MTH 111Z Lab Manual

Prepared by
Portland Community College
Mathematics Department
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Preface

This lab manual was designed with the intent of being used in the following manner. In each section:

- Students will complete the Preparation Exercises before receiving instruction on the content. Some instructors will have students do this before coming to class, while others might do it as a warmup in class.
- There will be some instructor-led presentation of the content. This may be a formal class lecture, a discovery activity, a video lecture, or something else.
- Students will then engage in Practice Exercises in a group setting to reinforce their initial understanding of the foundational concepts.
- An instructor will then assign one or more group activities. Because many of these activities are web-based and instructors can choose to use different activities, we have not included any of the possible activities in this document.
- The Definitions are meant to provide a single location for all definitions and some key concepts. Students can use this as a resource after having covered the topics in class.
- Students will complete some or all of the Exit Exercises, as decided by their instructor, to summarize their understanding of key concepts.
- In general, not every section of this lab manual will take the same amount of class time. Some sections might be half-day topics, while others could be two-day topics.
- Each instructor will identify for their class which exercises will be submitted as part of the course grade.

Course Resource Links:
Algebra and Trigonometry 2e: http://tiny.cc/111Z-Textbook
MTH 111 Supplement: http://tiny.cc/111Z-Supplement

Chapter 1

Functions

Contents

1.1 Functions and Function Notation

In this section, we'll develop our understanding of functions and function notation, whether the function is presented as a set of ordered pairs, a table of values, a graph, or an equation. We'll also learn how to evaluate a function given an input value and to solve for an input given a function's output value.

Textbook Reference: This relates to content in §3.1 and §3.2 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:
Many of us have at least one restaurant we would love to go to for a meal.
Go online and find a menu for a restaurant you'd choose to eat at.

a) What is the name of the restaurant and what type of food do they serve?

b) What is a dish you would like to get and how much does it cost?

c) Are there any other dishes on the menu that costs the same as your dish?
If so, what are they?

d) If I ask you the price of any specific dish from the menu, do you know how much it costs?

e) If you know how much I paid for a dish, do you know exactly what I ordered based just on the price? Explain your answer.

Practice Exercises

Practice 1:

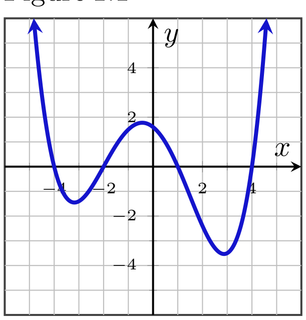
Determine if each of the following relations show y as a function of x.
Explain your reasoning by referencing the definition of a function. If a relation is not a function, provide a specific example why the definition was not satisfied.
Assume all ordered pairs are of the form (x, y).

a) $\{(\text{red}, \text{pepper}),\ (\text{green}, \text{pear}),\ (\text{purple}, \text{grape}),\ (\text{orange}, \text{orange}),\ (\text{yellow}, \text{pepper}),\ (\text{red}, \text{onion})\}$

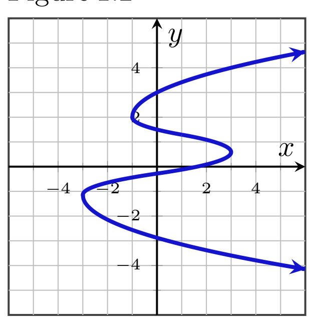
b) Table 1.1

x	-5	3	-1	2	4	6
y	9	-4	2	-4	-5	-6

c) Figure 1.1



d) Figure 1.2



Practice 2:

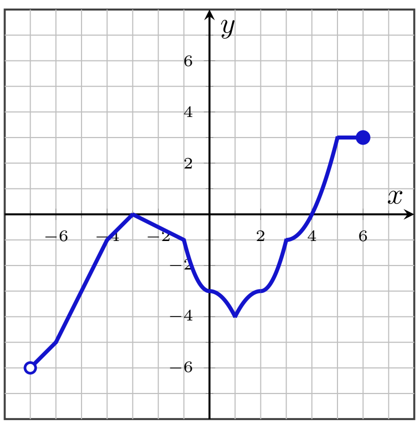
a) Let $y = g(x)$ be defined by the set of (x, y) ordered pairs:
$\{(-60, 5),\ (-23, 4),\ (-4, 3),\ (3, 2),\ (4, 1),\ (5, 0),\ (12, -1)\}$.

i. Find $g(5)$.

ii. Solve $g(x) = 3$.

b) Let $y = h(x)$ be defined by the graph in Figure 1.3 below.

Figure 1.3: $y = h(x)$



i. Find $h(-4)$.

ii. Solve $h(x) = -3$.

c) Let $f(x) = x^2 - 3$.

i. Find $f(-4)$.

ii. Solve $f(x) = 46$.

Definitions

Relation:

A **relation** is a set of (x, y) ordered pairs.

The variable x is called the **independent variable** or **input variable**.
Each individual x-value is referred to as an **input** or **input value**.

The variable y is called the **dependent variable** or **output variable**.
Each individual y-value is referred to as an **output** or **output value**.

Example: The set $\{(0,-2),\ (1,-1),\ (2,0),\ (1,-3),\ (3,-4)\}$ is a relation.

Function:

A **function** is a relation where each possible input value is paired with *exactly one* output value.
We say, "The output is a function of the input," and often write this algebraically as $y = f(x)$.

Example: The set $\{(0,-2),\ (1,-1),\ (4,0),\ (9,1),\ (16,2)\}$ is a function.
Example: The set $\{(0,-2),\ (\mathbf{1},-1),\ (2,0),\ (\mathbf{1},-3),\ (3,-4)\}$ is a relation, but *not* a function.

Domain and Range:

The **domain** of a relation or function is the set of all possible input values.
The **range** of a relation or function is the set of all possible output values.

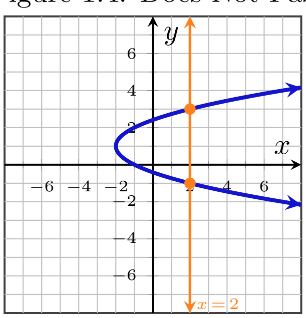
Example: Given the function $\{(0,-2),\ (4,0),\ (1,-1),\ (9,1),\ (16,2)\}$,
the domain of the function is $\{0,\ 4,\ 1,\ 9,\ 16\}$ and the range of the function is $\{-2,\ 0,\ -1,\ 1,\ 2\}$.

Vertical Line Test:

If a vertical line can be drawn that intersects the graph more than once, the graph is not the graph of a function with x as the independent variable and y as the dependent variable.

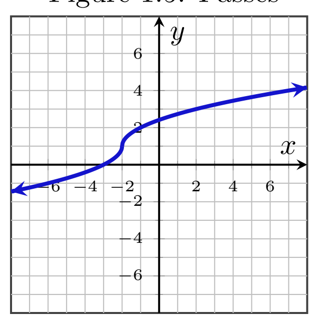
Example:

Figure 1.4: Does Not Pass



Example:

Figure 1.5: Passes



Exit Exercises

Exit 1:

a) What is the definition of a function?

b) What are two examples of functions that you use in your daily life outside of school? Explain how these are functions, referencing the definition of a function.

c) What do you look for in the graph of a relation to determine if the graph is the graph of a function or not? Fully explain your answer.

Exit 2:
If $f(x) = 2x^2 - 7x$, evaluate $f(-3)$.

Exit 3:
If $g(x) = x^2 + 6x$, solve $g(x) = 16$.

On a scale of 1 - 5, how are you feeling with the concepts related to functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

1.2 Domain, Range, and Behaviors of Functions

In this section, we'll learn to identify the domain and range of functions given in various forms, as well as determine when a function exhibits important behaviors.

Textbook Reference: This relates to content in §3.2 and §3.3 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:

a) For any real number k other than 0, what is $\dfrac{0}{k}$ and why?

b) For any real number k other than 0, what is $\dfrac{k}{0}$ and why?

c) Given $f(x) = \sqrt{x+2}$, evaluate $f(23)$ and $f(-18)$.

Preparation 2:

a) Draw $x > -2$ on a number line and write the set of values in both interval and set-builder notations.
Hint: Here is a review video of these two notations.

b) Draw $x \leq 6$ on a number line and write the set of values in both interval and set-builder notations.

Practice Exercises

Practice 1:

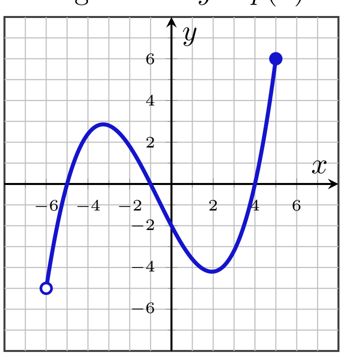
Algebraically find the domain of the following functions.
State the domains in both interval notation and set-builder notation.

a) $f(x) = \sqrt{-2x+18}$ b) $g(t) = \sqrt[3]{3t-24}$ c) $h(k) = \dfrac{k+3}{k-9}$

Practice 2:

Find the domain and range of the function p graphed in Figure 1.6.
State both in interval notation and set-builder notation.

Figure 1.6: $y = p(x)$



Practice 3:
Below are the graphs of $y = r(t)$ in Figure 1.7 and $y = s(t)$ in Figure 1.8.

Figure 1.7: $y = r(t)$

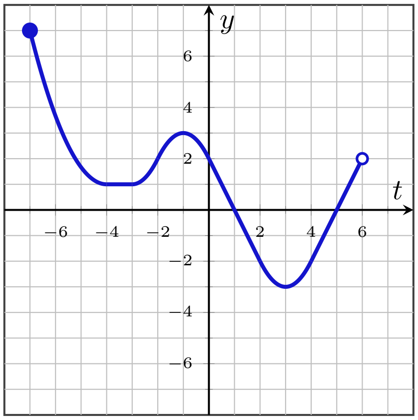
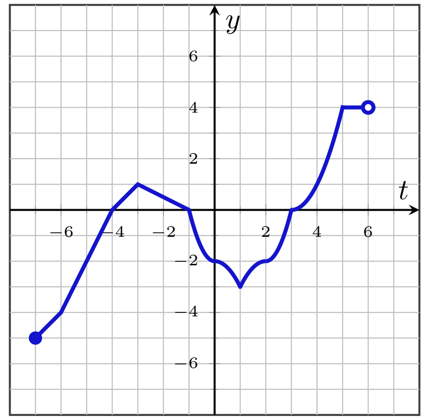


Figure 1.8: $y = s(t)$



a) Over what intervals is r increasing?

b) Over what intervals is s negative?

c) What is the absolute maximum value of r?

d) State any local minimum points of s.

e) Over what intervals is r constant?

f) Over what intervals is s decreasing?

g) Over what interval is r positive?

h) What is the absolute minimum value of s?

Definitions

Domain and Range:
The **domain** of a function is the set of all possible input values for the function.
The **range** of a function is the set of all possible output values for the function.
The domain and range are commonly stated using interval notation or set-builder notation.

Example: View this Desmos graph to see an interactive example of these definitions.

Positive and Negative:
A function f is **positive** if the output values are greater than 0. f is positive when $f(x) > 0$.
A function f is **negative** if the output values are less than 0. f is negative when $f(x) < 0$.

Example: View this Desmos graph to see an interactive example of these definitions.

Increasing, Decreasing, and Constant:
Let f be a function that is defined on an open interval I, with a and b in I and $b > a$.

f is **increasing** on I if $f(b) > f(a)$ for all a and b in I.
In other words, as you move left-to-right on the interval I, your y-values increase.

f is **decreasing** on I if $f(b) < f(a)$ for all a and b in I.
In other words, as you move left-to-right on the interval I, your y-values decrease.

f is **constant** on I if $f(b) = f(a)$ for all a and b in I.
In other words, as you move left-to-right on the interval I, your y-values do not change.

Example: View this Desmos graph to see an interactive example of these definitions.

Local Minimum or Maximum:
Given a function f that is defined on an open interval I, with c in I.

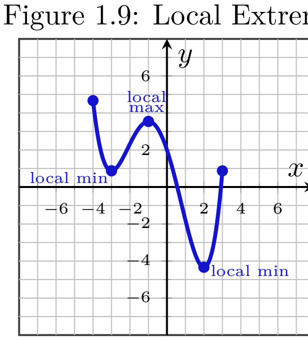
f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in I.
The **local maximum value** of f is the output $f(c)$.

f has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in I.
The **local minimum value** of f is the output $f(c)$.

Example: In Figure 1.9, the function has two local minimum points and one local maximum point.
The local minimum value of about 0.9 occurs at $x = -3$.
The local minimum value of about -4.3 occurs at $x = 2$.
The local maximum value of about 3.5 occurs at $x = -1$.

Example:

Figure 1.9: Local Extrema



Absolute Minimum or Maximum:

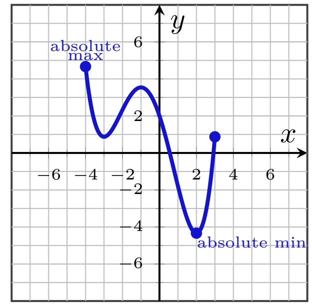
f has an **absolute maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f.
The **absolute maximum value** of f is the output $f(c)$.

f has an **absolute minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in the domain of f.
The **absolute minimum value** of f is the output $f(c)$.

Example: In Figure 1.10, the function has an absolute minimum point and an absolute maximum point.
The absolute minimum value of about -4.3 occurs at $x = 2$.
The absolute maximum value of about 4.8 occurs at $x = -4$.

Example:

Figure 1.10: Absolute Extrema



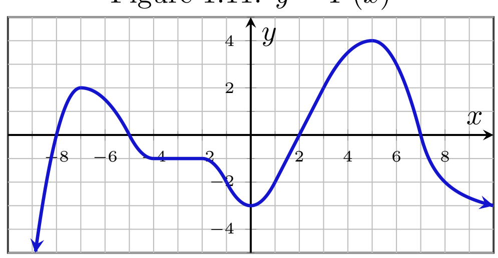
Exit Exercises

Exit 1:
What is the domain of $f(x) = \sqrt{x}$? What is the domain of $g(x) = \sqrt[3]{x}$? Why are these domains different?

Exit 2:
Graphically speaking, what is the difference between a function being negative and a function decreasing?

Exit 3:
For the function F graphed in Figure 1.11, answer the following.

Figure 1.11: $y = F(x)$



a) Over what intervals is F increasing?

b) What is the range of F?

c) Over what intervals is F negative?

d) What are any local minimum points on F?

e) Over what intervals is F constant?

f) What is the absolute maximum value of F?

On a scale of 1 - 5, how are you feeling with the concepts related to the graphical behaviors of functions? Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

1.3 Average Rates of Change and The Difference Quotient

In this section, we'll learn how to calculate the average rate of change of a function's output between two specific inputs, as well as evaluate the difference quotient, which is a general form of the average rate of change for a function.

Textbook Reference: This relates to content in §3.3 of *Algebra and Trigonometry 2e.*

Preparation Exercises

Preparation 1:
Suppose you're driving south on I-5 in Oregon and you pass mile marker 294 in Portland at 1:35 PM. Later, you pass mile marker 194 in Eugene at 3:05 PM.

a) What was your average speed (in miles per hour) of the trip from Portland to Eugene?

b) What was your speed at any particular moment, say as you drove past mile marker 256 in Salem?

Preparation 2:
Let $f(x) = x^2 - 2x$. Evaluate and simplify the following:

a) $f(4)$

b) $f(-6)$

c) $f(a)$

d) $f(a + h)$

Practice Exercises

Practice 1:

a) The function m in Table 1.2 below shows the cost of movie tickets[1] in the U.S. in the year t.

Table 1.2: Price of Movie Tickets in the U.S.

t (year)	$m(t)$ (in dollars)
1995	4.35
1999	5.06
2003	6.03
2009	7.50
2013	8.13
2017	8.97
2021	10.17

(a) What is the unit of the average rate of change in the price of a movie over any time period?

(b) What is the average rate of change in the price of a movie ticket from 2003 to 2021?

b) Given $f(n) = \frac{1}{3}n^2 - 1$, find the average rate of change of f on the interval $[3, 9]$.

[1] Costs obtained from https://www.natoonline.org/data/ticket-price

Practice 2:
Find and simplify the difference quotient for each of the following functions.

a) $f(x) = -6x + 8$

b) $g(x) = -2x^2 - 5x$

Definitions

Rate of Change:

A **rate of change** describes how the output values change in relation to a change in the input values. The unit for the rate of change is “output unit(s) per input unit.”

Average Rate of Change:

The **average rate of change** for a function f between two input values x_1 and x_2 is the difference in their output values divided by the difference in the two input values. The average rate of change is calculated using the formula

$$\text{average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1},\ x_1 \neq x_2$$

The average rate of change is the slope of the line between the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Example: The function $E(x)$ gives the cost of a dozen eggs (in dollars) x year after 2010. If we know $E(19) = 1.362$ and $E(23) = 2.666$, we can find average rate of change as

$$\begin{aligned}\frac{E(23) - E(19)}{23 \text{ years} - 19 \text{ years}} &= \frac{\$2.666 - \$1.362}{4 \text{ years}} \\ &= \frac{\$1.304}{4 \text{ years}} \\ &\approx \$0.33/\text{year}\end{aligned}$$

This shows that between 2019 and 2023, the cost of a dozen eggs increased on average by about \$0.33/year.

Difference Quotient:

The **difference quotient** for a function f is given by the formula

$$\frac{f(x+h) - f(x)}{h},\ h \neq 0$$

The difference quotient is the average rate of change between the two points $(x, f(x))$ and $(x + h, f(x + h))$.

Example: Given the function $f(x) = 3x^2 - 4x$, the difference quotient would be evaluated as

$$\begin{aligned}\frac{\boldsymbol{f(x+h)} - f(x)}{h} &= \frac{\boldsymbol{3(x+h)^2 - 4(x+h)} - (3x^2 - 4x)}{h} \\ &= \frac{\boldsymbol{3x^2 + 6xh + 3h^2 - 4x - 4h} - 3x^2 + 4x}{h} \\ &= \frac{6xh + 3h^2 - 4h}{h} \\ &= \frac{h(6x + 3h - 4)}{h} \\ &= 6x + 3h - 4,\ \text{for } h \neq 0\end{aligned}$$

Exit Exercises

Exit 1:

a) What are two situations in your daily life that involve an average rate of change?
What are the units for these rates of change?

b) What is the formula for the difference quotient for the function k that has an input variable p?

c) If you have a function m that gives the price of a gallon of milk in the year t, what would be the unit for the average rate of change for m?

Exit 2:
Find and simplify the difference quotient for the function $f(t) = \dfrac{3}{t-6}$.

On a scale of 1 - 5, how are you feeling with the concepts related to the difference quotient?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

1.4 Piecewise-Defined Functions

In this section, we'll explore piecewise-defined functions, which are functions constructed from pieces of several other functions. We'll find the domain and range of these types of functions, as well as graph, evaluate, solve them. And given the graph of a piecewise-defined function, we'll construct the formula for the graph's function.

Textbook Reference: This relates to content in §3.2 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:
A family event charges $4/person, with a maximum of $20 for any single family.

a) How much will a family of three pay?

b) How much will a family of seven pay?

c) At what number of people does the calculation change from being per person to a single charge for the whole family?

Preparation 2:
In November 2022, Portland General Electric set the rates for basic residential service as a function of the number of kilowatt-hours (kWh) of energy used. The rates in November 2022 were $0.0642/kWh when up to 1000 kWh (kilowatt-hours) are used and if greater than 1000 kWh are used, then the first 1000 kWh are billed at the $0.0642/kWh rate and $0.07002/kWh is charged for the energy usage greater than the initial 1000 kWh.

a) What is the cost for using 740 kWh?

b) What is the cost for using 1320 kWh?

c) What is a formula to find the cost for using x kWh if x is greater than 1000 kWh?

Practice Exercises

Practice 1:

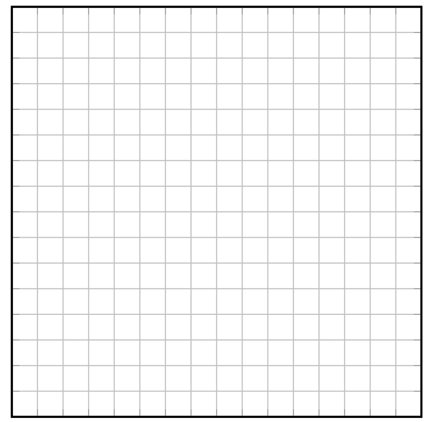
Let $f(x) = \begin{cases} x^2 - 4 & \text{if} \quad -2 \le x < 3 \\ \frac{2}{3}x - 1 & \text{if} \quad x \ge 3 \end{cases}$

a) Evaluate $f(5)$.

b) What is the domain of f?

c) Graph $y = f(x)$ in Figure 1.12.

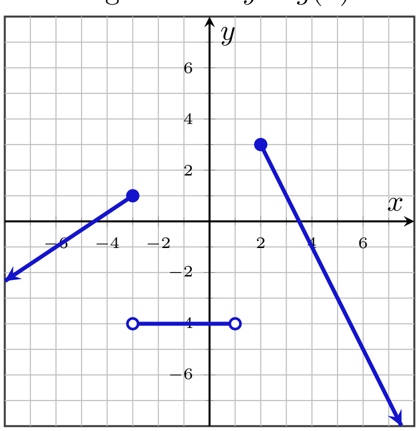
Figure 1.12: $y = f(x)$



Practice 2:

The graph of $y = g(x)$ is in Figure 1.13.

Figure 1.13: $y = g(x)$



a) Evaluate $g(-2)$.

b) Solve $g(x) = -1$.

c) What is the domain of g?

d) What is the range of g?

e) State the formula for the function g.

Definitions

Piecewise-Defined Function:

A **piecewise-defined function** is a function which uses different formulas for calculating the output on different intervals of its domain. Each formula is used on a distinct interval of the domain.

The notation we use to write a piecewise-defined function is:

$$f(x) = \begin{cases} \text{formula \#1} & \text{if} \quad x \text{ is in this part of the domain of } f \\ \text{formula \#2} & \text{if} \quad x \text{ is in this other part of the domain of } f \\ \text{etc.} & \text{if} \quad \text{etc.} \end{cases}$$

The domain of the function is the union of the intervals used by the separate formulas.

Example:

The function

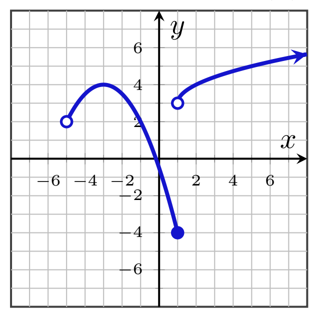
$$f(x) = \begin{cases} -\frac{1}{2}(x+3)^2 + 4 & \text{if} \quad -5 < x \leq 1 \\ \sqrt{x-1} + 3 & \text{if} \quad x > 1 \end{cases}$$

is a piece-wise defined function.

Its graph is in Figure 1.14 to the right.

Its domain is $(-5, \infty)$ and its range is $[-4, \infty)$.

Figure 1.14: $y = f(x)$



Exit Exercises

Exit 1:
Use $F(x) = \begin{cases} -\frac{1}{2}x^2 + 1 & \text{if} \quad x \leq 2 \\ -\frac{1}{2}x + 5 & \text{if} \quad x > 4 \end{cases}$ to answer the following.

a) Evaluate $F(8)$.

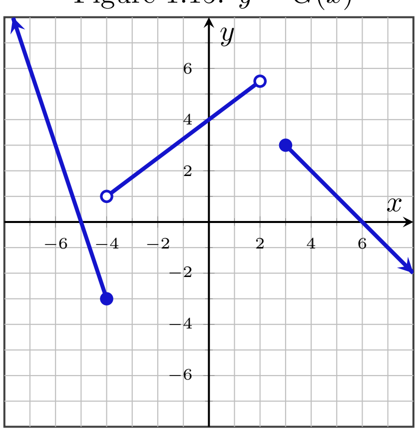
b) Evaluate $F(-6)$.

Exit 2:
In November 2022, Portland General Electric set the rates for basic residential service to be \$0.0642/kWh for up to 1000 kWh used and then \$0.07002/kWh for any usage greater than 1000 kWh. Find a piecewise-defined function that gives the cost of electricity used (in dollars) as a function of x, the amount of kWh used.

Exit 3:

$y = G(x)$ is graphed in Figure 1.15 below.

Figure 1.15: $y = G(x)$



a) State the formula for the function G.

b) Evaluate $G(-4)$ from the graph.

c) Solve $G(x) = 0$ from the graph.

On a scale of 1 - 5, how are you feeling with the concepts related to piecewise-defined functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

1.5 Algebraic Combinations of Functions and Function Composition

In this section, we'll learn about several ways we can combine functions algebraically, as well as use the output from one function as the input for another. As we've done before, we'll investigate these ideas with functions presented as graphs, as tables, and as formulas.

Textbook Reference: This relates to content in §3.4 of *Algebra and Trigonometry 2e.*

Preparation Exercises

Preparation 1:
Suppose it costs a bakery \$3,000 for rent and utilities each month and each loaf of bread costs \$2.15 to produce. The bakery sells each loaf of bread for \$6.49.

a) Find a function C that calculates the total cost (in dollars) each month to produce x loaves of bread.

b) Find a function R that calculates the total revenue (in dollars) each month from selling x loaves of bread.

c) Find a function P that calculates the profit (in dollars) from producing and selling x loaves of bread each month. (Note: The profit is the difference between the revenue and costs.)

Preparation 2:
The function $r = f(t) = 0.35t$ gives the radius (in inches) of the circular pattern formed when a drop of water hits a pond t seconds after the drop of water hits the pond's surface. The function $a = g(r) = \pi r^2$ gives the area of a circle (in square inches) when the circle has a radius of r inches.

How would you determine the area of the circle 9 seconds after a water drop lands on the pond?

Practice Exercises

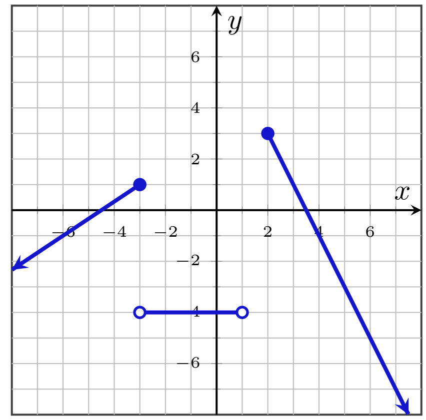
Practice 1:

Let $f(x) = x^2 - 4x$, $g(x) = \sqrt{3x+1}$, h be defined by Table 1.3, and k defined by Figure 1.16.

Table 1.3: $h(x)$

x	$h(x)$
-4	8
-2	-1
1	3
2	6
4	-7
8	-5

Figure 1.16: $y = k(x)$



a) Evaluate the following:

i) $(f - k)(-3)$

ii) $(g \cdot h)(1)$

b) Evaluate the following:

i) $(k \circ g)(5)$

ii) $(f \circ h)(8)$

Practice 2:

Let $F(x) = \dfrac{x+5}{x-3}$, $G(x) = x^2 - 4$, and $H(x) = \sqrt{2x+19}$

State the domain of the following functions:

a) $\dfrac{H}{G}$

b) $F \circ H$

Definitions

Algebraic Combinations of Functions

Two functions f and g can be combined using the operations of addition, subtraction, multiplication, or division as follows:

$\boldsymbol{f+g}$ is defined for all values of x in the domain of both f and g as

$$(f+g)(x) = f(x) + g(x)$$

$\boldsymbol{f-g}$ is defined for all values of x in the domain of both f and g as

$$(f-g)(x) = f(x) - g(x)$$

$\boldsymbol{f \cdot g}$ is defined for all values of x in the domain of both f and g as

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$\boldsymbol{\dfrac{f}{g}}$ is defined for all values of x in the domain of both f and g and where $g(x) \neq 0$ as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Function Composition

The **composition of functions**, $\boldsymbol{f \circ g}$ occurs when the output of one function g is used as the input of another function f and is defined as

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is all values of x in the domain of g where the values of $g(x)$ are in the domain of f.

Note that $f \cdot g$ is used for the product of two functions, while $f \circ g$ is used for composition.

Exit Exercises

Exit 1:
Answer the following in general for two functions f and g.

a) What is meant by $(f+g)(6)$? Explain both algebraically, as well as in written words.

b) For two functions f and g, how do you find the domain of $f+g$, $f-g$, or $f \cdot g$?
What else do you need to consider for $\dfrac{f}{g}$?

c) What is meant by $(f \circ g)(-4)$? Explain both algebraically, as well as in written words.

d) In general, does the order of composition matter? Does $(f \circ g)(x)$ yield the same thing as $(g \circ f)(x)$?

Exit 2:
Let $f(x) = x^2 + 7x$, $g(x) = \sqrt{5x-1}$, and $h(x) = \dfrac{x+1}{x-2}$.

a) Evaluate $(g-f)(10)$.

b) Evaluate $(h \circ f)(x)$.

On a scale of 1 - 5, how are you feeling with the concepts related to function algebra and function composition? Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

1.6 Graph Transformations and Symmetry

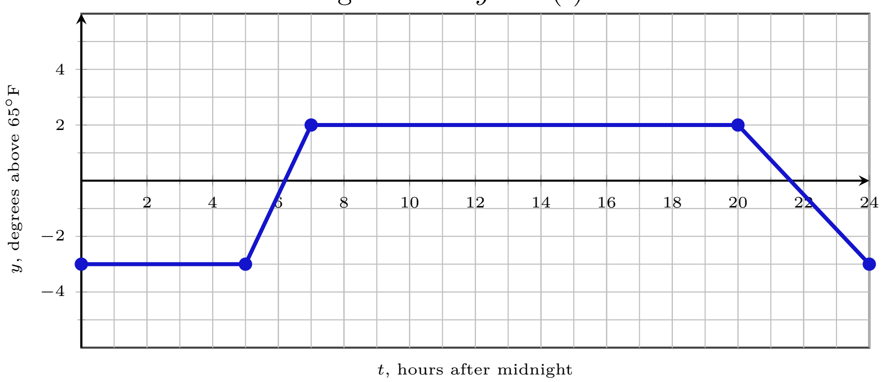
In this section, we'll look at ways we can algebraically manipulate the inputs and outputs of a function and see the impact this has on the overall function and its graph. We'll also look at the symmetry that some functions have related and how this symmetry relates to some of the transformations we'll see.

Textbook Reference: This relates to content in §3.5 of *Algebra and Trigonometry 2e.*

Preparation Exercises

Preparation 1:
Suppose the following graph represents the function A which gives the air temperature (in °F above 65°F) in a classroom t hours after midnight each weekday.

Figure 1.17: $y = A(t)$



a) Let W be a new function that has the same schedule as the room used by A, but is for a room that is always two degrees warmer than the room used by A. In Figure 1.17, sketch the graph of the function W.

b) What expression could we use for W to represent the function used to create the graph for the room that is always two degrees warmer than the room used for A?

c) If we were to vertically shift the graph of $y = A(t)$ down by one unit in the y direction to create a new function C, what would that mean in the context of the temperature of the room and what expression could we use to represent C?

Practice Exercises

Practice 1:

Let f be a function.

a) If $k_1(x) = -f(x+3)$, state the transformations that take the graph of $y = f(x)$ to the graph of $y = k_1(x)$.

b) If $k_2(x) = f(\frac{1}{5}x)+3$, state the transformations that take the graph of $y = f(x)$ to the graph of $y = k_2(x)$.

Practice 2:

Let g be a function for which we know that $g(3) = 5$.

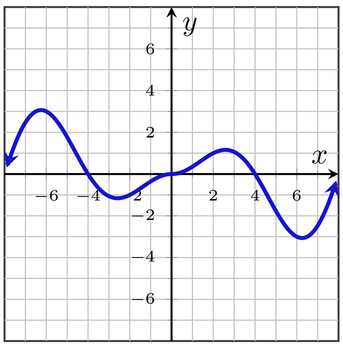
a) If $m(x) = -2g(x-4) + 7$, state a sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = m(x)$?

b) What point do you know is on the graph of $y = g(x)$?

c) What point do you know is on the graph of $y = m(x)$?

Practice 3:
Given in Figure 1.18 is the graph of $y = f(x)$.
Based on the graph, would you say that f is even, odd, or neither? Explain your answer.

Figure 1.18: $y = f(x)$



Practice 4:
Algebraically determine if the function $g(x) = -2x^2 - 3x$ is even, odd, or neither.

Practice 5:

a) If $k_3(x) = 4 \cdot f(6x - 2) + 5$, state the transformations that take the graph of $y = f(x)$ to the graph of $y = k_3(x)$.

b) If $k_4(x) = \frac{7}{3} \cdot f(\frac{1}{5}x - 1)$, state the transformations that take the graph of $y = f(x)$ to the graph of $y = k_4(x)$.

Definitions

Vertical Shift:
Given a function f, the graph of $g(x) = f(x) + k$ for some real number k is a **vertical shift** of the graph of $y = f(x)$.

If $k > 0$, g will be the graph of f shifted up by k units.
If $k < 0$, g will be the graph of f shifted down by k units.

Example: View this Desmos graph to see an interactive example of the definition.

Horizontal Shift:
Given a function f, the graph of $g(x) = f(x - h)$ for some real number h is a **horizontal shift** of the graph of $y = f(x)$.

If $h > 0$, g will be the graph of f shifted right by h units.
If $h < 0$, g will be the graph of f shifted left by h units.

Example: View this Desmos graph to see an interactive example of the definition.

Vertical Stretch/Compression:
Given a function f, the graph of $g(x) = a \cdot f(x)$ for some real number a, where $a \neq 0$, is a **vertical stretch** or a **vertical compression** of the graph of $y = f(x)$.

If $a > 1$, g will be the graph of f vertically stretched by a factor of a.
If $0 < a < 1$, g will be the graph of f vertically compressed by a factor of a.

If $a < 0$, g will be a combination of a vertical reflection over the x-axis *and* a vertical stretch or compression of f.

Example: View this Desmos graph to see an interactive example of the definition.

Horizontal Stretch/Compression:
Given a function f, the graph of $g(x) = f(b \cdot x)$ for some real number b, where $b \neq 0$, is a **horizontal stretch** or a **horizontal compression** of the graph of $y = f(x)$.

If $b > 1$, g will be the graph of f horizontally compressed by a factor of $\frac{1}{b}$.
If $0 < b < 1$, g will be the graph of f horizontally stretched by a factor of $\frac{1}{b}$.

If $b < 0$, g will be a combination of a horizontal reflection over the y-axis *and* a horizontal stretch or compression of f.

Example: View this Desmos graph to see an interactive example of the definition.

Vertical Reflection:
Given a function f, the graph of $y = -f(x)$ is a **vertical reflection** of the graph of $y = f(x)$ over the x-axis.

Example: View this Desmos graph to see an interactive example of the definition.

Horizontal Reflection:
Given a function f, the graph of $y = f(-x)$ is a **horizontal reflection** of the graph of $y = f(x)$ over the y-axis.

Example: View this Desmos graph to see an interactive example of the definition.

Combined Transformations:
Given a function f, the combined vertical transformations written in the form $y = a \cdot f(x) + k$, $a \neq 0$, would be applied in the order:

- a vertical reflection over the x-axis, if $a < 0$
- a vertical stretch or compression by a factor of $|a|$
- a vertical shift up or down by k units

Given a function f, the combined horizontal transformations written in the form $y = f\left(b \cdot (x - h)\right)$, $b \neq 0$, would be applied in the order:

- a horizontal reflection over the y-axis, if $b < 0$
- a horizontal stretch or compression by a factor of $\left|\frac{1}{b}\right|$
- a horizontal shift left or right by h units

Given a function f, the combined horizontal transformations written in the form $y = f(bx - h)$, $b \neq 0$, would be applied in the order:

- a horizontal shift left or right by h units
- a horizontal reflection over the y-axis, if $b < 0$
- a horizontal stretch or compression by a factor of $\left|\frac{1}{b}\right|$

Even Function:
Given a function f, if $f(-x) = f(x)$ for every input x, then f is an **even function**.
We describe even functions as being symmetrical about the y-axis.

Example: View this Desmos graph to see an interactive example of the definition.

Odd Function:
Given a function f, if $-f(-x) = f(x)$ for every input x, then f is an **odd function**.
We describe odd functions as being symmetrical about the origin.
Note: $f(x) = -f(-x)$ is equivalent to the statement $f(-x) = -f(x)$.

Example: View this Desmos graph to see an interactive example of the definition.

Exit Exercises

Exit 1:

a) What is meant by an “inside” change? How do inside changes impact the graph of a function?

b) What is meant by an “outside” change? How do outside changes impact the graph of a function?

c) What is the relationship between even and odd functions and transformations?

Exit 2:

f is a function and $f(-20) = 32$.

a) If $g(x) = -\frac{1}{8}f(-2x+10)+5$, list the sequence of transformations that take the graph of $y = f(x)$ to the graph of $y = g(x)$.

b) What point is on the graph of $y = f(x)$ and what point will be on the graph of $y = g(x)$?

On a scale of 1 - 5, how are you feeling with the concepts related to function transformations?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

1.7 Inverse Functions

In this section, we'll see what happens if you turn a function inside-out and make the output become the input and the input become the output. We'll also explore when doing this will result in a function and what it means if it does.

Textbook Reference: This relates to content in §3.7 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:
Consider the following set of ordered pairs:

$$\{(\text{Alaska}, 1),\ (\text{Washington}, 10),\ (\text{Oregon}, 6),\ (\text{Idaho}, 2),\ (\text{Nevada}, 4),\ (\text{Hawaii}, 2),\ (\text{California}, 52)\}$$

a) Does this set represent a function? Answer by referencing the definition of a function.

b) If we were to swap the x and y-values, would this new set be a function? Explain your answer.
$\{(1, \text{Alaska}),\ (10, \text{Washington}),\ (6, \text{Oregon}),\ (2, \text{Idaho}),\ (4, \text{Nevada}),\ (2, \text{Hawaii}),\ (52, \text{California})\}$

Preparation 2:
The formula to convert a temperature F in degrees Fahrenheit to a temperature C in degrees Celsius is given by the function

$$C = g(F) = \frac{5}{9}(F - 32)$$

a) For each temperature in degrees Fahrenheit, how many temperatures in degrees Celsius are produced by this formula?

b) For each temperature in degrees Celsius, how many temperatures in degrees Fahrenheit are produced by this formula?

c) Would it be true to state that C is a function of F and also that F is a function of C? Why or why not?

Practice Exercises

Practice 1:

f is defined in Table 1.4.

Table 1.4

x	-4	-2	0	2	4
$f(x)$	4	2	-2	-4	0

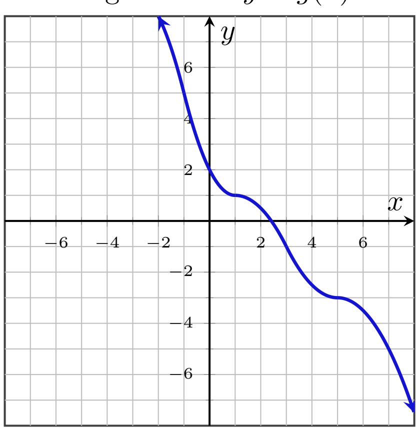
Evaluate the following.

a) $f(0)$ b) $f^{-1}(0)$ c) $f^{-1}(-4)$

Practice 2:

$y = g(x)$ is defined in Figure 1.19.

Figure 1.19: $y = g(x)$



Evaluate the following.

a) $g(5)$

b) $g^{-1}(5)$

c) $g^{-1}(-1)$

Practice 3:

Graph the function $f(x) = 4 + \sqrt[3]{x-1}$ in a graphing utility (such as Desmos) to confirm that it is a one-to-one function and then find the formula for f^{-1}.

Practice 4:
Graph the function $g(x) = 2 - \sqrt{x+3}$ in a graphing utility.

a) Is g a one-to-one function? Why or why not?

b) State the domain and range of g.

c) Algebraically find the formula for g^{-1}.

d) State the domain and range of g^{-1}.

Definitions

One-to-One Function:

A function f is said to be **one-to-one** if for every possible output (y-value) in the range of f, there is exactly one input (x-value) in the domain of f.

In other words, in a one-to-one function, each possible input is paired with exactly one output **AND** each possible output is paired with exactly one input.

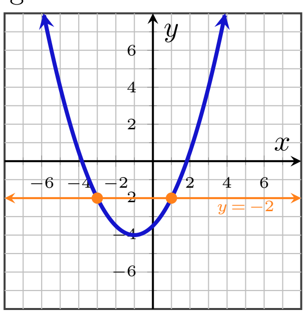
Example: The set $\{(0,-2),\ (1,\mathbf{-1}),\ (4,0),\ (9,\mathbf{-1}),\ (16,-3)\}$ is a function, but it is *not* one-to-one.
Example: The set $\{(0,-2),\ (1,-1),\ (2,0),\ (3,1),\ (4,2)\}$ is a function and *is* one-to-one.

Horizontal Line Test:

If a horizontal line can be drawn that intersects the graph of a function more than once, the graph is not the graph of a one-to-one function.

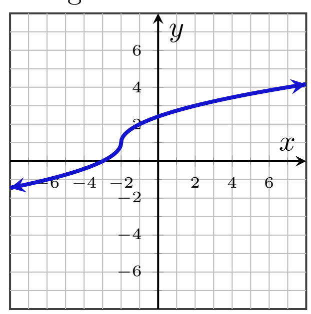
Example:

Figure 1.20: Does Not Pass



Example:

Figure 1.21: Passes



Inverse Function:

If a function f is one-to-one, then the function has an **inverse**, f^{-1}.

Two functions, f and f^{-1} are **inverse functions** if and only if both of the following are true:

- $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1}.
- $f^{-1}\left(f(x)\right) = x$ for all x in the domain of f.

The domain of a function f is the range of the inverse function f^{-1}.
The range of a function f is the domain of the inverse function f^{-1}.

Example:
If f is defined by the set $\{(0,-2),\ (1,-1),\ (2,0),\ (3,1),\ (4,2)\}$,
then f^{-1} is the set $\{(-2,0),\ (-1,1),\ (0,2),\ (1,3),\ (2,4)\}$.

The domain of f is $\{0,\ 1,\ 2,\ 3,\ 4\}$ and the range is $\{-2,\ -1,\ 0,\ 1,\ 2\}$.

The domain of f^{-1} is $\{-2,\ -1,\ 0,\ 1,\ 2\}$ and the range is $\{0,\ 1,\ 2,\ 3,\ 4\}$.

Exit Exercises

Exit 1:

a) Explain what is meant by the phrase "one-to-one" and how you can tell from the graph of the function if the function is one-to-one?

b) What is the relationship of the domain and range of a function f and its inverse function f^{-1}?

c) What happens when you compose two functions that are inverses of each other?

d) Why is $g(x) = (4x - 1)^3$ invertible and $h(x) = (4x - 1)^2$ is not?

e) Given $m(x) = 19 + (-3x + 1)^5$, find m^{-1}.

On a scale of 1 - 5, how are you feeling with the concepts related to inverse functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

Chapter 2

Exponential and Logarithmic Functions

Contents

2.1 Exponential Functions

In this section, we'll investigate a new type of function that has a constant *percent* rate of change, whether it's a constant percent of increase or a constant percent of decrease.

Textbook Reference: This relates to content in §6.1 and §6.2 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:
Suppose you have a water filter that can remove 60% of the contaminants in the water each time the water passes through the filter.

a) If 60% of the contaminants are removed with each pass, what percent still remains after a single pass?

b) If you start with 156.25 mcg of contaminants in some water, how much will be left after you pass the water through the filter once?

c) How much will be left after a second pass through the filter?

d) How much will be left after a third pass through the filter?

e) Will the filter ever remove all of the contaminants from the water? Why or why not?

Practice Exercises

Practice 1:
Which of the following functions are exponential functions?
For the exponential functions, identify if they represent exponential growth or decay.

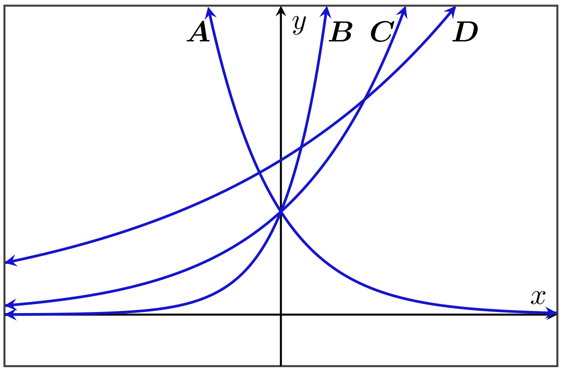
a) $f(x) = 2\left(\frac{3}{4}\right)^x$ b) $g(x) = 3\,(-5)^x$ c) $h(x) = 4\,(x)^7$ d) $j(x) = 59^{x+2}$

Practice 2:
A baseball card was worth \$50 in 1995 and its value has increased by 7% per year every year since then. Find a formula for a function V that models the value of the baseball card t years after 1995.

Evaluate $V(21)$ and explain its meaning in context.

Practice 3:
Match each function with one of the graphs in Figure 2.1.

Figure 2.1



- $F(x) = 2(3)^x$ is graph _____
- $G(x) = 2(0.5)^x$ is graph _____
- $H(x) = 3(1.2)^x$ is graph _____
- $J(x) = 2(1.5)^x$ is graph _____

Definitions

Exponential Function:

For any real number x, an **exponential function** is a function of the form $\boldsymbol{f(x) = a \cdot b^x}$ where

- a is a non-zero real number
- b is any positive real number, where $b \neq 1$

Exponential functions grow or decay with a constant *percent* rate of change.

Key Characteristics of Exponential Functions:

For an exponential function $f(x) = a \cdot b^x$, with $a > 0$ and $b > 0, b \neq 1$, we have the following:

- $(0, a)$ is the vertical intercept.
- There is no horizontal intercept.
- The domain of f is $(-\infty, \infty)$.
- The range of f is $(0, \infty)$.
- f is a one-to-one function.
- The horizontal asymptote is $y = 0$.
- If $b > 1$, then f is an increasing function and
  - as $x \to \infty$, $f(x) \to \infty$, and
  - as $x \to -\infty$, $f(x) \to 0$.
- If $0 < b < 1$, then f is a decreasing function and
  - as $x \to \infty$, $f(x) \to 0$, and
  - as $x \to -\infty$, $f(x) \to \infty$.

Figure 2.2: $y = a \cdot b^x, b > 1$

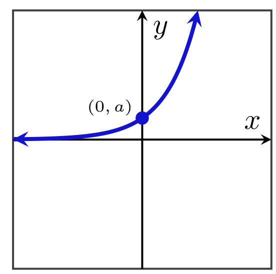
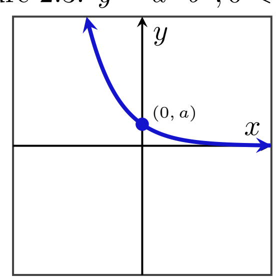


Figure 2.3: $y = a \cdot b^x, 0 < b < 1$



Example: View this Desmos graph to see an interactive example of a exponential function.

ExpFunction)

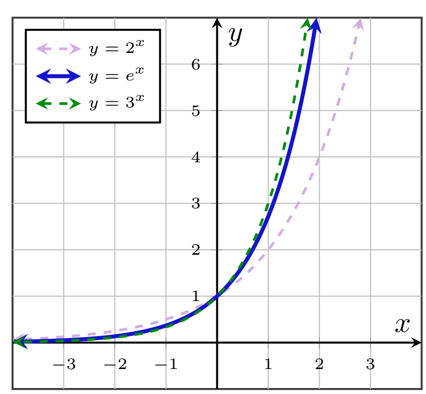
The Number $\boldsymbol{e}$:

The number $\boldsymbol{e}$ was discovered in the late 1600's by Jacob Bernoulli. Later in the 1700's, Leonard Euler discovered many of its interesting properties.

$\boldsymbol{e}$ can be approximated by 2.718281828459, though its decimal form does not end and does not repeat. It is an irrational number.

The graph of the function given by $f(x) = e^x$ looks a lot like the graphs of the functions given by $g(x) = 2^x$ and $h(x) = 3^x$, as shown in Figure 2.4.

Figure 2.4



Exit Exercises

Exit 1:
Given the formula for an exponential function, you should be able to look at the formula and identify if the function will represent exponential growth or exponential decay. How can you do this?

Give an example of a function for each, one exponential growth and one exponential decay, as part of your explanation.

Exit 2:
At the start of an experiment, a population of bacteria has 5 million bacteria and the population is decreasing by 13% every 6 hours. Find a formula for the function B that gives the number of bacteria (in millions) remaining after n 6-hour time intervals.

Evaluate $B(8)$ and explain its meaning in context.

Exit 3:
Find an exponential function f that satisfies $f(2) = \dfrac{3}{8}$ and $f(-1) = \dfrac{8}{9}$.

On a scale of 1 - 5, how are you feeling with the concepts related to exponential functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

2.2 Logarithmic Functions

In this section, we'll look at the inverse of exponential functions in general, as well as focus on the inverses of a few exponential functions with commonly used bases.

Textbook Reference: This relates to content in §6.3 and §6.4 of *Algebra and Trigonometry 2e*.

Preparation Exercises

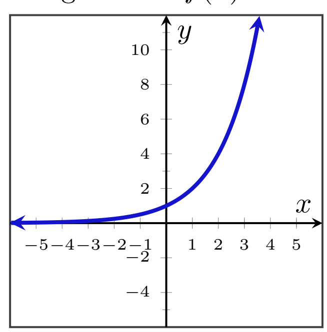
Preparation 1:

a) Rewrite $\sqrt[7]{2187} = 3$ as an exponential statement.

b) Rewrite $2^{10} = 1024$ as a radical statement.

Preparation 2:
Let $f(x) = 2^x$. The graph of $y = f(x)$ is in Figure 2.5.

Figure 2.5: $f(x) = 2^x$



a) Evaluate $f(5)$.

b) Evaluate $f(-2)$.

c) Solve $f(x) = 8$.

d) f is a one-to-one function. How do you know and what does this mean?

e) What is $f^{-1}(32)$?

f) What is $f^{-1}(\frac{1}{4})$?

g) What is $f^{-1}(1)$?

Practice Exercises

Practice 1:
Convert each exponential statement into a logarithmic statement.

a) $3^4 = 81$ b) $5^{-2} = \dfrac{1}{25}$ c) $7^0 = 1$

Practice 2:
Evaluate each logarithm without a calculator.

a) $\log(10000)$ b) $\log_{25}(5)$ c) $\log_4\left(\frac{1}{64}\right)$

Practice 3:
Let $f(x) = 7^x$ and $g(x) = \log_7(x)$.

a) What is the domain of f?

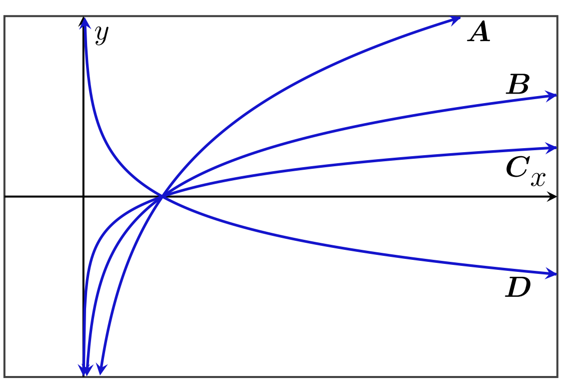
b) What is the range of f?

c) What is the domain of g?

d) What is the range of g?

Practice 4:
Match each function with one of the graphs in Figure 2.6.

Figure 2.6



- $f(x) = \log_{1.7}(x)$ is graph _____
- $g(x) = \log_{1/2}(x)$ is graph _____
- $h(x) = \log_{1.3}(x)$ is graph _____
- $j(x) = \log_3(x)$ is graph _____

Definitions

Logarithm:

For any real number $x > 0$, the **logarithm with base b of x**, where $b > 0$ and $b \neq 1$, is denoted by $\mathbf{log_b(x)}$ and is defined by

$$y = \log_b(x) \text{ if and only if } x = b^y$$

Common Logarithm:

The **common logarithm**, $\mathbf{log(x)}$, is a logarithm with base 10 and satisfies

$$y = \log(x) \text{ is equivalent to } 10^y = x, \text{for } x > 0$$

Natural Logarithm:

The **natural logarithm**, $\mathbf{ln(x)}$, is a logarithm with base e and satisfies the following:

$$y = \ln(x) \text{ is equivalent to } e^y = x, \text{for } x > 0$$

Key Characteristics of Logarithmic Functions:

For a logarthmic function $f(x) = \log_b(x)$, with $b > 0$, $b \neq 1$, and $x > 0$, we have the following:

- $(1, 0)$ is the horizontal intercept.
- There is no vertical intercept.
- The domain of f is $(0, \infty)$.
- The range of f is $(-\infty, \infty)$.
- f is a one-to-one function.
- The vertical asymptote is $x = 0$.
- If $b > 1$, then f is an increasing function and
  - as $x \to \infty$, $f(x) \to \infty$, and
  - as $x \to 0^+$, $f(x) \to -\infty$.
- If $0 < b < 1$, then f is a decreasing function and
  - as $x \to \infty$, $f(x) \to -\infty$, and
  - as $x \to 0^+$, $f(x) \to \infty$.

Figure 2.7: $y = \log_b(x), b > 1$

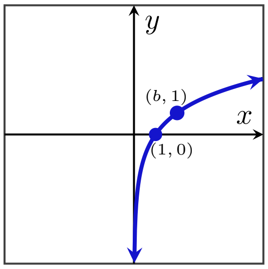
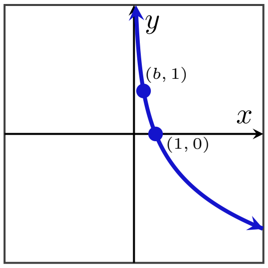


Figure 2.8: $y = \log_b(x), 0 < b < 1$



Example: View this Desmos graph to see an interactive example of a logarithmic function.

Exit Exercises

Exit 1:

a) Why do we call b the base of the logarithm $\log_b(x)$? Evaluate $\log_2(16)$ and use this in your answer.

b) Evaluate $\log_9(3)$ and then restate your logarithmic statement as an exponential statement.

c) For any $b > 0$, where $b \neq 1$, what is the domain of $f(x) = \log_b(x)$ and why is this the domain of f?

d) Does a logarithmic function have a horizontal or vertical asymptote and why?

e) Given $f(x) = \log_8(x)$ and $g(x) = -3 \cdot \log_8(x + 3) - 2$, state the sequence of transformations that takes the graph of $y = f(x)$ to the graph of $y = g(x)$ and also state the domain of g.

On a scale of 1 - 5, how are you feeling with the concepts related to logarithmic functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

2.3 Properties of Logarithms

In this section, we'll investigate some important properties of logarithms, which will help us to be able to solve exponential and logarithmic equations.

Textbook Reference: This relates to content in §6.5 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:
Simplify each expression in the left column without a calculator.
State the corresponding exponent rule in the right column.

a) 8^1 b^1

b) 7^0 b^0

c) 6^{-2} b^{-n}

d) $5^2 \cdot 5^9$ $b^n \cdot b^m$

e) $\dfrac{4^7}{4^3}$ $\dfrac{b^n}{b^m}$

f) $\left(3^2\right)^5$ $\left(b^n\right)^m$

Preparation 2:
Evaluate each logarithm without a calculator.

a) $\log_5(1)$

b) $\log_2(2)$

c) $\log(1)$

d) $\ln(e)$

Practice Exercises

Practice 1:
Expand $\log_6(5x^3y)$ as much as possible by rewriting the expression as a sum, difference, or product of logs or constant factors.

Practice 2:
Rewrite $\frac{1}{2}\ln(x+5) - 6\ln(x)$ as a single logarithm.

Practice 3:
Find the exact value of $\log_5(75) - \log_5(3) + \log_2(16)$ without using a calculator.

Practice 4:
Saskia and José disagree. Saskia says that $\log_3(x+2) - \log_3(x+1) = \dfrac{\log_3(x+2)}{\log_3(x+1)}$, but José says that's wrong. Who is correct and why?

Properties

Four Properties of Logarithms

Given any real number x and any positive real number b, with $b \neq 1$,

$$\log_b(1) = 0 \qquad \log_b(b^x) = x$$

$$\log_b(b) = 1 \qquad b^{\log_b(x)} = x$$

Product Rule for Logarithms

Given any positive real numbers M, N, and b, with $b \neq 1$,

$$\log_b(M \cdot N) = \log_b(M) + \log_b(N)$$

Quotient Rule for Logarithms

Given any positive real numbers M, N, and b, with $b \neq 1$,

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

Power Rule for Logarithms

Given any real number n, positive real numbers M and b, with $b \neq 1$,

$$\log_b(M^n) = n \cdot \log_b(M)$$

Change of Base Formula

Given positive real numbers M, n, and b, with $b \neq 1$ and $n \neq 1$,

$$\log_b(M) = \frac{\log_n(M)}{\log_n(b)}$$

Exit Exercises

Exit 1:

a) Only one of these is true. Which one and why?

i) $\log_2(8) + \log_2(16) = \log_2(8 \cdot 16)$

ii) $\log_2(8) \cdot \log_2(16) = \log_2(8 + 16)$

b) Only one of these is true. Which one and why?

i) $\dfrac{\log_3(27)}{\log_3(9)} = \log_3(27 - 9)$

ii) $\log_3(27) - \log_3(9) = \log_3\left(\frac{27}{9}\right)$

c) Rewrite each of these as a single logarithm.

i) $\log(10) - \log(2) + \log(3)$

ii) $\ln(a) - \ln(b) - \ln(c)$

d) Fully expand each of the following.

i) $\log_2\left(\dfrac{x^2}{y^3 z}\right)$

ii) $\ln(e^5 n^3 \sqrt[4]{k})$

On a scale of 1 - 5, how are you feeling with the concepts related to properties of logarithms?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

2.4 Exponential and Logarithmic Equations

In this section, we'll now combine what we know about exponential and logarithmic functions, as well as the properties of logarithms, in order to solve exponential and logarithmic equations.

Textbook Reference: This relates to content in §6.6 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:

a) Rewrite each logarithmic statement as an exponential statement.

(a) $\log_3(243) = 5$

(b) $\log_{16}(4) = \frac{1}{2}$

b) Solve $\log_5(x) = 3$ by first rewriting it as an exponential statement.

c) If $f(x) = x + 8$, what is the inverse function of f?

d) If $f(x) = 5x$, what is the inverse function of f?

e) How do inverse operations or inverse functions help us solve equations?

Practice Exercises

Practice 1:

Solve each of the following algebraically. Use a calculator to approximate any irrational solutions.

a) $3^{5x-6} = 81$

b) $9^{x-11} = 7$

c) $e^{x+3} + 4 = 19$

d) $5^{x-6} = 3^{2x+7}$

Practice 2:
Solve each of the following algebraically. Be sure to confirm any solutions are not extraneous.

a) $\log_6(3x+1) = \log_6(x-9)$

b) $\log_7(x-4) + 3 = 5$

c) $\log_3(x-2) = 1 - \log_3(x-4)$

Definitions and Properties

Logarithm:

For any real number $x > 0$, the **logarithm with base b of x**, where $b > 0$ and $b \neq 1$, is denoted by $\boldsymbol{\log_b(x)}$ and is defined by

$$y = \log_b(x) \text{ if and only if } x = b^y$$

One-to-One Property of Exponential Functions

For any algebraic expressions S and T, and any positive real number b, with $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T$$

One-to-One Property of Logarithmic Functions

For any algebraic expressions $S > 0$, $T > 0$, and any positive real number b, where $b \neq 1$,

$$\log_b(S) = \log_b(T) \text{ if and only if } S = T$$

Note: Because $\log_b(x)$ has the domain $(0, \infty)$ for all $b > 0, b \neq 1$, when we solve an equation involving logarithms, **we must always check** to see if the solution we've found is valid or if it is an extraneous solution.

Exit Exercises

Exit 1:

a) What's the general process for solving exponential equations that have one exponential expression in them?

b) Why can logarithmic equations have extraneous solutions and how can an extraneous solution be recognized?

c) Solve $4e^{2k+1} + 3 = 27$.

d) Solve $\log_8(5x + 12) - \log_8(x) = \log_8(2)$.

On a scale of 1 - 5, how are you feeling with the concepts related to exponential and logarithmic equations? Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

2.5 Exponential Functions and Compound Interest

In this section, we'll build upon our understanding of the general exponent function $f(x) = a \cdot b^x$ and see how variations of this formula are used in financial situations.

Textbook Reference: This relates to content in §6.1 of *Algebra and Trigonometry 2e.*

Preparation Exercises

Preparation 1:
The function $V(t) = 52.8(1.093)^t$ represents the value (in thousands of dollars) of a collectable car t years after June 1st, 2015.

a) What was the value of the car on June 1st, 2015?

b) Is the value of the car increasing or decreasing and at what rate is the value of the car changing?

Preparation 2:
Simple Interest is calculated using the formula $I = Prt$, where I is the interest earned, P is the principal or initial amount of money, r is the interest rate, and t is the amount of time that the interest is earned.

a) How much simple interest is owed after 7 years on a \$4,000 loan that earns 6% annual simple interest? How much will need to be paid back in total at the end of 7 years?

b) How much simple interest is earned after 6 years on a \$3,000 investment that earns 4% annual simple interest?

Practice Exercises

Practice 1:

How much is owed at the end of a 6 years if \$12,000 is borrowed at 6.4% annual interest, compounded quarterly?

Practice 2:

Maya said the formula she set up for an exercise to calculate the amount (in dollars) owed on a loan after some number of years is $A = 10250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the principal of the loan, the nominal interest rate, the number of compounds per year, and the number of years of the loan?

Practice 3:

If Marshon invests \$5000 at 5.8% annual interest compounded continuously, how much will he have in the account in 13 years?

Definitions

Simple Interest

The formula to calculate **simple interest** is

$$I = Prt$$

where I is the amount of interest earned,
P is the principal or initial amount of money,
r is the annual interest rate, and
t is the number of years the interest is earned

Compound Interest

The formula to calculate **compound interest** is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where A is total amount of money owed or earned,
P is the principal,
r is the nominal annual interest rate,
n is the number of times the interest is compounded per year, and
t is the number of years that the interest is earned.
Note: This formula is used where there is a finite number of compounding per year.

The Number *e*

e, also known as **Euler's number**, is the irrational number that results from

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n}$$

Example: You can watch this Numberphile YouTube video about e.

Continuously Compounded Interest

The formula to calculate **continuously compounded interest** is

$$A = Pe^{rt}$$

where A is total amount of money owed or earned,
P is the principal,
r is the nominal annual interest rate, and
t is the number of years that the interest is earned.
Note: This formula is used when the compounding happens continuously.

Effective Interest Rate

The **effective interest rate,** r_e is the equivalent interest rate that, if compounded annually, would yield the same result after 1 year as compounding the stated nominal rate n times per year or compounding the nominal rate continuously.

When compounding n times per year, $r_e = \left(1 + \frac{r}{n}\right)^n - 1$, where r is the nominal annual interest rate.
When compounding continuously, $r_e = e^r - 1$, where r is the nominal annual interest rate.

Exit Exercises

Exit 1:
In the year 2001, Alyssa opened a retirement account that earns a nominal interest rate 7.25% per year. Her initial deposit was $13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

Exit 2:
Kyoko received a $10,000 scholarship that she gets to invest for college. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $15,000 by the time she finishes high school in 6 years. To the nearest hundredth of a percent, what should her minimum nominal interest rate be in order to reach her goal?

Exit 3:
A small business is planning on building a new facility in 9 years. They can invest money at 7% annual interest, compounded daily, right now. If they need $600,000 to build the new facility in 9 years, what is the minimum they need to invest to ensure they have $600,000 in 9 years?

On a scale of 1 - 5, how are you feeling with the concepts related to exponential functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

2.6 Exponential and Logarithmic Models

In this section, we'll investigate a few ways that exponential and logarithmic functions are used to model the world around us, as well as how logarithms help us to quantify sound levels, acidity, and earthquakes.

Textbook Reference: This relates to content in §6.7 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:
Find the formula for an exponential function f that satisfies $f(-2) = \dfrac{16}{27}$ and $f(3) = \dfrac{9}{2}$.

Preparation 2:
The substance Einsteinium-253 decays at a continuous rate of about 3.3862% per day. If a scientist starts with a 60 mg sample of Einsteinium-253, how long until there are only 30 mg remaining?

Preparation 3:
The function $V(t) = 32.8(1.093)^t$ represents the value (in thousands of dollars) of a collectable car t years after June 1st, 2015. How long will it take for the value of the car to reach $65,600?

Practice Exercises

Practice 1:
A doctor prescribes 175 milligrams of a drug that decays by 20% each hour.

a) Write an exponential model representing D, the amount of the drug (in mg) remaining in the patient's system, t hours after having the drug administered. (Note: 20% per hour is not a continuous rate.)

b) To the nearest hour, what is the half-life of the drug?

c) When will there be 25 mg of the drug remaining in the patient's system?

Practice 2:
In 2022, Canada's population[1] grew by about 2.7%. If Canada's population were to maintain that same growth rate, how long would it take for Canada's population to double?

Practice 3:
Cobalt-60[2] is used for radiotherapy and has a half-life of 5.26 years. Find the continuous annual rate of decay.

[1] Growth rate obtained from https://www.statcan.gc.ca/en/subjects-start/population_and_demography/40-million
[2] Half-life value obtained from https://www.britannica.com/science/cobalt-60

Definitions

Doubling Time

The amount of time it takes an exponential growth model to increase to double the starting value.

Half-Life

The amount of time it takes an exponential decay model to decay to half of the starting value.

Exponential Function of the Form $f(t) = ae^{kt}$:

For any real number t, an **exponential function with the form $f(t) = ae^{kt}$** where

- a is a non-zero real number
- k is any non-zero real number and is the continuous growth rate.

Exponential functions of the form $f(t) = ae^{kt}$ are often referred to as a **continuous growth model**.

Key Characteristics of $f(t) = ae^{kt}$:

For an exponential function $f(t) = ae^{kt}$, with $a > 0$, we have the following:

- $(0, a)$ is the vertical intercept.
- There is no horizontal intercept.
- The domain of f is $(-\infty, \infty)$.
- The range of f is $(0, \infty)$.
- f is a one-to-one function.
- The horizontal asymptote is $y = 0$.
- If $k > 0$, then f is an increasing function and
  - as $t \to \infty$, $f(t) \to \infty$, and
  - as $t \to -\infty$, $f(t) \to 0$.
- If $k < 0$, then f is a decreasing function and
  - as $t \to \infty$, $f(t) \to 0$, and
  - as $t \to -\infty$, $f(t) \to \infty$.

Figure 2.9: $y = ae^{kt}, k > 0$

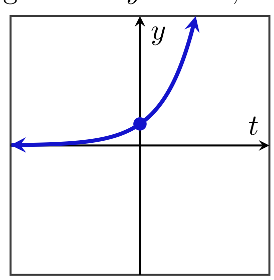
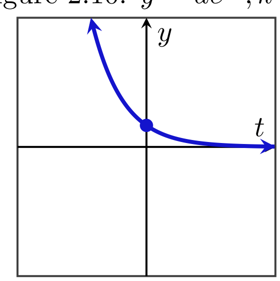


Figure 2.10: $y = ae^{kt}, k < 0$



Decibels

The loudness $L(x)$, measured in **decibels (dB)**, of a sound of intensity x, measured in watts/m^2, is defined by

$$L(x) = 10 \log\left(\frac{x}{I_0}\right),$$

where $I_0 = 10^{-12}$ watts/m^2 and approximately represents the least intense sound that a human ear can detect.

pH

The **pH of a chemical solution** is used to measure the acidity or alkalinity of the solution.
The formula used to calculate the pH of a solution is

$$\mathrm{pH} = -\log\left[\mathrm{H}^+\right],$$

where $\left[\mathrm{H}^+\right]$ is the concentration of hydrogen ions in moles per liter.
pH values range from 0 (acidic) to 14 (basic), with 7 being neutral.

Richter scale

The Richter scale is one way of converting seismographic readings into numbers that provide a reference for measuring the magnitude M of an earthquake. All earthquakes are compared to a zero-level earthquake whose seismographic reading measures 0.001 millimeter at a distance of 100 kilometers from the epicenter.
An earthquake whose seismographic reading measures x millimeters has a magnitude $M(x)$, give by

$$M(x) = \log\left(\frac{x}{x_0}\right),$$

where $x_0 = 10^{-3}$ is the reading of a zero-level earthquake the same distance from the epicenter.

Exit Exercises

Exit 1:

a) For what type of exponential models would we discuss the half-life? What does the half-life measure?

b) For what type of exponential models would we discuss the doubling time? What does the doubling time measure?

Exit 2:

A patient receives an injection of 20 mg of a medicine that decays exponentially. 45 minutes after the injection, there are 8 mg of medicine left in her body. What is the half-life of this medication?

Exit 3:

A research student is working with a culture of bacteria that doubles in size every 13 minutes. The initial population count was 1350 bacteria. Find an exponential function that expresses the bacteria population, P, as a function of t, the number of **hours** after the experiment began.

On a scale of 1 - 5, how are you feeling with the concepts related to exponential functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

Chapter 3

Polynomial and Rational Functions

Contents

3.1 Polynomial Functions and Some Key Characteristics

In this section, we'll learn about the long-term behaviors of polynomial functions and how to identify this, as well as other key attributes of polynomial functions, based on their algebraic formula.

Textbook Reference: This relates to content in §5.2 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:

a) Completely factor $2x^2 + 2x - 12$.

b) Completely factor $2x^2 + 3x - 2$.

c) Algebraically find and state the x-intercepts of the graph of $f(x) = 2x^2 + 2x - 12$.

d) Algebraically find and state the x-intercepts of the graph of $f(x) = 2x^2 + 3x - 2$.

e) Algebraically find and state the y-intercept of the graph of $f(x) = 2x^2 + 3x - 2$.

Practice Exercises

Practice 1:

Let $f(x) = \frac{1}{2}x^4 + 3x^3 - 16x$, which in factored form is $f(x) = \frac{1}{2}x(x-2)(x+4)^2$.

a) Describe the end behavior of the graph of $y = f(x)$.

b) At most how many turning points does the graph of $y = f(x)$ have?

c) What is the y-intercept of $y = f(x)$?

d) What are the x-intercept(s) of $y = f(x)$?

Practice 2:

Let $g(x) = 2x^3 - 5x^2 - 4x + 12$, which in factored form is $g(x) = (2x+3)(x-2)^2$.

a) Describe the end behavior of the graph of $y = g(x)$.

b) At most how many turning points does the graph of $y = g(x)$ have?

c) What is the y-intercept of $y = g(x)$?

d) What are the x-intercept(s) of $y = g(x)$?

Definitions

Power Function

A **power function** is a function that can be represented in the form

$$f(x) = kx^p,$$

where k and p are real numbers. k is called the coefficient.

Polynomial Function

A **polynomial function** is of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n,\ a_{n-1},\ \ldots,\ a_1,\ a_0$ are real numbers, $a_n \neq 0$, and n is a non-negative integer.
The **leading term** is the highest degree term, $a_n x^n$.
The **degree** of the polynomial is n.
The **leading coefficient** is the coefficient of the leading term, a_n.

Example: The polynomial $5x^2 - 8x + 4$ has a leading term of $5x^2$, is second degree polynomial, and has a leading coefficient of 5

x-intercept or Horizontal Intercept

A **horizontal intercept** or **x-intercept** of a graph is a point where the graph intersects the horizontal or x-axis. This occurs when the function has an output value of 0.

Example: We can find the horizontal intercepts of the graph of $f(x) = 5x^2 - 8x + 4$ by solving $f(x) = 0$.

y-intercept or Vertical Intercept

A **vertical intercept** or **y-intercept** of a graph is a point where the graph intersects the vertical or y-axis. This occurs when the function has an input value of 0.

Example: We can find the vertical intercept of the graph of $f(x) = 5x^2 - 8x + 4$ by evaluating $f(0)$.

End Behavior or Long-Term Behavior

The **end behavior** or **long-term behavior** of a polynomial function is determined by its leading term. The long-term behavior of the polynomial function will be consistent with the power function that is the leading term of the polynomial.

Example: The end behavior of $f(x) = \mathbf{5x^4} - 7x^2 + 8x - 9$ will be the same as the end behavior of $y = 5x^4$.

Turning Point

A **turning point** of a polynomial function is a point at which the graph changes from increasing to decreasing or from decreasing to increasing.

Example: The graph of $f(x) = 5x^4 - 7x^2 + 8x - 9$ has at most 3 turning points.

Root or Zero

A **root** or **zero** of a polynomial function is a value r for which $f(r) = 0$.
r is a zero of a polynomial function f if and only if $(x - r)$ is a factor of f.

Example: If 7 is a zero of a polynomial, then $(x - 7)$ is a factor of the polynomial.
Example: If $(x + 3)$ is a factor of a polynomial, then -3 is a zero of the polynomial.

Exit Exercises

Exit 1:

a) What does the degree of a polynomial function tell you about the graph of the function?

b) What does the leading coefficient of a polynomial function tell you about the graph of the function?

c) If you know the graph of a polynomial function has 7 turning points, what can you say about the degree of the function?

d) What do we know about a polynomial function's formula if we know the function has the following behavior?
As $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$.

Exit 2:

State the degree, leading coefficient, long-term behavior, y-intercept, and the x-intercepts for function $f(x) = \frac{1}{3}(x-4)(x+5)^2(x+1)$.

On a scale of 1 - 5, how are you feeling with the concepts related to polynomial functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

😟 1 2 3 4 5 😃

3.2 Graphs of Polynomial Functions

In this section, we'll focus on the short-term behaviors of polynomial functions and the location and behavior of the x-intercepts can be identified from their factored algebraic form. We'll then use this knowledge to both graph polynomial functions, as well as construct polynomial functions based on a graph.

Textbook Reference: This relates to content in §5.3 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:
Let $f(x) = -2(x+3)(x-2)^2(x+1)$.

a) Describe the end behavior of the graph of $y = f(x)$.

b) What are the x-intercept(s) of $y = f(x)$?

c) What is the y-intercept of $y = f(x)$?

d) At most how many turning points does the graph of $y = f(x)$ have?

Preparation 2:
Let $g(x) = -2(x+3)^2(x-2)(x+1)^3$.

a) Describe the end behavior of the graph of $y = g(x)$.

b) What are the x-intercept(s) of $y = g(x)$?

c) What is the y-intercept of $y = g(x)$?

d) At most how many turning points does the graph of $y = g(x)$ have?

Practice Exercises

Practice 1:

Let $f(x) = \frac{1}{2}(x-2)(x+1)^2(x-4)$.

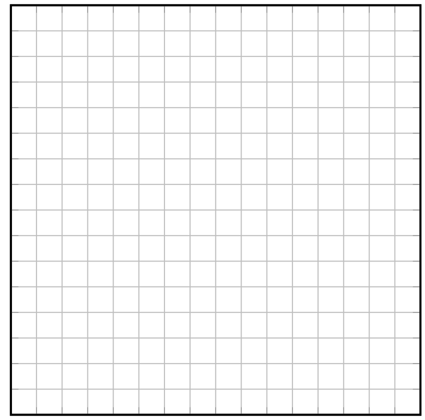
a) What is the end behavior of the graph of $y = f(x)$? Why?

b) What are the x-intercepts and their behaviors? Why?

c) What is the y-intercept?

d) Sketch a graph of $y = f(x)$ in Figure 3.1 to the right.

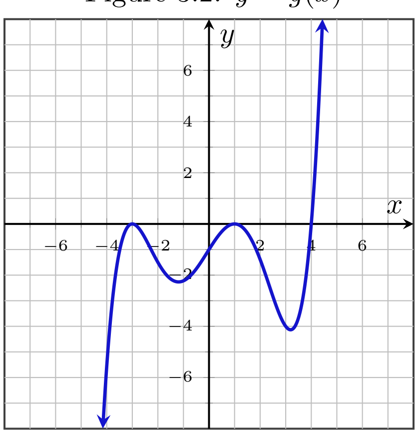
Figure 3.1: $y = f(x)$



Practice 2:

Given in Figure 3.2 is the graph of a polynomial function g.

Figure 3.2: $y = g(x)$



a) Based on the end behavior and other characteristics of the graph of $y = g(x)$, what are the possibilities for the degree of g?

b) Based on the end behavior of the graph of $y = g(x)$, is the leading coefficient of g positive or negative?

c) Based on the x-intercepts, what are the linear factors of g and their powers?

d) What is a possible formula for the function g?

Definitions

Multiplicity

The **multiplicity** of a factor is the number of times that factor occurs in the factored version of the polynomial. We will also refer to the **multiplicity of the zero** for the zero associated with this factor.

Example: For $f(x) = 3(x-5)(x+4)^2$, the multiplicity of 5 is one and the multiplicity of -4 is two.

The sum of the multiplicities of the real roots for a polynomial function is less than or equal to the degree of the polynomial.

Even Multiplicity

When a root or zero has even multiplicity, then the graphical behavior at the associated x-intercept is that the graph will touch, but not cross, the x-axis at that x-intercept.

Odd Multiplicity

When a root or zero has odd multiplicity, then the graphical behavior at the associated x-intercept is that the graph will cross over the x-axis at that x-intercept.

When the multiplicity of a root is one, the graph will cross through the x-intercept in a somewhat linear manner.

When the multiplicity of a root is a larger odd number, the graph will cross through the x-intercept by flattening out as it crosses.

Example: View this Desmos graph to see an interactive example how the multiplicity of a root of a polynomial function impacts the behavior of the related x-intercept.

Exit Exercises

Exit 1:

a) What is the difference between a zero and an x-intercept of a polynomial function?

b) How is the factored version of a polynomial function useful when graphing the function? What does the factored version help us to be able to quickly identify?

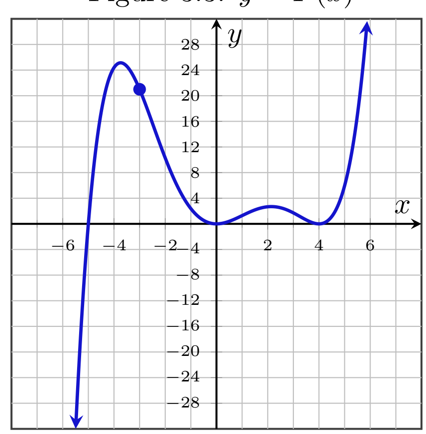
c) How is the expanded (non-factored) version of a polynomial function useful when graphing the function? What does the non-factored version help us to be able to quickly identify?

d) If the graph of a polynomial function touches, but doesn't cross, the x-axis at a point $(k, 0)$, what do we know about the factored form of that function?

Exit 2:

Identify a possible formula for the polynomial function F whose graph is in Figure 3.3.
There is a point on the graph at $(-3, 21)$.

Figure 3.3: $y = F(x)$



On a scale of 1 - 5, how are you feeling with the concepts related to polynomial functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

3.3 Rational Functions and Some Key Characteristics

In this section, we'll look at rational functions and see how the concepts we explored with polynomial functions carry over to a function that is a fraction of two polynomials.

Textbook Reference: This relates to content in §5.6 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:

a) For any real number k other than 0, what is $\dfrac{0}{k}$ and why?

b) For any real number k other than 0, what is $\dfrac{k}{0}$ and why?

c) What is the domain of $f(x) = \dfrac{3x^2 - 3x - 36}{2x^2 + 2x - 60}$?

Preparation 2:

Let $g(x) = -7(x+2)^2(x-1)^3(x+5)$.

a) What are the x-intercepts and their behaviors?

b) What is the y-intercept?

Practice Exercises

Practice 1:
Let $r(x) = \dfrac{p(x)}{q(x)}$.

a) What usually happens on the graph of $r(x)$ when $p(x) = 0$?

b) What usually happens on the graph of $r(x)$ when $q(x) = 0$?

Practice 2:
Let $R(x) = \dfrac{5(x+3)(x-4)^2}{(x-2)^2(x+1)}$.

a) What is the domain of R?

b) What is the y-intercept of R?

c) What are the zeros of R?

d) What are the x-intercepts of R?

e) What happens when $x = -1$?

f) What happens when $x = 2$?

Practice 3:
When working with a rational function,

a) What is the behavior of a vertical asymptote that comes from a root of the denominator with odd multiplicity? Include at least two sketches of possible examples in your answer.

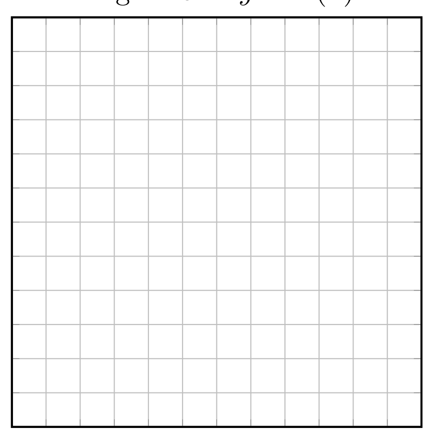
b) What is the behavior of a vertical asymptote that comes from a root of the denominator with even multiplicity? Include at least two sketches of possible examples in your answer.

Practice 4:

Sketch a graph of a function $y = F(x)$ in Figure 3.4 that has:

- a vertical asymptote at $x = -2$ with even multiplicity,
- a y-intercept at $(0, -2)$,
- an x-intercept at $(-4, 0)$ with even multiplicity,
- an x-intercept at $(3, 0)$ with odd multiplicity, and
- there are no other vertical asymptotes or x-intercepts.

Figure 3.4: $y = F(x)$



Definitions

Rational Function

A **rational function** is a function that can be written as the quotient of two polynomial functions, where the denominator is not 0.

$$r(x) = \frac{p(x)}{q(x)}, \text{where } p \text{ and } q \text{ are polynomial functions and } q(x) \neq 0$$

Vertical Asymptote

Given a function f and a real number a, a **vertical asymptote** of the graph of $y = f(x)$ is a vertical line $x = a$ where $f(x)$ tends toward positive or negative infinity as x approaches a from either the left or the right. We write this as:

$$\text{As } x \to a^-,\ f(x) \to \pm\infty \text{ or as } x \to a^+,\ f(x) \to \pm\infty.$$

Example: $x = 1$ is a vertical asymptote of $f(x) = \dfrac{x-5}{x-1}$ due to the following:

- As x approaches $x = 1$ from the left, the y-values increase towards ∞. Mathematically, we write this: As $x \to 1^-$, $f(x) \to \infty$.
- As x approaches $x = 1$ from the right, the y-values decrease towards $-\infty$. Mathematically, we write this: As $x \to 1^+$, $f(x) \to -\infty$.

View this Desmos graph to see an interactive version of this example.

Multiplicity and Vertical Asymptotes

The multiplicity of a root of the denominator of a rational function impacts the behavior of the vertical asymptote.

If the rational function is factored and reduced to its simplest terms, **a root of the denominator with even multiplicity** will produce a vertical asymptote that approaches ∞ on both sides or that approaches $-\infty$ on both sides.

If the rational function is factored and reduced to its simplest terms, **a root of the denominator with odd multiplicity** will produce a vertical asymptote that approaches ∞ on one side and $-\infty$ on the other.

Example: View this Desmos graph to see an interactive example of this concept.

Horizontal Asymptote

Given a function f and a real number L, a **horizontal asymptote** of a graph of $y = f(x)$ is a horizontal line $y = L$ where $f(x)$ tends toward L as x approaches ∞ or as x approaches $-\infty$. We write this as:

$$\text{As } x \to \infty \text{ or as } x \to -\infty,\ f(x) \to L.$$

Example: View this Desmos graph to see an interactive example of the definition.

Exit Exercises

Exit 1:

a) Explain when and why rational functions have vertical asymptotes?

b) If $(-5, 0)$ is an x-intercept of a rational function R, what do you know about the formula for $R(x)$?

c) If $x = 4$ is a vertical asymptote of a rational function R, what do you know about the formula for $R(x)$?

Exit 2:

Let $R(x) = \dfrac{4x(x-7)^2(x+4)}{5(x+7)(x+1)}$.

a) What is the domain of R? Answer using both interval and set-builder notations.

b) What are the x-intercepts of the graph of R?

c) What are the vertical asymptotes of the graph of R?

On a scale of 1 - 5, how are you feeling with the concepts related to rational functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

1 2 3 4 5

3.4 Graphs of Rational Functions

In this section, we'll continue working with rational functions. We'll examine the long-term or end behavior and see what happens when there is a common root of the numerator and denominator. We'll then use everything we've learned about rational functions to graph them, as well as construct rational functions based on a given graph.

Textbook Reference: This relates to content in §5.6 of *Algebra and Trigonometry 2e*.

Preparation Exercises

Preparation 1:
Suppose a bakery has daily fixed costs of \$150. To produce a single loaf of bread costs an additional \$2.75 for labor and materials.

a) Find a function that calculates the daily cost per loaf, C in dollars, to produce x loaves of bread.
Hint: To find the cost per loaf, you need to divide the total costs by the number of loaves of bread.

b) If the bakery produce 5 loaves of bread, what is the cost per loaf?

c) If the bakery produce 50 loaves of bread, what is the cost per loaf?

d) If the bakery produce 500 loaves of bread, what is the cost per loaf?

e) What is happening to the per loaf cost as the number of loaves made increases?

f) Is there a limit to the how low the cost per loaf can go?

Practice Exercises

Practice 1:
The function $r(x) = \dfrac{-6(x+5)^2(x-4)}{(x-7)}$ does not have a horizontal asymptote. How could you change the formula for this function so that it does have a horizontal asymptote?

Practice 2:
Let $R(x) = \dfrac{3(x+2)(x-4)}{(x+2)(x+3)}$. Answer the following without using a calculator.

a) What is the domain of R?

b) What is the y-intercept of graph of R?

c) What are the x-intercepts of the graph of R?

d) What are the vertical asymptotes of the graph of R?

e) Does the graph of R have any holes? Why or why not?

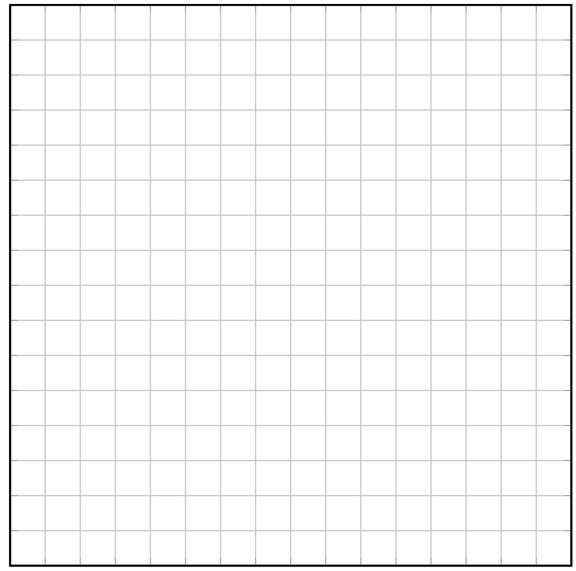
f) If the graph of R has any holes, find the coordinates of the hole(s).

g) Does the graph of R have a horizontal asymptote? Why or why not?

h) If the graph of R has a horizontal asymptote, state its equation.

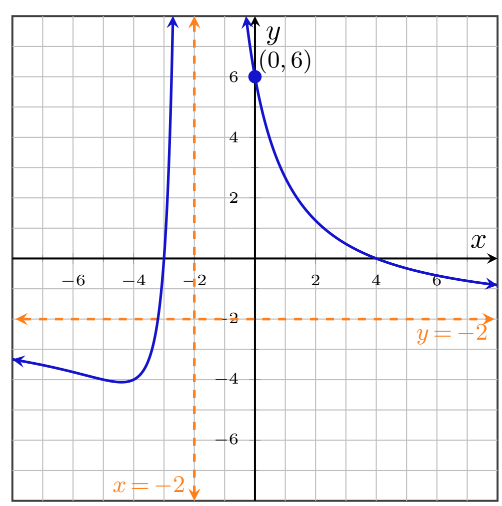
Practice 3:
Graph $R(x) = \dfrac{3(x+2)(x-4)}{(x+2)(x+3)}$ in Figure 3.5. Clearly identify all vertical asymptotes, x-intercepts, the y-intercept, any holes, and any horizontal asymptote.

Figure 3.5: $y = r(x)$



Practice 4:
Find a possible formula for the rational function graphed in Figure 3.6. Take into account the vertical asymptote(s), the x-intercept(s), the y-intercept, and whether or not there is a horizontal asymptote.

Figure 3.6: $y = r(x)$



Definitions

Holes or Removable Discontinuities

A **hole** or **removable discontinuity** for a rational function r occurs at an x-value that is a root of both the numerator and denominator and whose multiplicity in the numerator is greater than or equal to its multiplicity in the denominator.
Note: If the multiplicity of the root is greater in the denominator, then the root will create a vertical asymptote.

Example: There is a removable discontinuity at $x = 1$ for $f(x) = \dfrac{(x-1)(x-5)}{(x-1)(x+3)}$ and $g(x) = \dfrac{(x-1)^2(x-5)}{(x-1)(x+3)}$, but $x = 1$ is a vertical asymptote for $h(x) = \dfrac{(x-1)(x-5)}{(x-1)^2(x+3)}$.

Horizontal Asymptotes of Rational Functions

Whether a rational function has is a horizontal asymptote can be determined by comparing the degrees of the numerator and denominator.

- If the degree of the denominator is greater than that of the numerator, the function will have a horizontal asymptote of $y = 0$.
- If the degree of the denominator is equal to that of the numerator, the function will have a horizontal asymptote. The equation of the horizontal asymptote will be based on the ratio of the leading coefficients.
- If the degree of the denominator is less than that of the numerator, the function will not have a horizontal asymptote.

Example: $f(x) = \dfrac{6x^5 - 8x^3 + 2}{3x^6 - 9x + 5}$ has a horizontal asymptote of $y = 0$.

$g(x) = \dfrac{6x^6 - 8x^3 + 2}{3x^6 - 9x + 5}$ has a horizontal asymptote of $y = 2$.

$h(x) = \dfrac{6x^7 - 8x^3 + 2}{3x^6 - 9x + 5}$ has no horizontal asymptote.

Exit Exercises

Exit 1:
Let $R(x) = \dfrac{5(x+4)^2(x-2)}{2(x+4)(x+1)^2}$.

a) What is the domain of R? Answer using both interval and set-builder notations.

b) What are the x-intercepts of the graph of R?

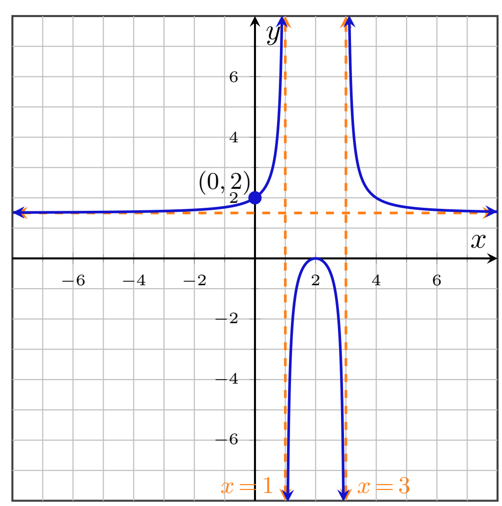
c) What are the vertical asymptotes of the graph of R?

d) Does the graph of R have any holes? Why or why not?

e) Does the graph of R have a horizontal asymptote? If it does, what is the horizontal asymptote?

Exit 2:
Find a possible formula for the rational function graphed in Figure 3.7. Take into account the vertical asymptote(s), the x-intercept(s), the y-intercept, and whether or not there is a horizontal asymptote.

Figure 3.7: $y = r(x)$



On a scale of 1 - 5, how are you feeling with the concepts related to rational functions?
Place a mark on the scale below to indicate your overall comfort with the topics we saw in this section.

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Chapter 4

Appendix

Contents

4.1 Direct Links

In the event that some TinyCC links from the MTH 111Z Lab Manual break, here are the direct links used at the time of publication.

Please note that while the direct links to Desmos graphs might no longer be the most current version of the Desmos graphs, any older graphs should still be reasonably useful.

Preface Links

- *Algebra and Trigonometry* 2e: https://openstax.org/details/books/algebra-and-trigonometry-2e
- MTH 111Z Supplement: https://spot.pcc.edu/math/mth111-112-supplement-landing.html

Chapter 1 Links

- §1.2 Interval and Set-Builder Notations: https://youtu.be/aLvRu8Int4M
- §1.2 Domain and Range: https://www.desmos.com/calculator/dgrur8be3f
- §1.2 Positive and Negative: https://www.desmos.com/calculator/v8qsikcufe
- §1.2 Increasing, Decreasing, and Constant: https://www.desmos.com/calculator/ondcewmuy0
- §1.6 Vertical Shift: https://www.desmos.com/calculator/sv2boowrlu
- §1.6 Horizontal Shift: https://www.desmos.com/calculator/eloexg8kaz
- §1.6 Vertical Stretch/Compression: https://www.desmos.com/calculator/m6uervj5h6
- §1.6 Horizontal Stretch/Compression: https://www.desmos.com/calculator/0he8y9sftj
- §1.6 Vertical Reflection: https://www.desmos.com/calculator/ow1t0ajggh
- §1.6 Horizontal Reflection: https://www.desmos.com/calculator/0pgds68h5c
- §1.6 Even Function: https://www.desmos.com/calculator/aqintrlh6h
- §1.6 Odd Function: https://www.desmos.com/calculator/qtpfhmydes

Chapter 2 Links

- §2.1 Exponential Function: https://www.desmos.com/calculator/zcp3gaj5ws
- §2.2 Logarithmic Function: https://www.desmos.com/calculator/9bqm1kbl4i
- §2.5 The Number e: https://youtu.be/AuA2EAgAegE

Chapter 3 Links

- §3.2 Multiplicity and Polynomials: https://www.desmos.com/calculator/b3pmc7es7z
- §3.3 Vertical Asymptote: https://www.desmos.com/calculator/t10efozl3i
- §3.3 Multiplicity and Vertical Asymptotes: https://www.desmos.com/calculator/u1ed06auqc
- §3.3 Horizontal Asymptote: https://www.desmos.com/calculator/9bqtdmnl4w