 2.1.5 The angles $\frac{\pi}{3}$ and $30°$ are their own reference angles since they are acute angles; seen in Figure 2.1.6 and Figure 2.1.7.

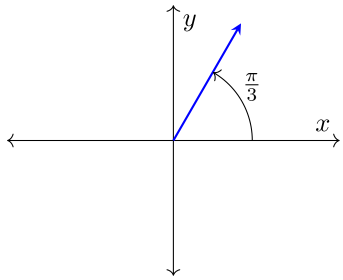


Figure 2.1.6

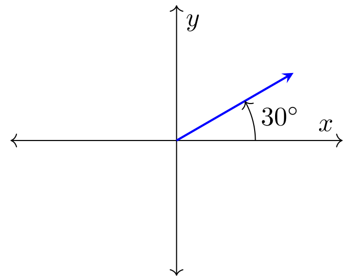


Figure 2.1.7

□

Example 2.1.8 The reference angle for $\frac{2\pi}{3}$ is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$ (see Figure 2.1.9), while the reference angle for $150°$ is $180° - 150° = 30°$ (see Figure 2.1.10).

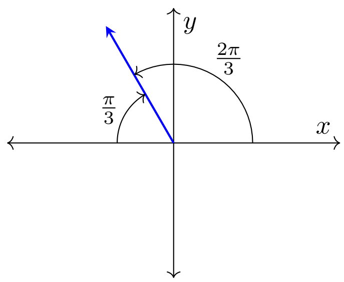


Figure 2.1.9

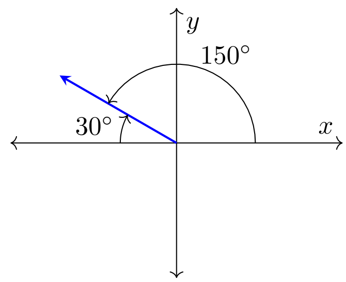


Figure 2.1.10

□

Example 2.1.11 The reference angle for $\frac{4\pi}{3}$ is $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$ (see Figure 2.1.12), while the reference angle for $210°$ is $210° - 180° = 30°$ (see Figure 2.1.13).

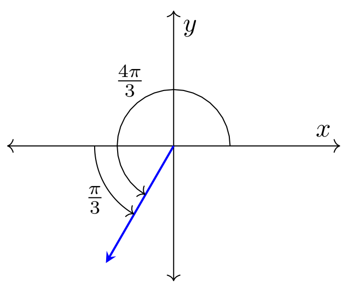


Figure 2.1.12

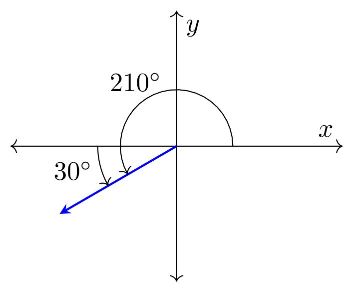


Figure 2.1.13

□

Example 2.1.14 The reference angle for $\frac{5\pi}{3}$ is $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$ (see Figure 2.1.15), while the reference angle for $330°$ is $360° - 330° = 30°$ (see Figure 2.1.16).

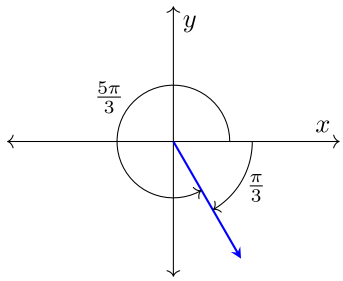


Figure 2.1.15

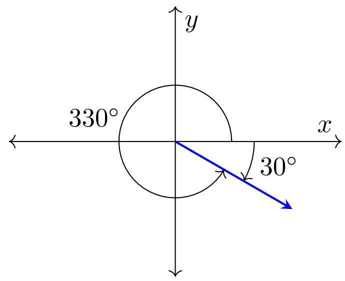


Figure 2.1.16

□

Example 2.1.17 The reference angle for 7.5 radians is $7.5 - 2\pi \approx 1.2$ radians (see Figure 2.1.18), and the reference angle for $-137°$ is $180° + (-137°) = 43°$ (see Figure 2.1.19).

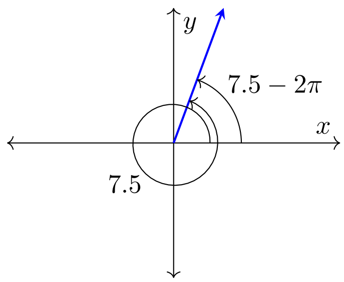


Figure 2.1.18

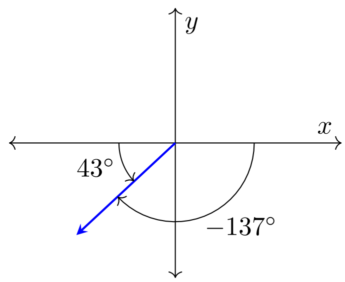


Figure 2.1.19

□

2.1.3 Degrees, Minutes, and Seconds

When measuring angles in degrees, fractions of a degree can be represented in minutes and seconds.

Definition 2.1.20 One **minute** is $\frac{1}{60}$ of a degree, so 60 minutes is equal to one degree (written $60' = 1°$).

One **second** is $\frac{1}{60}$ of a minute, i.e., $\frac{1}{3600}$ of a degree, so 60 seconds is equal to one minute (written $60'' = 1'$) and 3600 seconds is equal to one degree (written $3600'' = 1°$). ◊

When we write an angle's measure in the form $D°M'S''$ where (D, M, and S are real numbers), then the angle's measure is D degrees plus M minutes plus S seconds.

Example 2.1.21 Convert $34°15'27''$ into decimal form.

Solution.

$$\begin{aligned}34°15'27'' &= 34° + 15'\left(\frac{1°}{60'}\right) + 27''\left(\frac{1°}{3600''}\right)\\ &= 34° + 0.25° + 0.0075°\\ &= 34.2575°\end{aligned}$$

□

Example 2.1.22 Convert $61.72°$ into $D°M'S''$ form.

Solution.

$$61.72° = 61° + 0.72°$$

$$
\begin{aligned}
&= 61^\circ + 0.72^\circ \cdot \left(\frac{60'}{1^\circ}\right) \\
&= 61^\circ + 43.2' \\
&= 61^\circ 43' + 0.2' \\
&= 61^\circ 43' + 0.2' \cdot \left(\frac{60''}{1'}\right) \\
&= 61^\circ 43' 12''
\end{aligned}
$$

□

2.1.4 Exercises

Coterminal Angles. In Exercises 1–3, find both a positive and negative angle that is coterminal angle with the following angles.

1. 63° **2.** $\frac{\pi}{9}$ **3.** $\frac{13\pi}{8}$

Reference Angles. In Exercises 4–12, find the reference angle for the following angles.

4. 120° **5.** $\frac{5\pi}{4}$ **6.** 400°

7. $\frac{13\pi}{8}$ **8.** 2 **9.** $\frac{10\pi}{11}$

10. 2000° **11.** $-\frac{9\pi}{5}$ **12.** -100°

Convert to Decimal Form. In Exercises 13–16, convert the angle measure to decimal form (round your answers to the nearest thousandth when necessary).

13. $243^\circ 10'$ **14.** $3^\circ 25'$

15. $-23^\circ 3'$ **16.** $75^\circ 32' 17''$

Convert to $D^\circ M' S''$ Form. In Exercises 17–20, convert the angle measure to $D^\circ M' S''$ form.

17. 12.4° **18.** 1.53°

19. -144.9° **20.** 0.416°

2.2 Generalized Definitions of Trigonometric Functions

We can generalize the definitions of the trigonometric functions so that they are applicable to circles of any size.

Definition 2.2.1 If the point $P = (x, y)$ is specified by the angle θ on the circumeference of a circle of radius r, then $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$.

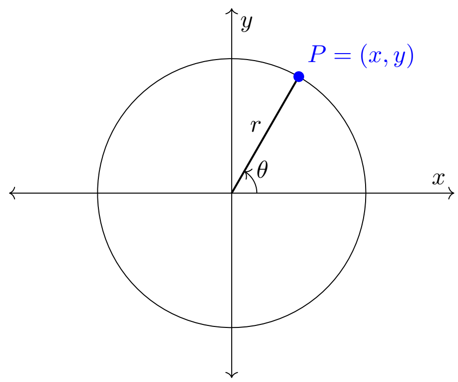


Figure 2.2.2

◊

Notice that if we are on a unit circle, where $r = 1$, then these definitions for $\cos(\theta)$ and $\sin(\theta)$ simplifly accordingly:

$$\cos(\theta) = \frac{x}{r} = \frac{x}{1} = x$$
$$\sin(\theta) = \frac{y}{r} = \frac{y}{1} = y$$

We can use Definition 2.2.1 to express the other four trigonometric functions in terms of x, y, and r.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y/r}{x/r} = \frac{y}{x}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{y/r} = \frac{r}{y}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{x/r} = \frac{r}{x}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{y/x} = \frac{x}{y}$$

We summarize this in the following definition.

Definition 2.2.3 If the point $P = (x, y)$ is specified by the angle θ on the circumeference of a circle of radius r, then we can define the six trigonemtric functions as follows.

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r} \qquad \tan(\theta) = \frac{y}{x}$$
$$\sec(\theta) = \frac{r}{x} \qquad \csc(\theta) = \frac{r}{y} \qquad \cot(\theta) = \frac{x}{y}$$

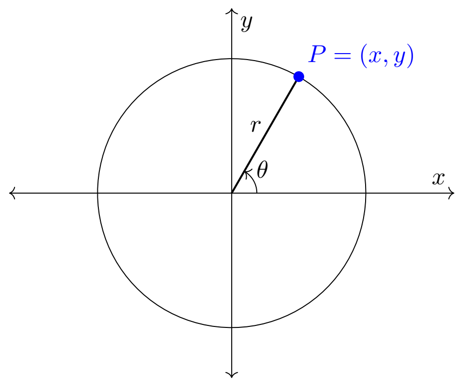


Figure 2.2.4

◊

Example 2.2.5 Find the exact value of each of the six trigonometric functions of an angle θ if $(-1, 2)$ is a point on its terminal side.

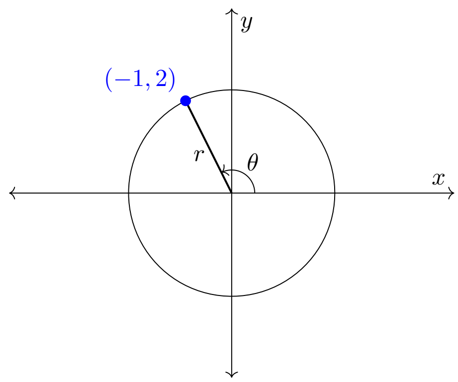
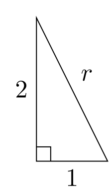


Figure 2.2.6

Solution. We need the values of x, y, and r to determine the exact value of each of the trigonemtric function. We are given x and y, but will need to find the value of r.

We can think of r as the hypotenuse of a right triangle whose horiztonal leg has a length of $|-1| = 1$ unit and vertical leg has a length of 2 units. We can use the Pythagorean Theorem to solve for r:

$$r^2 = 1^2 + 2^2$$
$$r^2 = 5$$
$$r = \sqrt{5}$$



Now that we have values for x, y, and r, we can use Definition 2.2.3 to state the exact values of each trigonmetric function.

$$\begin{aligned}\cos(\theta) &= \frac{x}{r} \\ &= \frac{-1}{\sqrt{5}} \\ &= \frac{-1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} \\ &= -\frac{\sqrt{5}}{5}\end{aligned} \qquad \begin{aligned}\sin(\theta) &= \frac{y}{r} \\ &= \frac{2}{\sqrt{5}} \\ &= \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} \\ &= \frac{2\sqrt{5}}{5}\end{aligned}$$

$$\begin{aligned}\tan(\theta) &= \frac{y}{x} \\ &= \frac{2}{-1} \\ &= -2\end{aligned} \qquad \begin{aligned}\sec(\theta) &= \frac{r}{x} \\ &= \frac{\sqrt{5}}{-1} \\ &= -\sqrt{5}\end{aligned}$$

$$\begin{aligned}\csc(\theta) &= \frac{r}{y} \\ &= \frac{\sqrt{5}}{2}\end{aligned} \qquad \begin{aligned}\cot(\theta) &= \frac{x}{y} \\ &= \frac{-1}{2} \\ &= -\frac{1}{2}\end{aligned}$$

□

We can use Definition 2.2.1 to express the coordinates of point P in Figure 2.2.2. We do this by solving the equations $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$ for x and y, respectively:

$$\cos(\theta) = \frac{x}{r} \implies x = r\cos(\theta)$$
$$\sin(\theta) = \frac{y}{r} \implies y = r\sin(\theta)$$

We summarize this below.

Definition 2.2.7 If the point $P = (x, y)$ is specified by the angle θ on the circumeference of a circle of radius r, then $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

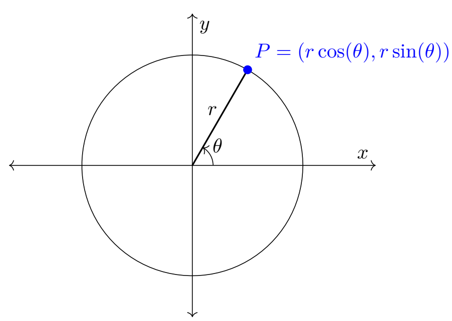


Figure 2.2.8

◊

Example 2.2.9 Find the coordinates of the point on a circle with a radius of 8 units corresponding to an angle of $\frac{7\pi}{4}$.

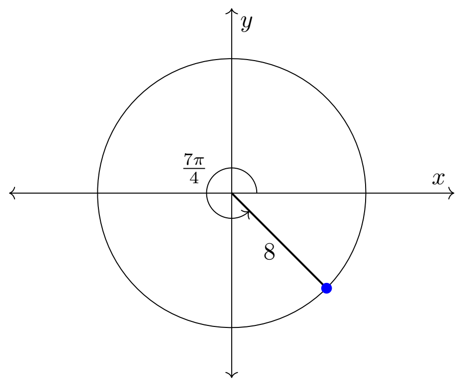


Figure 2.2.10

Solution. We are given the values of r and θ, so we can use Definition 2.2.7 to determine the coordinate values.

Find x:

$$\begin{aligned} x &= r\cos(\theta) \\ &= 8\cos\left(\frac{7\pi}{4}\right) \\ &= 8\cdot\frac{\sqrt{2}}{2} \\ &= 4\sqrt{2} \end{aligned}$$

Find y:

$$\begin{aligned} y &= r\sin(\theta) \\ &= 8\sin\left(\frac{7\pi}{4}\right) \\ &= 8\left(-\frac{\sqrt{2}}{2}\right) \\ &= -4\sqrt{2} \end{aligned}$$

So the coordinates of the point on a circle with a radius of 8 units corresponding to an angle of $\frac{7\pi}{4}$ are $(4\sqrt{2}, -4\sqrt{2})$. □

Exercises

Six Trigonemtric Function Values. In Exercises 1–2, find the exact value of each of the six trigonometric functions of an angle θ if the given point is on its terminal side.

1. $(3, 4)$

2. $(-2, -6)$

Find the coordinates. In Exercises 3–4, find the coordinates of the point on a circle with the given radius corresponding to the given angle.

3. $r = 3, \theta = \frac{7\pi}{6}$

4. $r = 10, \theta = \frac{\pi}{3}$

2.3 Graphing Sinusoidal Functions: Phase Shift vs. Horizontal Shift

Let's consider the function $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$. Using what we study in MTH 111 about graph transformations, it should be apparent that the graph of $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ can be obtained by transforming the graph of $g(x) = \sin(x)$. (To confirm this, notice that $g(x)$ can be expressed in terms of $f(x) = \sin(x)$, as $g(x) = f\left(2x - \frac{2\pi}{3}\right)$.) Since the constants "2" and "$\frac{2\pi}{3}$" are multiplied by and subtracted from the input variable, x, what we study in MTH 111 tells us that these constants represent a horizontal stretch/compression and a horizontal shift, respectively.

It is often recommended in MTH 111 that we factor-out the horizontal stretching/compressing factor before transforming the graph, i.e., it's often recommended that we first re-write $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ as $g(x) = \sin\left(2\left(x - \frac{\pi}{3}\right)\right)$.

After writing g in this format, we can draw its graph by performing the following sequence of transformations of the "base function" $f(x) = \sin(x)$:

1. Compress horizontally by a factor of $\frac{1}{2}$.
2. Shift $\frac{\pi}{3}$ units to the right.

The advantage of this method is that the y-intercept of $f(x) = \sin(x)$, $(0, 0)$, ends-up exactly where the horizontal shift suggests: when we compress the graph by a factor of $\frac{1}{2}$, the the y-interceot of the graph doesn't move since $\frac{1}{2} \cdot 0 = 0$; then, when we shift the graph $\frac{\pi}{3}$ units to the right, the point $(0, 0)$ ends up at $\left(\frac{\pi}{3}, 0\right)$; so the y-intercept ends up moving $\frac{\pi}{3}$ units to the right, exactly how far we shifted.

Compare this with the alternative method: we can leave $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ as-is and skip factoring-out the horizontal stretching/compressing factor, but then we need the following sequence to transform $f(x) = \sin(x)$ into the graph of g:

1. Shift $\frac{2\pi}{3}$ units to the right.
2. Compress horizontally by a factor of $\frac{1}{2}$.

The disadvantage of this method is that the y-intercept of $f(x) = \sin(x)$ *doesn't* end-up where the horizontal shift suggests: when we shift the graph of $f(x) = \sin(x)$ to the right by $\frac{2\pi}{3}$ units, the y-intercept moves from $(0, 0)$ to $\left(\frac{2\pi}{3}, 0\right)$; then, when we compress the graph by a factor of $\frac{1}{2}$, it moves to $\left(\frac{\pi}{3}, 0\right)$, so the y-intercept *doesn't* end up shifted $\frac{2\pi}{3}$ units to the right.

Figure 2.3.1 shows the graphs of $y = f(x)$ and $y = g(x)$. Notice that the behavior of $y = g(x)$ at $x = \frac{\pi}{3}$ is like the behavior of $y = f(x)$ at $x = 0$, i.e., $y = g(x)$ appears to have been shifted $\frac{\pi}{3}$ units to the right. For this reason, $\frac{\pi}{3}$ is called the horzontal shift of $g(x) = \sin\left(2x - \frac{2\pi}{3}\right) = \sin\left(2\left(x - \frac{\pi}{3}\right)\right)$.

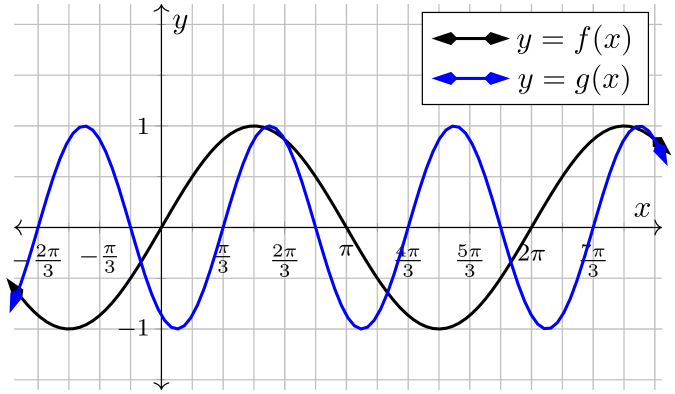


Figure 2.3.1 $y = g(x)$ with $f(x) = \sin(x)$

The constant $\frac{2\pi}{3}$ is given a different name, phase shift, since it can be used to determine how far "out-of-phase" a sinusoidal function is in comparison with $y = \sin(x)$ or $y = \cos(x)$. To determine how far out-of-phase a sinusoidal function is, we can determine the ratio of the phase shift and 2π. (We use 2π because it's the period of $y = \sin(x)$ and $y = \cos(x)$.) Since $\frac{2\pi}{3}$ is the phase shift for $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$, the graph of $y = g(x)$ is out-of-phase $\frac{2\pi/3}{2\pi} = \frac{1}{3}$ of a period. (Since this number is positive, it represents a horizontal shift to the right $\frac{1}{3}$ of a period.)

Definition 2.3.2 Given a sinusoidal function of the form $y = A\sin(wx - C) + k$ or $y = A\cos(wx - C) + k$, the **phase shift** is C and $\frac{|C|}{2\pi}$ represents the fraction of a period that the graph has been shifted (shift to the right if C is positive or to the left if C is negative). ♢

Definition 2.3.3 If we re-write the function as $y = A\sin\left(w\left(x - \frac{C}{w}\right)\right) + k$ or $y = A\cos\left(w\left(x - \frac{C}{w}\right)\right) + k$, we can see that the **horizontal shift** is $\frac{C}{w}$ units (shift to the right if $\frac{C}{w}$ is positive or to the left if $\frac{C}{w}$ is negative). ♢

Example 2.3.4 Identify the phase shift and horizontal shift of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$.

Solution. The phase shift of $g(x) = \cos\left(3x - \frac{\pi}{4}\right)$ is $\frac{\pi}{4}$. This tells us that the graph of $y = g(x)$ is out of phase $\frac{|\pi/4|}{2\pi} = \frac{1}{8}$ of a period, i.e., compared with $y = \cos(x)$, the graph of $g(x) = \cos(3x - \frac{\pi}{4})$ has been shifted one-eighth of a period to the right.

To find the horizontal shift, we need to factor-out 3 from $3x - \frac{\pi}{4}$.

$$\begin{aligned} g(x) &= \cos\left(3x - \frac{\pi}{4}\right) \\ &= \cos\left(3\left(x - \frac{\pi}{3 \cdot 4}\right)\right) \\ &= \cos\left(3\left(x - \frac{\pi}{12}\right)\right) \end{aligned}$$

So the horizontal shift is $\frac{\pi}{12}$. This tells us, that compared with $y = \cos(x)$, the graph of $g(x) = \cos(3x - \frac{\pi}{4})$ has been shifted $\frac{\pi}{12}$ to the right.

Notice that the period of $g(x) = \cos(3x - \frac{\pi}{4})$ is $2\pi \cdot \frac{1}{3} = \frac{2\pi}{3}$, and one-eighth of $\frac{2\pi}{3}$ is $\frac{2\pi}{3} \cdot \frac{1}{8} = \frac{\pi}{12}$, so a shift of one-eighth of a period is the same as a shift of $\frac{\pi}{12}$ units! □

Example 2.3.5 Draw a graph $q(t) = 2\sin(4t + \pi) + 1$. First, find it's amplitude, period, midline, phase shift, and horiontal shift.

Solution. Amplitude: $|A| = |2| = 2$
Period: $P = 2\pi \cdot \frac{1}{|w|} = \frac{2\pi}{4} = \frac{\pi}{2}$
Midline: $y = 1$
Phase shift: $-\pi$ (this tells is that the graph is out-of-phase $\frac{|-\pi|}{2\pi} = \frac{1}{2}$ of a period)
Horizontal shift: $\frac{\pi}{4}$ units to the left since:

$$\begin{aligned} q(t) &= 2\sin(4t+\pi)+1 \\ &= 2\sin\Big(4\Big(t+\frac{\pi}{4}\Big)\Big)+1 \\ &= 2\sin\Big(4\Big(t-\Big(-\frac{\pi}{4}\Big)\Big)\Big)+1 \end{aligned}$$

Now we can draw a graph of $q(t) = 2\sin(4t+\pi)+1$ by drawing a sinusoidal function with the necessary features; see Figure 2.3.6.

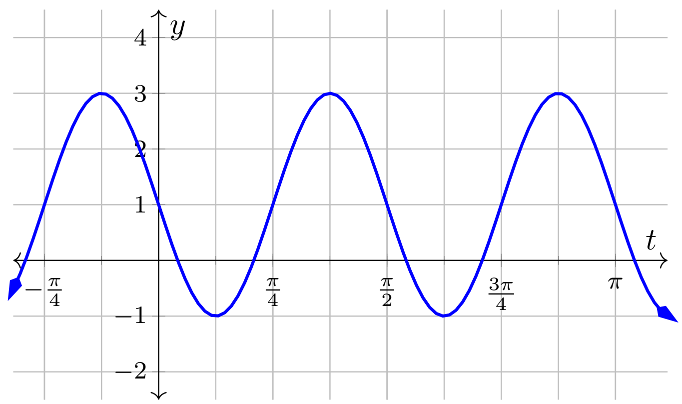


Figure 2.3.6 $y = q(t)$

□

Exercises

Sketch Sinusoidal Graphs. In Exercises 1–4, draw a graph of each of the following functions. List the amplitude, midline, period, phase shift, and horizontal shift.

1. $f(x) = 3\sin\big(3x - \frac{\pi}{2}\big)$

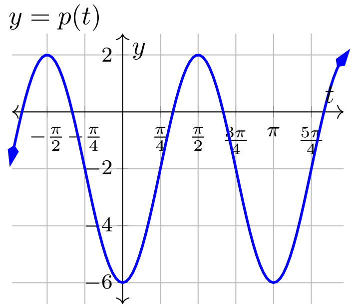
2. $g(t) = \cos(4t+\pi)+3$

3. $m(\theta) = 2\cos(2\pi\theta - \pi)+4$

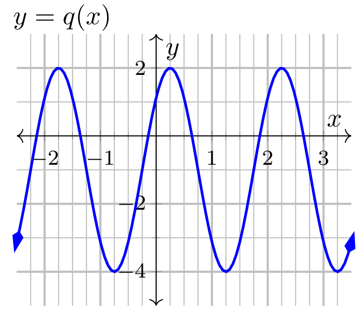
4. $n(x) = -4\sin\big(\pi x + \frac{\pi}{4}\big) - 2$

Find the Formula. In Exercises 5–6, find two algebraic rules (one involving sine and one involving cosine) for each of the functions graphed below.

5. $y = p(t)$



6. $y = q(x)$



2.4 Complex Numbers and Polar Coordinates

2.4.1 Forms of Complex Numbers

Recall that a complex number has the form $a+bi$ where $a, b \in \mathbb{R}$ and $i=\sqrt{-1}$. Complex numbers have two parts: a real part and an imaginary part. For the number $a+bi$, the real part is a and the imaginary part is b. Because they have two parts, we can use the two dimensional rectangular coordinate plane to represent complex numbers. We use the horizontal axis to represent the real part and the vertical axis to represent the complex part. Thus, the complex number $a+bi$ can be represented by the point (a, b) on the rectangular coordinate plane; see Figure 2.4.1.

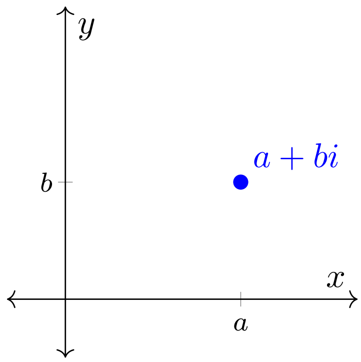


Figure 2.4.1

As we've studied in this course, the rectangular ordered pair (a, b) can be represented in polar coordinates (r, θ) where r represents the distance the point is from the origin and θ represents the angle between the positive x-axis and the segment connecting the origin and the point; see Figure 2.4.2.

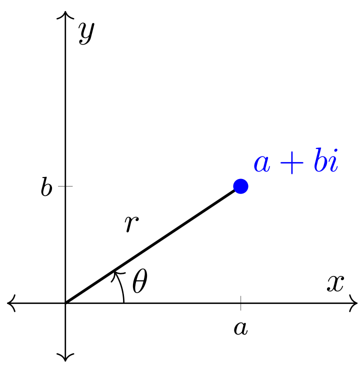


Figure 2.4.2

We know that if the rectangular pair (a, b) represents the same point as the polar pair (r, θ), then $a=r\cos(\theta)$ and $b=r\sin(\theta)$. Thus,

$$\begin{aligned} a+bi &= r\cos(\theta)+r\sin(\theta)\cdot i \\ &= r(\cos(\theta)+i\cdot\sin(\theta)) \end{aligned}$$

i.e., we can express a complex number using the "polar information" r and θ.

The expression "$r(\cos(\theta)+i\cdot\sin(\theta))$" is what a textbook might decribe as the "polar form of a complex number." But a more appropriate expression to label as "the polar form of a complex number" involves Euler's Formula. Euler's Formula is an identity that establishes a surprising connection between the exponential function e^x and complex numbers.

This is Euler's Formula. For reference purposes, we state this in a theorem.

Theorem 2.4.3 Euler's Formula.

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$$

Notice that if we multiply both sides of Euler's formula by r, we obtain a formula that allows us to write any complex number in polar form:

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i \cdot \sin(\theta) \\ \implies r \cdot e^{i\theta} &= r \cdot (\cos(\theta) + i \cdot \sin(\theta)) \\ \implies r(e^{i\theta}) &= r\cos(\theta) + r\sin(\theta) \cdot i \end{aligned}$$

Definition 2.4.4 The **polar form** of the complex number $z = r\cos(\theta) + r\sin(\theta) \cdot i$ is $z = re^{i\theta}$. ♢

Let's review what we've established: First, we observed that we can write a complex number of the form "$a + bi$" in the form "$r \cdot (\cos(\theta) + i \cdot \sin(\theta))$". Then we noticed that we can write an expression of the form "$r \cdot (\cos(\theta) + i \cdot \sin(\theta))$" in the form "$re^{i\theta}$". Finally, we realized that we can write a complex number "$a + bi$" in the form "$re^{i\theta}$" so we defined "$re^{i\theta}$" as being the polar form of the complex number $a + bi$.

Example 2.4.5 Express in "rectangular form" (i.e. in the form $z = a + bi$) the complex number $z = 6e^{\frac{5\pi}{6} \cdot i}$, given in polar form.

Solution.

$$\begin{aligned} z &= 6e^{\frac{5\pi}{6} \cdot i} \\ &= 6\cos\left(\frac{5\pi}{6}\right) + 6\sin\left(\frac{5\pi}{6}\right) \cdot i \\ &= 6\left(-\frac{\sqrt{3}}{2}\right) + 6\left(\frac{1}{2}\right) \cdot i \\ &= -3\sqrt{3} + 3i \end{aligned}$$

Thus, the complex number $z = 6e^{\frac{5\pi}{6} \cdot i}$ can be expressed in "rectangular form" as $z = -3\sqrt{3} + 3i$. □

Example 2.4.6 Express in polar form (i.e. in the form $z = re^{i\theta}$) the complex number $z = 3 - 3i$, given in "rectangular form."

Solution. We can associate the complex number $z = 3 - 3i$ with the rectangular ordered pair $(3, -3)$, and then translate this ordered pair into polar coordinates (r, θ), and finally use the polar ordered pair to obtain the polar form $z = re^{i\theta}$. First, let's find r:

$$\begin{aligned} r &= \sqrt{(3)^2 + (-3)^2} \\ &= \sqrt{9 + 9} \\ &= 3\sqrt{2}. \end{aligned}$$

Now, let's find θ:

$$\begin{aligned} \tan(\theta) &= \frac{-3}{3} \\ \implies \theta &= \tan^{-1}(-1) \\ \implies \theta &= -\frac{\pi}{4} \end{aligned}$$

Thus, the complex number $z = 3 - 3i$ can be expressed in polar form as $z = 3\sqrt{2}e^{-\frac{\pi}{4} \cdot i}$. □

2.4.2 Using the Polar Form to Find Complex Roots

Example 2.4.7 Find the two square roots of $-1+\sqrt{3}i$ using the polar form of $-1+\sqrt{3}i$.

Solution. Recall that there are two distinct square roots of any positive real number (e.g., the two square roots of 4 are 2 and -2). The same is true for any complex number. We can find two different square roots of a complex number by using two difference polar forms of the number.

To find polar forms of $-1+\sqrt{3}i$, we first associate the number with the rectangular ordered pair $\left(-1,\sqrt{3}\right)$, and then translate it into polar coordinates (r,θ). First let's find r:

$$\begin{aligned} r &= \sqrt{(-1)^2+\left(\sqrt{3}\right)^2} \\ &= \sqrt{1+3} \\ &= 2. \end{aligned}$$

Now, let's find θ. $\tan(\theta)=-\sqrt{3}$ with θ in Quadrant II.

Both $\theta=\frac{2\pi}{3}$ and $\theta=-\frac{4\pi}{3}$ satisfy the condition, so we'll use these two angles to obtain two polar forms of $-1+\sqrt{3}i$:

$$-1+\sqrt{3}i=2e^{\frac{2\pi}{3}\cdot i}\text{and}-1+\sqrt{3}i=2e^{-\frac{4\pi}{3}\cdot i}$$

Therefore,

$$\begin{aligned} \left(-1+\sqrt{3}i\right)^{1/2} &= (2e^{\frac{2\pi}{3}\cdot i})^{1/2} \\ &= 2^{\frac{1}{2}}e^{\frac{2\pi}{3}\cdot\frac{1}{2}i} \\ &= \sqrt{2}e^{\frac{\pi}{3}\cdot i} \\ &= \sqrt{2}\left(\cos\left(\frac{\pi}{3}\right)+i\cdot\sin\left(\frac{\pi}{3}\right)\right) \\ &= \sqrt{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\ &= \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i \end{aligned}$$

and

$$\begin{aligned} \left(-1+\sqrt{3}i\right)^{1/2} &= \left(2e^{-\frac{4\pi}{3}\cdot i}\right)^{1/2} \\ &= 2^{\frac{1}{2}}e^{-\frac{4\pi}{3}\cdot\frac{1}{2}i} \\ &= \sqrt{2}e^{-\frac{2\pi}{3}\cdot i} \\ &= \sqrt{2}\left(\cos\left(-\frac{2\pi}{3}\right)+i\cdot\sin\left(-\frac{2\pi}{3}\right)\right) \\ &= \sqrt{2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) \\ &= -\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i. \end{aligned}$$

So both $\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i$ and $-\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i$ are square roots of $-1+\sqrt{3}i$. But just as 2, not -2, is called the principal square root of 4, only one of thes two square roots that we found is the principal square root of $-1+\sqrt{3}i$. The principal square root (or principal nth root) of a complex number is the root with the greatest real component. And if there is a tie between two roots for having the greatest real component, the one with positive imaginary component is the principal root. So the first root we found (i.e., the one we found using $\theta = \frac{2\pi}{3}$) is the principal square root of $-1+\sqrt{3}i$, because its real part is $\frac{\sqrt{2}}{2}$ which is greater than the other root's real part, $-\frac{\sqrt{2}}{2}$. The princial root is the one represented by the radical symbol, so we can write

$$\sqrt{-1+\sqrt{3}i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i.$$

□

Example 2.4.8 Find $\sqrt[3]{-4\sqrt{2}+4\sqrt{2}i}$ using the polar form of $-4\sqrt{2}+4\sqrt{2}i$.

Solution. To find polar forms of $-4\sqrt{2}+4\sqrt{2}i$, we first associate the number with the rectangular ordered pair $\left(-4\sqrt{2}, 4\sqrt{2}\right)$, and then translate it into polar coordinates (r, θ). First let's find r:

$$\begin{aligned} r &= \sqrt{\left(-4\sqrt{2}\right)^2 + \left(4\sqrt{2}\right)^2} \\ &= \sqrt{4^2 \cdot 2 + 4^2 \cdot 2} \\ &= 4\sqrt{2+2} \\ &= 8 \end{aligned}$$

Now, let's find θ:

$$\begin{aligned} &\tan(\theta) = \frac{4\sqrt{2}}{-4\sqrt{2}} \\ \implies\ &\theta = \tan^{-1}(-1) + \pi \quad \text{(add } \pi \text{ since } \left(-4\sqrt{2}, 4\sqrt{2}\right) \text{ is in Quadrant II)} \\ \implies\ &\theta = -\frac{\pi}{4} + \pi \\ \implies\ &\theta = \frac{3\pi}{4} \end{aligned}$$

Note: we add π since $\left(-4\sqrt{2}, 4\sqrt{2}\right)$ is in Quadrant II, outside the range of arctangent.

So the polar form of $-4\sqrt{2}+4\sqrt{2}i$ is $z = 8e^{\frac{3\pi}{4}\cdot i}$. Therefore,

$$\begin{aligned} \sqrt[3]{-4\sqrt{2}+4\sqrt{2}i} &= \left(8e^{\frac{3\pi}{4}\cdot i}\right)^{1/3} \\ &= 8^{\frac{1}{3}} e^{\frac{3\pi}{4}\cdot\frac{1}{3}i} \\ &= 2e^{\frac{\pi}{4}\cdot i} \\ &= 2\left(\cos\left(\frac{\pi}{4}\right) + i\cdot\sin\left(\frac{\pi}{4}\right)\right) \\ &= 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) \\ &= \sqrt{2} + \sqrt{2}i \end{aligned}$$

□

2.4.3 Exercises

Polar Form. In Exercises 1–3, find the polar form $z = re^{i\theta}$ of the following complex numbers given in rectangular form.

1. $z = 6 + 6\sqrt{3} \cdot i$

2. $z = -2\sqrt{3} + 2i$

3. $z = 5\sqrt{2} - 5\sqrt{2} \cdot i$

Rectangular Form. In Exercises 4–6, find the rectangular form $z = a + bi$ of the following complex numbers given in polar form.

4. $z = 8e^{\frac{\pi}{6} \cdot i}$

5. $z = 4e^{\pi \cdot i}$

6. $z = 5e^{\frac{4\pi}{3} \cdot i}$

Principal Roots. In Exercises 7–10, find the following principal roots by first converting to the polar form of each complex number.

7. $\sqrt{18 - 18\sqrt{3} \cdot i}$

8. $\sqrt[3]{-16 + 16i}$

9. $\sqrt{-i}$

10. $\sqrt[5]{-16\sqrt{3} - 16i}$

11. Find all three cube roots of $27i$.

12. Find both of the square roots of $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

13. Find all three solutions to the equation of $z^3 + 1 = 0$.

Appendix A

Answers and Solutions to Exercises

1 · MTH 111 Supplement

1.1 · Graph Transformations

· Exercises

One Function in Terms of Another.

1.1.1. Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

1.1.2. Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

1.1.3. Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

1.1.4. Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

1.1.5. Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f(\frac{1}{2}x)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations.

1.1.6. Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

1.1.7. Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

1.1.8. Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

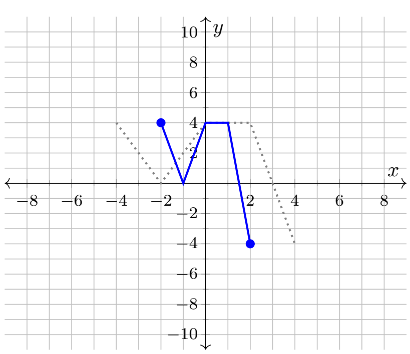
1.1.9. Solution.

$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

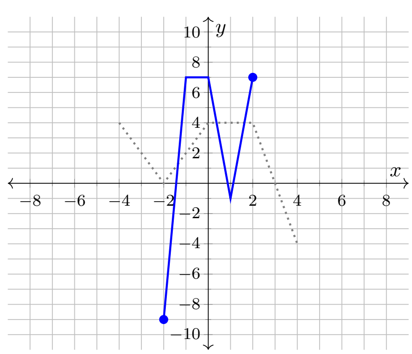
So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

Sketch Transformations.

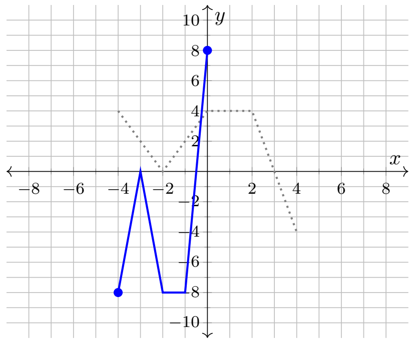
1.1.10. Answer.



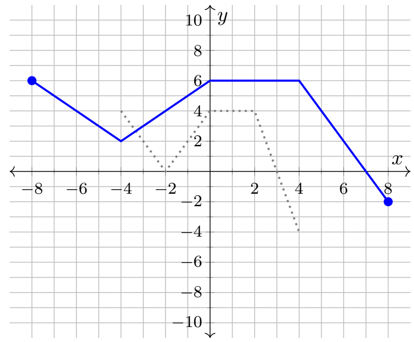
1.1.11. Answer.



1.1.12. Answer.



1.1.13. Answer.



1.2 · Inverse Functions

· Exercises

1.2.1. Solution. m is an invertible function since it is one-to-one, i.e., each output corresponds to exactly one input. Here is a table-of-values for m^{-1}.

x	0	5	10	15	20
$m^{-1}(x)$	1	2	3	4	5

1.2.2. Solution. p isn't an invertible function since it isn't one-to-one. Notice how the output 0 corresponds to two distinct output values.

1.3 · Exponential Functions

· Exercises

Find the Formula.

1.3.1. Answer. $f(x) = 50 \cdot 2^x$

1.3.2. Answer. $f(x) = 4 \cdot \left(\frac{1}{2}\right)^x$

1.3.3. Answer. $f(x) = 2 \cdot 3^x$

1.3.4. Answer. $f(x) = 10 \cdot \left(\frac{4}{5}\right)^x$

1.3.5. Answer. $f(x) = 5 \cdot \left(\frac{1}{5}\right)^x$

1.3.6. Answer. $f(x) = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^x$

1.4 · Logarithmic Functions

· Exercises

1.4.1. Answer. $a = \sqrt{5}$

Find the Base.

1.4.2. Answer. $a = 20$

1.4.3. Answer. $a = \sqrt{3}$

2 · MTH 112 Supplement

2.1 · Angles

2.1.4 · Exercises

Coterminal Angles.

2.1.4.1.
Answer. $423°$ and $-297°$ are coterminal with $63°$.

2.1.4.2.
Answer. $\frac{19\pi}{9}$ and $-\frac{17\pi}{9}$ are coterminal with $\frac{\pi}{9}$.

2.1.4.3.
Answer. $\frac{29\pi}{8}$ and $-\frac{3\pi}{8}$ are coterminal with $\frac{13\pi}{8}$.

Reference Angles.

2.1.4.4. **Answer.** $60°$

2.1.4.5. **Answer.** $\frac{\pi}{4}$

2.1.4.6. **Answer.** $40°$

2.1.4.7. **Answer.** $\frac{3\pi}{8}$

2.1.4.8. **Answer.** $\pi - 2 \approx 1.14$

2.1.4.9. **Answer.** $\frac{\pi}{11}$

2.1.4.10. **Answer.** $20°$

2.1.4.11. **Answer.** $\frac{\pi}{5}$

2.1.4.12. **Answer.** $80°$

Convert to Decimal Form.

2.1.4.13. **Answer.** $243°10' \approx 243.167°$

2.1.4.14. **Answer.** $3°25' \approx 3.417°$

2.1.4.15. **Answer.** $-23°3' = -23.05°$

2.1.4.16. **Answer.** $75°32'17'' \approx 75.538°$

Convert to $D°M'S''$ Form.

2.1.4.17. **Answer.** $12.4° = 12°24'$

2.1.4.18. **Answer.** $1.53° = 1°31'48''$

2.1.4.19. **Answer.** $-144.9° = -144°54'$

2.1.4.20. **Answer.** $0.416° = 0°24'57.6''$

2.2 · Generalized Definitions of Trigonometric Functions

· Exercises

Six Trigonemtric Function Values.

2.2.1. **Answer.**

$$\cos(\theta) = \frac{3}{5}$$
$$\sin(\theta) = \frac{4}{5}$$
$$\tan(\theta) = \frac{4}{3}$$
$$\sec(\theta) = \frac{5}{3}$$
$$\csc(\theta) = \frac{5}{4}$$
$$\cot(\theta) = \frac{3}{4}$$

2.2.2. **Answer.**

$$\sin(\theta) = -\frac{\sqrt{10}}{10}$$
$$\cos(\theta) = -\frac{3\sqrt{10}}{10}$$
$$\tan(\theta) = 3$$
$$\sec(\theta) = -\sqrt{10}$$
$$\csc(\theta) = -\frac{\sqrt{10}}{3}$$
$$\cot(\theta) = \frac{1}{3}$$

Find the coordinates.

2.2.3. **Answer.** $\left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

2.2.4. **Answer.** $(5, 5\sqrt{3})$

2.3 · Graphing Sinusoidal Functions: Phase Shift vs. Horizontal Shift

· Exercises

Sketch Sinusoidal Graphs.

2.3.1. Answer.

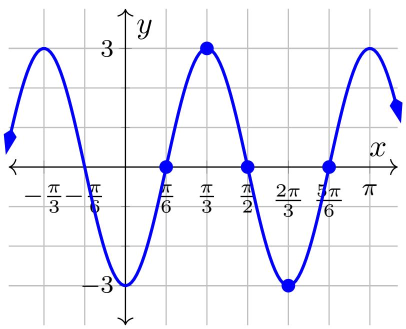
Amplitude 3 units

Period $\frac{2\pi}{3}$ units

Midline $y = 0$

Phase shift $\frac{\pi}{2}$

Horizontal Shift $\frac{\pi}{6}$ units to the right



2.3.2. Answer.

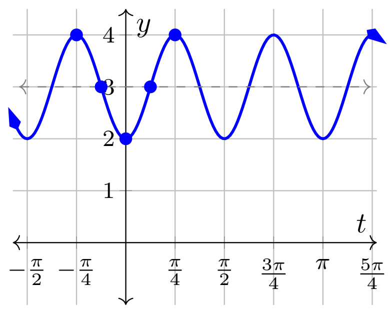
Amplitude 1 unit

Period $\frac{\pi}{2}$ units

Midline $y = 3$

Phase shift $-\pi$

Horizontal Shift $\frac{\pi}{4}$ units to the left



2.3.3. Answer.

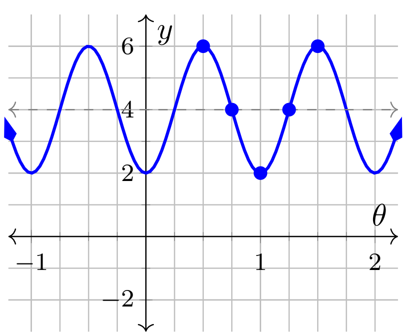
Amplitude 2 units

Period 1 unit

Midline $y = 4$

Phase shift π

Horizontal Shift $\frac{1}{2}$ of a unit to the right



2.3.4. Answer.

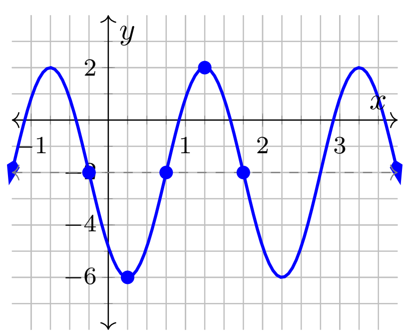
Amplitude 4 units

Period 2 units

Midline $y = -2$

Phase shift $-\frac{\pi}{4}$

Horizontal Shift $\frac{1}{4}$ of a unit to the left



Find the Formula.

2.3.5. Answer.

$$p(x) = 4\sin\left(2\left(x - \frac{\pi}{4}\right)\right) - 2$$

$$p(x) = 4\cos\left(2\left(x - \frac{\pi}{2}\right)\right) - 2$$

2.3.6. Answer.

$$q(x) = 3\sin\left(\pi\left(x + \frac{1}{4}\right)\right) - 1$$

$$q(x) = 3\cos\left(\pi\left(x - \frac{1}{4}\right)\right) - 1$$

2.4 · Complex Numbers and Polar Coordinates
2.4.3 · Exercises

Polar Form.

2.4.3.1. Answer. $z = 12e^{\frac{\pi}{3}\cdot i}$

2.4.3.2. Answer. $z = 4e^{\frac{5\pi}{6}\cdot i}$

2.4.3.3. Answer. $z = 10e^{-\frac{\pi}{4}\cdot i}$

Rectangular Form.

2.4.3.4. Answer. $z = 4\sqrt{3} + 4i$

2.4.3.5. Answer. $z = -4$

2.4.3.6.
Answer. $z = -\frac{5}{2} - \frac{5\sqrt{3}}{2} \cdot i$

Principal Roots.

2.4.3.7.
Answer. $\sqrt{18 - 18\sqrt{3} \cdot i} = 3\sqrt{3} - 3i$ (and the non-principal root is $-3\sqrt{3} + 3i$)

2.4.3.8.
Answer. $\sqrt[3]{-16 + 16i} = 2 + 2i$ (and the non-principal roots are $\left(-1 - \sqrt{3}\right) + \left(-1 + \sqrt{3}\right) i$ and $\left(-1 + \sqrt{3}\right) + \left(-1 - \sqrt{3}\right) i$)

2.4.3.9.
Answer. $\sqrt{-i} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ (and the non-principal root is $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$)

2.4.3.10.
Answer. $\sqrt[5]{-16\sqrt{3} - 16i} = \sqrt{3} - i$ (and the non-principal roots are $2\cos\left(\frac{7\pi}{30}\right) + 2i\sin\left(\frac{7\pi}{30}\right)$, $2\cos\left(\frac{19\pi}{30}\right) + 2i\sin\left(\frac{19\pi}{30}\right)$, $2\cos\left(\frac{29\pi}{30}\right) - 2i\sin\left(\frac{29\pi}{30}\right)$, and $2\cos\left(\frac{17\pi}{30}\right) - 2i\sin\left(\frac{17\pi}{30}\right)$)

2.4.3.11. Answer. $\frac{3\sqrt{3}}{2} + \frac{3}{2}i$, $-3i$, and $-\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

2.4.3.12. Answer. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

2.4.3.13. Answer. $\left\{-1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i\right\}$