

Portland Community College  
MTH 251 Lab Manual

PRINCIPIA  
MATHEMATICA.

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# Calculus Lab Manual

MTH 251 at Portland Community College



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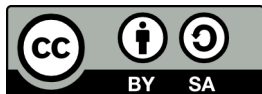
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# Acknowledgements

In 2012, Steve Simonds wrote most of this lab manual as a Word/PDF document with some support from PCC curriculum development grants.

In 2015, Alex Jordan converted the lab manual to PreT<sub>E</sub>Xt (enabling the HTML output alongside a PDF) with some support from PCC's professional development grants.

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## To All

Calculus I is taught at Portland Community College using a lecture/lab format. The laboratory time is set aside for students to investigate the topics and practice the skills that are covered during their lecture periods. This lab manual serves as a guide for the laboratory component of the course.

**HTML and PDF** This manual has been released with several synchronous versions that offer different features. The essential content of each version is the same as for all others.

- A web version is available at <http://spot.pcc.edu/math/clm>, and this version is recommended. The web version offers full walk-through solutions to supplemental problems. More importantly, it offers interactive elements and easier navigation than print versions.
- A PDF-for-print is available at <http://spot.pcc.edu/math/clm/clm-print.pdf>. This version is designed to be printed, not read on a screen. To save on printing expense, this version is mostly black-and-white, and only offers short answers to the supplemental exercises (as opposed to full solutions).
- A “for-printing-with-color” PDF is available at <http://spot.pcc.edu/math/clm/clm-print-color.pdf>. This version is like the above, except links and graphs retain their color.

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- In the MathJax contextual menu, you may set options for triggering a zoom effect on math content, and also by what factor the zoom will be.
- If you change the MathJax renderer to MathML, then a screen reader will generally have success verbalizing the math content.

**Tablets and Smartphones** PreT<sub>E</sub>Xt documents like this lab manual are “mobile-friendly”. When you view the HTML version, the display adapts to whatever screen size or window size you are using. A math teacher will always recommend that you do not study from the small screen on a phone, but if it’s necessary, this manual gives you that option.

## To the Student

The lab activities have been written under the presumption that students will be working in groups and will be actively discussing the examples and problems included in each activity. Many of the exercises and problems lend themselves quite naturally to discussion. Some of the more algebraic problems are not so much discussion problems as they are “practice and help” problems.

You do not need to fully understand an example before starting on the associated problems. The intent is that your understanding of the material will grow while you work on the problems.

While working through the lab activities, the students in a given group should be working on the same activity at the same time. Sometimes this means an individual student will have to go a little more slowly than he or she may like

and sometimes it means an individual student will need to move on to the next activity before he or she fully grasps the current activity.

Many instructors will want you to focus some of your energy on the way you write your mathematics. It is important that you do not rush through the activities. Write your solutions as if they are going to be graded; that way you will know during lab time if you understand the proper way to write and organize your work.

If your lab section meets more than once a week, *you should not work on lab activities between lab sections that occur during the same week*. It is OK to work on lab activities outside of class once the entire classroom time allotted for that lab has passed.

Lab 8 on related rates uses a strategy for related rates exercises that your instructor may not use in lecture. It is a multi-step process that may take some time to learn. We recommend that you read the examples from this lab *before* coming to lab. Too much lab time should not be spent silently reading to yourself.

There are no written solutions for the lab activity problems. Between your group mates, your instructor, and (if you have one) your lab assistant, you should know whether or not you have the correct answers and proper writing strategies for these problems.

Each lab has a section of supplementary exercises; these exercises are fully keyed with solutions. The supplementary exercises are not simply copies of the problems in the lab activities. While some questions will look familiar, many others will challenge you to apply the material covered in the lab to a new type of problem. These questions are meant to supplement your textbook homework, not replace your textbook homework.

## To the Instructor

This version of the calculus lab manual is significantly different from versions prior to 2012. The topics have been arranged in a developmental order. Because of this, students who work each activity in the order they appear may not get to all of the topics covered in a particular week.

Additionally, a few changes have been introduced with the release of the PreT<sub>E</sub>Xt version. The most notable changes are:

- The numbering scheme does not match the earlier numbering scheme. This was a necessary consequence of converting to PreT<sub>E</sub>Xt.
- The related rates lab has been mostly rewritten using DREDS.
- An activity on logarithmic differentiation has been added.

- The printed version only contains short answers to the supplemental questions rather than complete walk-through solutions. However, complete solutions may still be found in the HTML version.

It is strongly recommended that you pick and choose what you consider to be the most vital activities for a given week, and that you have your students work those activities first. For some activities you might also want to have the students only work selected problems in the activity. Students who complete the high priority activities and problems can then go back and work the activities that they initially skipped. There are also fully keyed problems in the supplementary exercises that the students could work on both during lab time and outside of class. (The full solutions are only in the HTML version.)

A suggested schedule for the labs is shown in the table below. Again, you should choose what you feel to be the most relevant activities and problems for a given week, and have the students work those activities and problems first.

***(Students should consult their syllabus for their schedule.)***

**Table 0.0.1:** Suggested Schedule

Week	Labs (Lab Activities)	Supplementary Exercises
1	Velocity through Limits	Supplement 1.4
2	Limits Laws through Piecewise-Defined Functions	Supplement 2.14
3	Instantaneous Velocity through Derivative Units	Supplement 3.5
4	Graph Features through Nondifferentiability	Supplement 4.7
5	Higher Order Derivatives through Graph Features from Derivatives	Supplement 4.7
6	Leibniz Notation through Derivative Formulas and Function Behavior	Supplement 5.12
7	Introduction to the Chain Rule through Chain Rule with Leibniz Notation	Supplement 6.5
8	General Implicit Differentiation through Logarithmic Differentiation	Supplement 7.4
9, 10	Introduction to Related Rates† through Making Graphs	Supplement 8.3, Supplement 9.8

† Students should read the related rates examples *before* coming to lab.

# Contents

<b>Acknowledgements</b>	<b>v</b>
<b>To All</b>	<b>vii</b>
<b>To the Student</b>	<b>viii</b>
<b>To the Instructor</b>	<b>ix</b>
<b>1 Rates Of Change</b>	<b>1</b>
1.1 Velocity . . . . .	1
1.2 Secant Line to a Curve . . . . .	3
1.3 The Difference Quotient . . . . .	5
1.4 Supplement . . . . .	9
<b>2 Limits and Continuity</b>	<b>11</b>
2.1 Limits . . . . .	11
2.2 Limits Laws . . . . .	14
2.3 Indeterminate Limits . . . . .	15
2.4 Limits at Infinity . . . . .	17
2.5 Limits at Infinity Tending to Zero . . . . .	19
2.6 Ratios of Infinities . . . . .	20
2.7 Non-existent Limits . . . . .	21
2.8 Vertical Asymptotes . . . . .	22
2.9 Continuity . . . . .	25
2.10 Discontinuities . . . . .	26
2.11 Continuity on an Interval . . . . .	27

2.12	Discontinuous Formulas . . . . .	29
2.13	Piecewise-Defined Functions . . . . .	30
2.14	Supplement . . . . .	33
<b>3</b>	<b>Introduction to the First Derivative</b>	<b>37</b>
3.1	Instantaneous Velocity . . . . .	37
3.2	Tangent Lines . . . . .	38
3.3	The First Derivative . . . . .	40
3.4	Derivative Units . . . . .	43
3.5	Supplement . . . . .	45
<b>4</b>	<b>Functions, Derivatives, Antiderivatives</b>	<b>47</b>
4.1	Graph Features . . . . .	47
4.2	Graphical Derivatives . . . . .	49
4.3	Nondifferentiability . . . . .	55
4.4	Higher Order Derivatives . . . . .	56
4.5	Antiderivatives . . . . .	58
4.6	Graph Features from Derivatives . . . . .	62
4.7	Supplement . . . . .	68
<b>5</b>	<b>Derivative Formulas</b>	<b>75</b>
5.1	Leibniz Notation . . . . .	75
5.2	The Derivative Operator . . . . .	76
5.3	The Power Rule . . . . .	76
5.4	The Constant Factor Rule . . . . .	77
5.5	The Constant Divisor Rule . . . . .	78
5.6	The Sum and Difference Rules . . . . .	79
5.7	The Product Rule . . . . .	80
5.8	The Quotient Rule . . . . .	82
5.9	Simplification . . . . .	83
5.10	Product, Quotient Rules Together . . . . .	84
5.11	Derivative Formulas and Function Behavior . . . . .	85
5.12	Supplement . . . . .	87

<b>6</b>	<b>The Chain Rule</b>	<b>91</b>
6.1	Introduction to the Chain Rule . . . . .	91
6.2	Order to Apply Rules . . . . .	94
6.3	Not Simplifying First . . . . .	95
6.4	Chain Rule with Leibniz Notation . . . . .	96
6.5	Supplement . . . . .	99
<b>7</b>	<b>Implicit Differentiation</b>	<b>101</b>
7.1	General Implicit Differentiation . . . . .	101
7.2	Derivatives of Inverse Functions . . . . .	104
7.3	Logarithmic Differentiation . . . . .	105
7.4	Supplement . . . . .	107
<b>8</b>	<b>Related Rates</b>	<b>109</b>
8.1	Introduction to Related Rates . . . . .	109
8.2	Diagrams for Related Rates . . . . .	114
8.3	Supplement . . . . .	119
<b>9</b>	<b>Critical Numbers and Graphing</b>	<b>121</b>
9.1	Motivation . . . . .	121
9.2	Local Extrema . . . . .	122
9.3	Formal ID of Critical Numbers . . . . .	124
9.4	Sign Tables . . . . .	126
9.5	Inflection Points . . . . .	129
9.6	End Behavior . . . . .	130
9.7	Making Graphs . . . . .	133
9.8	Supplement . . . . .	136
<b>A</b>	<b>Limit Laws</b>	<b>139</b>
<b>B</b>	<b>Derivative Formulas</b>	<b>142</b>
<b>C</b>	<b>Some Useful Rules of Algebra</b>	<b>144</b>

<b>D Units of Measure</b>	<b>145</b>
<b>E Solutions to Supplemental Exercises</b>	<b>148</b>



# Lab 1

## Rates Of Change

### 1.1 Velocity

Motion is frequently modeled using calculus. A building block for this application is the concept of **average velocity**. Average velocity is defined to be net displacement divided by elapsed time. More precisely,

**Definition 1.1.1** (Average Velocity). If  $p$  is a position function for something moving along a numbered line, then we define the **average velocity** over the time interval  $[t_0, t_1]$  to be:

$$\frac{p(t_1) - p(t_0)}{t_1 - t_0}.$$

### Exercises

According to simplified Newtonian physics, if an object is dropped from an elevation<sup>1</sup> of 200 meters (m) and allowed to free fall to the ground, then the elevation of the object is given by the position function

$$p(t) = 200 \text{ m} - (4.9 \text{ m/s}^2) t^2$$

where  $t$  is the amount of time that has passed since the object was dropped. (The “200 m” in the formula represents the elevation from which the object was dropped, and the “ $4.9 \frac{\text{m}}{\text{s}^2}$ ” represents one half of the acceleration due to gravity near the surface of Earth.)

1. Calculate  $p(3 \text{ s})$  and  $p(5 \text{ s})$ . Make sure that you include the units specified in the formula for  $p(t)$ , and that you replace  $t$  with, respectively, 3 s and 5 s. Make sure that you simplify the units, as well as the numerical part of the function value.

---

<sup>1</sup>If you encounter an unfamiliar unit of measure while reading this lab manual, see Appendix D.

2. Calculate  $\frac{p(5\text{s})-p(3\text{s})}{5\text{s}-3\text{s}}$ ; include units while making the calculation. Note that you already calculated  $p(3\text{s})$  and  $p(5\text{s})$  in Exercise 1. What does the result tell you in the context of this problem?

**Remark 1.1.2** (Units in formulas). Thus far we have used a formula where every number that appears in the formula comes with its real world unit ( $p(t) = 200\text{ m} - (4.9\text{ m/s}^2)t^2$ ). A formula like this allows you to calculate  $p(5\text{s})$  or  $p(5\text{ min})$  as you please, as long as you track the units and reduce ratios appropriately.

However, all those units in the original formula might distract you from new calculus topics. We *could* just write  $p(t) = 200 - 4.9t^2$ , but if we do so, we *have* to specify that  $t$  is in seconds and  $p(t)$  is in meters. With this formula, we can calculate something like  $p(5)$ , but we need to recognize that in context, the 5 means that 5 *seconds* have passed. And when we arrive at the result,  $p(5) = 77.5$ , we need to realize that the position is 77.5 *meters*.

For the remainder of this lab manual, formulas will be written in this second way, without explicit units. Each variable will be specified with what unit it is intended to be measured in. To continue the current block of exercises, we redefine

$$p(t) = 200 - 4.9t^2,$$

where  $t$  is in s, and  $p(t)$  is in m.

3. Use Definition 1.1.1 to find a formula for the average velocity of this object over the general time interval  $[t_0, t_1]$ . The first couple of lines of this process are shown below. Copy these lines onto your paper and continue the simplification process.

$$\begin{aligned} \frac{p(t_1) - p(t_0)}{t_1 - t_0} &= \frac{[200 - 4.9t_1^2] - [200 - 4.9t_0^2]}{t_1 - t_0} \\ &= \frac{200 - 4.9t_1^2 - 200 + 4.9t_0^2}{t_1 - t_0} \\ &= \frac{-4.9t_1^2 + 4.9t_0^2}{t_1 - t_0} \\ &= \dots \end{aligned}$$

4. Check the formula you derived in Exercise 3 using  $t_0 = 3$  and  $t_1 = 5$ ; that is, compare the value generated by the formula to that you found in Exercise 2.

5. Now explore some average velocities in tabular form.

Using the formula found in Exercise 3, replace  $t_0$  with 3 but leave  $t_1$  as a variable; simplify the result. Then copy Table 1.1.3 onto your paper and fill in the missing entries.

$t_1$ (s)	$\frac{p(t_1)-p(3)}{t_1-3}$ ( $\frac{\text{m}}{\text{s}}$ )
2.9	
2.99	
2.999	
3.001	
3.01	
3.1	

**Table 1.1.3:** Values of  $\frac{p(t_1)-p(3)}{t_1-3}$

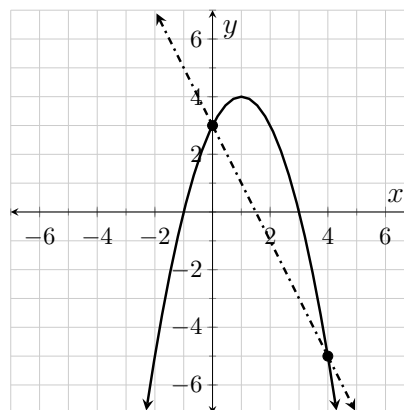
6. As the value of  $t_1$  gets closer to 3, the values in the  $y$ -column of Table 1.1.3 appear to be converging on a single quantity; what is this quantity? What does it mean in the context of this problem?

## 1.2 Secant Line to a Curve

One of the building blocks in differential calculus is the *secant line to a curve*. The only requirement for a line to be considered a secant line to a curve is that the line must intersect the curve in at least two points.

In Figure 1.2.1, we see a secant line to the curve  $y = f(x)$  through the points  $(0, 3)$  and  $(4, -5)$ . Verify that the slope of this line is  $-2$ .

The formula for  $f$  is  $f(x) = 3 + 2x - x^2$ . We can use this formula to come up with a generalized formula for the slope of secant lines to this curve. Specifically, the slope of the line connecting the point  $(x_0, f(x_0))$  to the point  $(x_1, f(x_1))$  is derived in the following example.



**Figure 1.2.1:** A secant line to  $y = f(x) = 3 + 2x - x^2$

**Example 1.2.2** (Calculating Secant Slope).

$$\begin{aligned}
 m_{\text{sec}} &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
 &= \frac{(3 + 2x_1 - x_1^2) - (3 + 2x_0 - x_0^2)}{x_1 - x_0} \\
 &= \frac{3 + 2x_1 - x_1^2 - 3 - 2x_0 + x_0^2}{x_1 - x_0} \\
 &= \frac{(2x_1 - 2x_0) - (x_1^2 - x_0^2)}{x_1 - x_0} \\
 &= \frac{2(x_1 - x_0) - (x_1 + x_0)(x_1 - x_0)}{x_1 - x_0} \\
 &= \frac{[2 - (x_1 + x_0)](x_1 - x_0)}{x_1 - x_0} \\
 &= 2 - x_1 - x_0 \text{ for } x_1 \neq x_0
 \end{aligned}$$

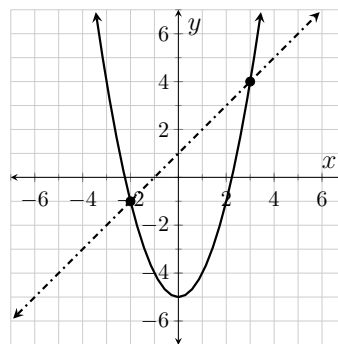
We can check our formula using the line in Figure 1.2.1. If we let  $x_0 = 0$  and  $x_1 = 4$  then our simplified slope formula gives us:  $2 - x_1 - x_0 = 2 - 4 - 0$ , which simplifies to  $-2$  as we expected.

## Exercises

Let  $g(x) = x^2 - 5$ .

1. Following Example 1.2.2, find a formula for the slope of the secant line connecting the points  $(x_0, g(x_0))$  and  $(x_1, g(x_1))$ .
2. Check your slope formula using the two points indicated in Figure 1.2.3.

That is, use the graph to find the slope between the two points and then use your formula to find the slope; make sure that the two values agree!



**Figure 1.2.3:**  
 $y = g(x) = x^2 - 5$  and  
 a secant line

## 1.3 The Difference Quotient

While it's easy to see that the formula  $\frac{f(x_1)-f(x_0)}{x_1-x_0}$  gives the slope of the line connecting two points on the function  $f$ , the resultant expression can at times be awkward to work with. We actually already saw that in Example 1.2.2.

The algebra associated with secant lines (and average velocities) can sometimes be simplified if we designate the variable  $h$  to be the run between the two points (or the length of the time interval). With this designation we have  $x_1 - x_0 = h$  which gives us  $x_1 = x_0 + h$ . Making these substitutions we get (1.3.1). The expression on the right side of (1.3.1) is called the **difference quotient** for  $f$ .

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h} \quad (1.3.1)$$

Let's revisit the function  $f(x) = 3 + 2x - x^2$  from Example 1.2.2. The difference quotient for this function is derived in Example 1.3.1.

**Example 1.3.1** (Calculating a Difference Quotient).

$$\begin{aligned} \frac{f(x_0 + h) - f(x_0)}{h} &= \frac{[3 + 2(x_0 + h) - (x_0 + h)^2] - [3 + 2x_0 - x_0^2]}{h} \\ &= \frac{3 + 2x_0 + 2h - x_0^2 - 2x_0h - h^2 - 3 - 2x_0 + x_0^2}{h} \\ &= \frac{2h - 2x_0h - h^2}{h} \\ &= \frac{h(2 - 2x_0 - h)}{h} \\ &= 2 - 2x_0 - h \text{ for } h \neq 0 \end{aligned}$$

Please notice that all of the terms without a factor of  $h$  subtracted to zero. Please notice, too, that we avoided all of the tricky factoring that appeared in Example 1.2.2!

For simplicity's sake, we generally drop the variable subscript when applying the difference quotient. So for future reference we will define the difference quotient as follows:

**Definition 1.3.2** (The Difference Quotient). The **difference quotient** for the function  $y = f(x)$  is the expression  $\frac{f(x+h)-f(x)}{h}$ .

## Exercises

Completely simplify the difference quotient for each of the following functions. Please note that the template for the difference quotient needs to be adapted to the function name and independent variable in each given equation. For example, the difference quotient for the function in Exercise 1 is  $\frac{v(t+h)-v(t)}{h}$ .

Please make sure that you lay out your work in a manner consistent with the way the work is shown in Example 1.3.1 (excluding the subscripts, of course).

1.  $v(t) = 2.5t^2 - 7.5t$

2.  $g(x) = 3 - 7x$

3.  $w(x) = \frac{3}{x+2}$

Suppose that an object is tossed directly upward in the air in such a way that the elevation of the object (measured in ft) is given by the function  $f$  defined by  $f(t) = 40 + 40t - 16t^2$  where  $t$  is the amount of time that has passed since the object was tossed (measured in s).

4. Simplify the difference quotient for  $f$ .
5. Ignoring the unit, use the difference quotient to determine the average velocity over the interval  $[1.6, 2.8]$ .
6. Calculate  $f(1.6)$  and  $f(2.8)$ . What unit is associated with the values of these expressions? What is the contextual significance of these values?
7. Use the expression  $\frac{f(2.8)-f(1.6)}{2.8-1.6}$  and the answers to Exercise 6 to verify the value you found in Exercise 5. What unit is associated with the values of this expression? What is the contextual significance of this value?
8. Ignoring the unit, use the difference quotient to determine the average velocity over the interval  $[0.4, 2.4]$ .

Moose and squirrel were having casual conversation when suddenly, without any apparent provocation, Boris Badenov launched anti-moose missile in their direction. Fortunately, squirrel had ability to fly as well as great knowledge of missile technology, and he was able to disarm missile well before it hit ground.

The elevation (ft) of the tip of the missile  $t$  seconds after it was launched is given by the function  $h(t) = -16t^2 + 294.4t + 15$ .

9. Calculate the value of  $h(12)$ . What unit is associated with this value? What does the value tell you about the flight of the missile?

10. Calculate the value of  $\frac{h(10)-h(0)}{10}$ . What unit is associated with this value? What does this value tell you about the flight of the missile?
11. The velocity ( $\frac{\text{ft}}{\text{s}}$ ) function for the missile is  $v(t) = -32t + 294.4$ . Calculate the value of  $\frac{v(10)-v(0)}{10}$ . What unit is associated with this value? What does this value tell you about the flight of the missile?

Timmy lived a long life in the 19th century. When Timmy was seven he found a rock that weighed exactly half a stone. (Timmy lived in jolly old England, don't you know.) That rock sat on Timmy's window sill for the next 80 years and wouldn't you know the weight of that rock did not change even one smidge the entire time. In fact, the weight function for this rock was  $w(t) = 0.5$  where  $w(t)$  was the weight of the rock (stones) and  $t$  was the number of years that had passed since that day Timmy brought the rock home.

12. What was the average rate of change in the weight of the rock over the 80 years it sat on Timmy's window sill?
13. Ignoring the unit, simplify the expression  $\frac{w(t_1)-w(t_0)}{t_1-t_0}$ . Does the result make sense in the context of this problem?
14. Showing each step in the process and ignoring the unit, simplify the difference quotient for  $w$ . Does the result make sense in the context of this problem?

Truth be told, there was one day in 1842 when Timmy's mischievous son Nigel took that rock outside and chucked it into the air. The velocity of the rock ( $\frac{\text{ft}}{\text{s}}$ ) was given by  $v(t) = 60 - 32t$  where  $t$  was the number of seconds that had passed since Nigel chucked the rock.

15. What, *including unit*, are the values of  $v(0)$ ,  $v(1)$ , and  $v(2)$  and what do these values tell you in the context of this problem? Don't just write that the values tell you the velocity at certain times; explain what the velocity values tell you about the motion of the rock.
16. Ignoring the unit, simplify the difference quotient for  $v$ .
17. What is the unit for the difference quotient for  $v$ ? What does the value of the difference quotient (including unit) tell you in the context of this problem?

Suppose that a vat was undergoing a controlled drain and that the amount of fluid left in the vat (gal) was given by the formula  $V(t) = 100 - 2t^{3/2}$  where  $t$  is the number of minutes that had passed since the draining process began.

18. What, *including unit*, is the value of  $V(4)$  and what does that value tell you in the context of this problem?
19. Ignoring the unit, write down the formula you get for the difference quotient of  $V$  when  $t = 4$ .
20. Copy Table 1.3.3 onto your paper and fill in the missing values.

Look for a pattern in the output and write down enough digits for each entry so that the pattern is clearly illustrated; the first two entries in the output column have been given to help you understand what is meant by this direction.

$h$	$y$
-0.1	-5.962...
-0.01	-5.9962...
-0.001	
0.001	
0.01	
0.1	

**Table 1.3.3:**  $y = \frac{V(4+h)-V(4)}{h}$

21. What is the unit for the  $y$  values in Table 1.3.3? What do these values (with their unit) tell you in the context of this problem?
22. As the value of  $h$  gets closer to 0, the values in the  $y$  column of Table 1.3.3 appear to be converging to a single number; what is this number and what do you think that value (with its unit) tells you in the context of this problem?



# 1.4 Supplement

## Exercises

The function  $z$  shown in Figure 1.4.1 was generated by the formula  $z(x) = 2 + 4x - x^2$ .

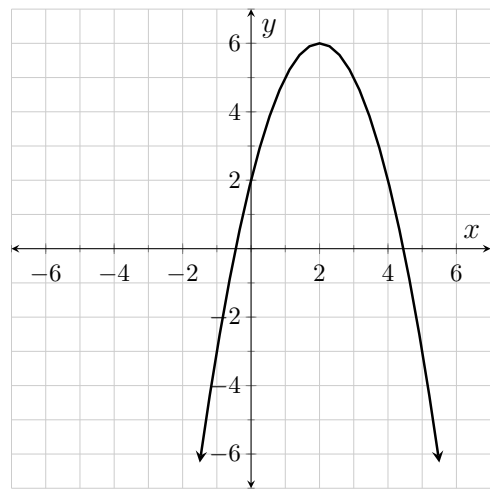


Figure 1.4.1:  $y = z(x) = 2 + 4x - x^2$

$h$	$\frac{z(4+h)-z(4)}{h}$
-0.1	-3.9
-0.01	
-0.001	
0.001	
0.01	
0.1	

Table 1.4.2:  $\frac{z(4+h)-z(4)}{h}$

1. Simplify the difference quotient for  $z$ .
2. Use the graph to find the slope of the secant line to  $z$  between the points where  $x = -1$  and  $x = 2$ . Check your simplified difference quotient for  $z$  by using it to find the slope of the same secant line.
3. Replace  $x$  with 4 in your difference quotient formula and simplify the result. Then copy Table 1.4.2 onto your paper and fill in the missing values.
4. As the value of  $h$  gets closer to 0, the values in the  $y$  column of Table 1.4.2 appear to be converging on a single number; what is this number?
5. The value found in Exercise 4 is called *the slope of the tangent line to  $z$  at 4*. Draw onto Figure 1.4.1 the line that passes through the point  $(4, 2)$  with this slope. The line you just drew is called *the tangent line to  $z$  at 4*.

Find the difference quotient for each function showing all relevant steps in an organized manner.

6.  $f(x) = 3 - 7x$

7.  $g(x) = \frac{7}{x+4}$

8.  $z(x) = \pi$

9.  $s(t) = t^3 - t - 9$

10.  $k(t) = \frac{(t-8)^2}{t}$

Suppose that an object is tossed directly upward in the air in such a way that the elevation of the object (measured in ft) is given by the function  $s(t) = 150 + 60t - 16t^2$  where  $t$  is the amount of time that has passed since the object was tossed (measure in seconds).

11. Find the difference quotient for  $s$ .

12. Use the difference quotient to determine the average velocity of the object over the interval  $[4, 4.2]$  and then verify the value by calculating  $\frac{s(4.2)-s(4)}{4.2-4}$ .

Several applied functions are given below. In each case, find the indicated quantity (including unit) and interpret the value in the context of the application.

13. The velocity,  $v$ , of a roller coaster (in  $\frac{\text{ft}}{\text{s}}$ ) is given by

$$v(t) = -100 \sin\left(\frac{\pi t}{15}\right)$$

where  $t$  is the amount of time (s) that has passed since the coaster topped the first hill. Find and interpret  $\frac{v(7.5)-v(0)}{7.5-0}$ .

14. The elevation of a ping pong ball relative to the table top (in m) is given by the function  $h(t) = 1.1 \left| \cos\left(\frac{2\pi t}{3}\right) \right|$  where  $t$  is the amount of time (s) that has passed since the ball went into play. Find and interpret  $\frac{h(3)-h(1.5)}{3-1.5}$ .

15. The period of a pendulum (s) is given by  $P(x) = \frac{6}{x+1}$  where  $x$  is the number of swings the pendulum has made. Find and interpret  $\frac{P(29)-P(1)}{29-1}$ .

16. The acceleration of a rocket ( $\frac{\text{mph}}{\text{s}}$ ) is given by  $a(t) = 0.02 + 0.13t$  where  $t$  is the amount of time (s) that has passed since lift-off. Find and interpret  $\frac{a(120)-a(60)}{120-60}$ .

# Lab 2

## Limits and Continuity

### 2.1 Limits

While working Exercises 18–22 from Activity 1.3 you completed Table 1.3.3. In the context of that problem the difference quotient being evaluated returned the average rate of change in the volume of fluid remaining in a vat between times  $t = 4$  and  $t = 4 + h$ . As the elapsed time closes in on 0 this average rate of change converges to  $-6$ . From that we deduce that the rate of change in the volume 4 minutes into the draining process must have been  $6 \frac{\text{gal}}{\text{min}}$ .

Please note that we could not deduce the rate of change 4 minutes into the process by replacing  $h$  with 0; in fact, there are at least two things preventing us from doing so. From a strictly mathematical perspective, we cannot replace  $h$  with 0 because that would lead to division by zero in the difference quotient. From a more physical perspective, replacing  $h$  with 0 would in essence stop the clock. If time is frozen, so is the amount of fluid in the vat and the entire concept of “rate of change” becomes moot.

It turns out that it is frequently more useful (not to mention interesting) to explore the *trend* in a function as the input variable *approaches* a number rather than the actual value of the function at that number. Mathematically we describe these trends using **limits**.

If we denote  $f(h) = \frac{V(4+h)-V(4)}{h}$  to be the difference quotient in Table 1.3.3, then we could describe the trend evidenced in the table by saying “the limit of  $f(h)$  as  $h$  approaches zero is  $-6$ .” Please note that as  $h$  changes value, the value of  $f(h)$  changes, not the value of the limit. The limit value is a fixed number to which the value of  $f(h)$  converges. Symbolically we write  $\lim_{h \rightarrow 0} f(h) = -6$ .

Most of the time the value of a function at the number  $a$  and the limit of the function as  $x$  approaches  $a$  are in fact the same number. When this occurs we say that the function is **continuous** at  $a$ . We will explore the concept of continuity more in depth at the end of this lab, once we have a handle on the idea of

limits. To help you better understand the concept of limit we need to have you confront situations where the function value and limit value are not equal to one another. Graphs can be useful for helping distinguish function values from limit values, so that is the perspective you are going to use in the first couple of problems in this lab.

## Exercises

1. Several function values and limit values for the function in Figure 2.1.1 are given below. You and your group mates should take turns reading the equations aloud. Make sure that you read the symbols correctly, that's part of what you are learning! Also, discuss why the values are what they are and make sure that you get help from your instructor to clear up any confusion.

$$f(-2) = 6, \text{ but } \lim_{x \rightarrow -2} f(x) = 3.$$

$$f(-4) \text{ is undefined, but } \lim_{x \rightarrow -4} f(x) = 2.$$

$$f(1) = -1, \text{ but } \lim_{x \rightarrow 1} f(x) \text{ does not exist.}$$

$\underbrace{\lim_{x \rightarrow 1^-} f(x)}_{\substack{\text{the limit of } f(x) \\ \text{as } x \text{ approaches } 1 \\ \text{from the left}}} = -3, \text{ but}$	$\underbrace{\lim_{x \rightarrow 1^+} f(x)}_{\substack{\text{the limit of } f(x) \\ \text{as } x \text{ approaches } 1 \\ \text{from the right}}} = -1.$
---	--

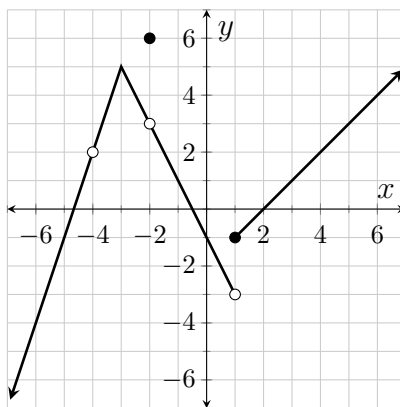


Figure 2.1.1:  $y = f(x)$

Copy each of the following expressions onto your paper and either state the value or state that the value is undefined or doesn't exist. Make sure that when discussing the values you use proper terminology. All expressions are in reference to the function  $g$  shown in Figure 2.1.2.

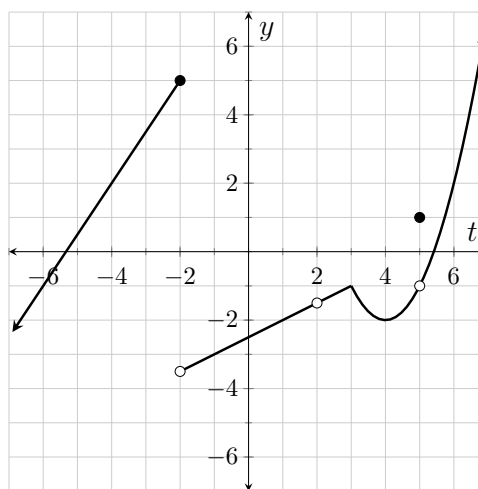


Figure 2.1.2:  $y = g(t)$

- |                                      |                                      |                                    |
|--------------------------------------|--------------------------------------|------------------------------------|
| 2. $g(5)$                            | 3. $\lim_{t \rightarrow 5} g(t)$     | 4. $g(3)$                          |
| 5. $\lim_{t \rightarrow 3^-} g(t)$   | 6. $\lim_{t \rightarrow 3^+} g(t)$   | 7. $\lim_{t \rightarrow 3} g(t)$   |
| 8. $g(2)$                            | 9. $\lim_{t \rightarrow 2} g(t)$     | 10. $g(-2)$                        |
| 11. $\lim_{t \rightarrow -2^-} g(t)$ | 12. $\lim_{t \rightarrow -2^+} g(t)$ | 13. $\lim_{t \rightarrow -2} g(t)$ |

Values of the function  $f$ , where  $f(x) = \frac{3x^2 - 16x + 5}{2x^2 - 13x + 15}$ , are shown in Table 2.1.3, and values of the function  $p$ , where  $p(t) = \sqrt{t - 12}$ , are shown in Table 2.1.4. These questions are in reference to these functions.

$x$	$f(x)$
4.99	2.0014
4.999	2.00014
4.9999	2.000014
5.0001	1.999986
5.001	1.99986
5.01	1.9986

Table 2.1.3:  $f(x) = \frac{3x^2 - 16x + 5}{2x^2 - 13x + 15}$

$t$	$p(t)$
20.9	2.98...
20.99	2.998...
20.999	2.9998...
21.001	3.0002...
21.01	3.002...
21.1	3.02...

Table 2.1.4:  $p(t) = \sqrt{t - 12}$

14. What is the value of  $f(5)$ ?
15. What is the value of  $\lim_{x \rightarrow 5} \frac{3x^2 - 16x + 5}{2x^2 - 13x + 15}$ ?

16. What is the value of  $p(21)$ ?
17. What is the value of  $\lim_{t \rightarrow 21} \sqrt{t - 12}$ ?

Create tables similar to Tables 2.1.3 and 2.1.4 from which you can deduce each of the following limit values. Make sure that you include table numbers, table captions, and meaningful column headings. Make sure that your input values follow patterns similar to those used in Tables 2.1.3 and 2.1.4. Make sure that you round your output values in such a way that a clear and compelling pattern in the output is clearly demonstrated by your stated values. Make sure that you state the limit value!

18.  $\lim_{t \rightarrow 6} \frac{t^2 - 10t + 24}{t - 6}$
19.  $\lim_{x \rightarrow -1^+} \frac{\sin(x + 1)}{3x + 3}$
20.  $\lim_{h \rightarrow 0^-} \frac{h}{4 - \sqrt{16 - h}}$

## 2.2 Limits Laws

When proving the value of a limit we frequently rely upon laws that are easy to prove using the technical definitions of limit. These laws can be found in Appendix A. The first of these type laws are called replacement laws. Replacement laws allow us to replace limit expressions with the actual values of the limits.

### Exercises

The value of each of the following limits can be established using one of the replacement laws. Copy each limit expression onto your own paper, state the value of the limit (e.g.  $\lim_{x \rightarrow 9} 5 = 5$ ), and state the replacement law (by number) that establishes the value of the limit.

1.  $\lim_{t \rightarrow \pi} t$
2.  $\lim_{x \rightarrow 14} 23$
3.  $\lim_{x \rightarrow 14} x$

4. (Applying Limit Laws) This exercise is fully worked out to serve as an example.

Find  $\lim_{x \rightarrow 7} (4x^2 + 3)$ .

$$\lim_{x \rightarrow 7} (4x^2 + 3) = \lim_{x \rightarrow 7} (4x^2) + \lim_{x \rightarrow 7} 3 \quad \text{Limit Law A1}$$

$$\begin{aligned}
&= 4 \lim_{x \rightarrow 7} x^2 + \lim_{x \rightarrow 7} 3 && \text{Limit Law A3} \\
&= 4 \left( \lim_{x \rightarrow 7} x \right)^2 + \lim_{x \rightarrow 7} 3 && \text{Limit Law A6} \\
&= 4 \cdot 7^2 + 3 && \text{Limit Law R1, R2} \\
&= 199
\end{aligned}$$

The algebraic limit laws allow us to replace limit expressions with equivalent limit expressions. When applying limit laws our first goal is to come up with an expression in which every limit in the expression can be replaced with its value based upon one of the replacement laws. This process is shown in Exercise 4. Please note that all replacement laws are saved for the second to last step and that each replacement is explicitly shown. Please note also that each limit law used is referenced by number.

Use the limit laws to establish the value of each of the following limits. Make sure that you use the step-by-step, vertical format shown in Exercise 4. Make sure that you cite the limit laws used in each step. To help you get started, the steps necessary in Exercise 5 are outlined as:

- (a) Apply Limit Law A6
- (b) Apply Limit Law A1
- (c) Apply Limit Law A3
- (d) Apply Limit Law R1 and Limit Law R2

$$\begin{array}{lll}
5. \lim_{t \rightarrow 4} \sqrt{6t + 1} & 6. \lim_{y \rightarrow 7} \frac{y + 3}{y - \sqrt{y + 9}} & 7. \lim_{x \rightarrow \pi} (x \cos(x))
\end{array}$$

## 2.3 Indeterminate Limits

Many limits have the form  $\frac{0}{0}$  which means the expressions in both the numerator and denominator limit to zero (e.g.  $\lim_{x \rightarrow 3} \frac{2x-6}{x-3}$ ). The form  $\frac{0}{0}$  is called **indeterminate** because we do not know the value of the limit (or even if it exists) so long as the limit has that form. When confronted with limits of form  $\frac{0}{0}$  we must first manipulate the expression so that common factors causing the zeros in the numerator and denominator are isolated. Limit law A7 can then be used to justify eliminating the common factors and once they are gone we may proceed with the application of the remaining limit laws. Examples 2.3.1 and 2.3.2 illustrate this situation.

**Example 2.3.1.**

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{x^2 - 8x + 15}{x - 3} &= \lim_{x \rightarrow 3} \frac{(x - 5)(x - 3)}{x - 3} \\
&= \lim_{x \rightarrow 3} (x - 5) && \text{Limit Law A7} \\
&= \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 5 && \text{Limit Law A2} \\
&= 3 - 5 && \text{Limit Law R1, R2} \\
&= -2
\end{aligned}$$

Note that the forms of the limits on either side of the first line are  $\frac{0}{0}$ , but the form of the limit in the second line is no longer indeterminate. So at the second line, we can begin applying limit laws.

**Example 2.3.2.**

$$\begin{aligned}
\lim_{\theta \rightarrow 0} \frac{1 - \cos(\theta)}{\sin^2(\theta)} &= \lim_{\theta \rightarrow 0} \left( \frac{1 - \cos(\theta)}{\sin^2(\theta)} \cdot \frac{1 + \cos(\theta)}{1 + \cos(\theta)} \right) \\
&= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2(\theta)}{\sin^2(\theta) \cdot (1 + \cos(\theta))} \\
&= \lim_{\theta \rightarrow 0} \frac{\sin^2(\theta)}{\sin^2(\theta) \cdot (1 + \cos(\theta))} \\
&= \lim_{\theta \rightarrow 0} \frac{1}{1 + \cos(\theta)} && \text{Limit Law A7} \\
&= \frac{\lim_{\theta \rightarrow 0} 1}{\lim_{\theta \rightarrow 0} (1 + \cos(\theta))} && \text{Limit Law A5} \\
&= \frac{\lim_{\theta \rightarrow 0} 1}{\lim_{\theta \rightarrow 0} 1 + \lim_{\theta \rightarrow 0} \cos(\theta)} && \text{Limit Law A1} \\
&= \frac{\lim_{\theta \rightarrow 0} 1}{\lim_{\theta \rightarrow 0} 1 + \cos(\lim_{\theta \rightarrow 0} \theta)} && \text{Limit Law A6} \\
&= \frac{1}{1 + \cos(0)} && \text{Limit Law R1, R2} \\
&= \frac{1}{2}
\end{aligned}$$

The form of each limit above was  $\frac{0}{0}$  until the fourth line, where we were able to begin applying limit laws.

As seen in Example 2.3.2, trigonometric identities can come into play while trying to eliminate the form  $\frac{0}{0}$ . Elementary rules of logarithms can also play a role in this process. Before you begin evaluating limits whose initial form is  $\frac{0}{0}$ , you need to make sure that you recall some of these basic rules. That is the purpose of Exercise 1.



## Exercises

1. Complete each of the following identities (over the real numbers). Make sure that you check with your lecture instructor so that you know which of these identities you are expected to memorize.

The following identities are valid for all values of  $x$  and  $y$ .

$$\begin{array}{ll} 1 - \cos^2(x) = \underline{\hspace{2cm}} & \tan^2(x) = \underline{\hspace{2cm}} \\ \sin(2x) = \underline{\hspace{2cm}} & \tan(2x) = \underline{\hspace{2cm}} \\ \sin(x + y) = \underline{\hspace{2cm}} & \cos(x + y) = \underline{\hspace{2cm}} \\ \sin\left(\frac{x}{2}\right) = \underline{\hspace{2cm}} & \cos\left(\frac{x}{2}\right) = \underline{\hspace{2cm}} \end{array}$$

There are three versions of the following identity; write them all.

$$\cos(2x) = \underline{\hspace{2cm}} \quad \cos(2x) = \underline{\hspace{2cm}} \quad \cos(2x) = \underline{\hspace{2cm}}$$

The following identities are valid for all positive values of  $x$  and  $y$  and all values of  $n$ .

$$\begin{array}{ll} \ln(xy) = \underline{\hspace{2cm}} & \ln\left(\frac{x}{y}\right) = \underline{\hspace{2cm}} \\ \ln(x^n) = \underline{\hspace{2cm}} & \ln(e^n) = \underline{\hspace{2cm}} \end{array}$$

Use the limit laws to establish the value of each of the following limits after first manipulating the expression so that it no longer has form  $\frac{0}{0}$ . Make sure that you use the step-by-step, vertical format shown in Examples 2.3.1 and Example 2.3.2. Make sure that you cite each limit law used.

$$2. \lim_{x \rightarrow -4} \frac{x + 4}{2x^2 + 5x - 12}$$

$$3. \lim_{x \rightarrow 0} \frac{\sin(2x)}{\sin(x)}$$

$$4. \lim_{\beta \rightarrow 0} \frac{\sin(\beta + \pi)}{\sin(\beta)}$$

$$5. \lim_{t \rightarrow 0} \frac{\cos(2t) - 1}{\cos(t) - 1}$$

$$6. \lim_{x \rightarrow 1} \frac{4 \ln(x) + 2 \ln(x^3)}{\ln(x) - \ln(\sqrt{x})}$$

$$7. \lim_{w \rightarrow 9} \frac{9 - w}{\sqrt{w} - 3}$$

## 2.4 Limits at Infinity

We are frequently interested in a function's "end behavior." That is, what is the behavior of the function as the input variable increases without bound or decreases without bound.

Many times a function will approach a horizontal asymptote as its end behavior. Assuming that the horizontal asymptote  $y = L$  represents the end behavior of the function  $f$  both as  $x$  increases without bound and as  $x$  decreases into the negative without bound, we write  $\lim_{x \rightarrow \infty} f(x) = L$  and  $\lim_{x \rightarrow -\infty} f(x) = L$ .

The formalistic way to read  $\lim_{x \rightarrow \infty} f(x) = L$  is “the limit of  $f(x)$  as  $x$  approaches infinity equals  $L$ .” When read that way, however, the words need to be taken *anything but literally*. In the first place,  $x$  isn’t approaching anything! The entire point is that  $x$  is increasing without any bound on how large its value becomes. Secondly, there is no place on the real number line called “infinity”; infinity is not a number. Hence  $x$  certainly can’t be approaching something that isn’t even there!

## Exercises

1. For the function in Figure 9.6.1 from Lab 9, we could (correctly) write down that  $\lim_{x \rightarrow \infty} f_1(x) = -2$  and  $\lim_{x \rightarrow -\infty} f_1(x) = -2$ . Go ahead and write (and say aloud) the analogous limits for the functions in Figures 9.6.2, 9.6.3, 9.6.4, and 9.6.5,.

Values of the function  $f$  defined by  $f(x) = \frac{3x^2 - 16x + 5}{2x^2 - 13x + 15}$  are shown in Table 2.4.1. Both of the questions below are in reference to this function.

$x$	$f(x)$
-1000	1.498...
-10,000	1.4998...
-100,000	1.49998...
-1,000,000	1.499998...

**Table 2.4.1:**  $f(x) = \frac{3x^2 - 16x + 5}{2x^2 - 13x + 15}$

2. Find  $\lim_{x \rightarrow -\infty} f(x)$ .

3. What is the equation of the horizontal asymptote for the graph of  $y = f(x)$ ?

Jorge and Vanessa were in a heated discussion about horizontal asymptotes. Jorge said that functions never cross horizontal asymptotes. Vanessa said Jorge was nuts. Vanessa whipped out her trusty calculator and generated the values in Table 2.4.2 to prove her point.

$t$	$g(t)$
$10^3$	1.008...
$10^4$	0.9997...
$10^5$	1.0000004...
$10^6$	0.9999997...
$10^7$	1.00000004...
$10^8$	1.000000009...
$10^9$	1.0000000005...
$10^{10}$	0.9999999995...

**Table 2.4.2:**  $g(t) = 1 + \frac{\sin(t)}{t}$

4. Find  $\lim_{t \rightarrow \infty} g(t)$ .
5. What is the equation of the horizontal asymptote for the graph of  $y = g(t)$ ?
6. How many times does the curve  $y = g(t)$  cross its horizontal asymptote?

## 2.5 Limits at Infinity Tending to Zero

When using limit laws to establish limit values as  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ , Limit Law A1–Limit Law A6 and Limit Law R2 are still in play (when applied in a valid manner), but Limit Law R1 cannot be applied. (The reason it cannot be applied is discussed in detail in Exercises 11–18 from Activity 2.8.)

There is a new replacement law that can only be applied when  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ ; this is Limit Law R3. Limit Law R3 essentially says that if the value of a function is increasing without any bound on how large it becomes or if the function is decreasing without any bound on how large its absolute value becomes, then the value of a constant divided by that function must be approaching zero. An analogy can be found in extremely poor party planning. Let's say that you plan to have a pizza party and you buy five pizzas. Suppose that as the hour of the party approaches more and more guests come in the door—in fact the guests never stop coming! Clearly as the number of guests continues to rise the amount of pizza each guest will receive quickly approaches zero (assuming the pizzas are equally divided among the guests).

## Exercises

1. Consider the function  $f$  defined by  $f(x) = \frac{12}{x}$ .

Complete Table 2.5.1 without the use of your calculator. What limit value and limit law are being illustrated in the table?

$x$	$f(x)$
1000	
10,000	
100,000	
1,000,000	

**Table 2.5.1:**  $f(x) = \frac{12}{x}$

## 2.6 Ratios of Infinities

Many limits have the form  $\frac{\infty}{\infty}$ , which we take to mean that the expressions in both the numerator and denominator are increasing or decreasing without bound. When confronted with a limit of type  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$  or  $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)}$  that has the form  $\frac{\infty}{\infty}$ , we can frequently resolve the limit if we first divide the dominant factor of the dominant term of the denominator from both the numerator and the denominator. When we do this, we need to completely simplify each of the resultant fractions and make sure that the resultant limit exists before we start to apply limit laws. We then apply the algebraic limit laws until all of the resultant limits can be replaced using Limit Law R2 and Limit Law R3. This process is illustrated in Example 2.6.1.

**Example 2.6.1.**

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \frac{3t^2 + 5t}{3 - 5t^2} &= \lim_{t \rightarrow \infty} \left( \frac{3t^2 + 5t}{3 - 5t^2} \cdot \frac{1/t^2}{1/t^2} \right) \\
 &= \lim_{t \rightarrow \infty} \frac{3 + 5/t}{3/t^2 - 5} && \text{No longer indeterminate form} \\
 &= \frac{\lim_{t \rightarrow \infty} (3 + 5/t)}{\lim_{t \rightarrow \infty} (3/t^2 - 5)} && \text{Limit Law A5} \\
 &= \frac{\lim_{t \rightarrow \infty} 3 + \lim_{t \rightarrow \infty} \frac{5}{t}}{\lim_{t \rightarrow \infty} \frac{3}{t^2} - \lim_{t \rightarrow \infty} 5} && \text{Limit Law A1, Limit Law A2} \\
 &= \frac{3 + 0}{0 - 5} && \text{Limit Law R2, Limit Law R3} \\
 &= -\frac{3}{5}
 \end{aligned}$$

## Exercises

Use the limit laws to establish the value of each limit after dividing the dominant term-factor in the denominator from both the numerator and denominator. Remember to simplify each resultant expression before you begin to apply the limit laws.

$$\begin{array}{lll} 1. \lim_{t \rightarrow -\infty} \frac{4t^2}{4t^2 + t^3} & 2. \lim_{t \rightarrow \infty} \frac{6e^t + 10e^{2t}}{2e^{2t}} & 3. \lim_{y \rightarrow \infty} \sqrt{\frac{4y + 5}{5 + 9y}} \end{array}$$

## 2.7 Non-existent Limits

Many limit values do not exist. Sometimes the non-existence is caused by the function value either increasing without bound or decreasing without bound. In these special cases we use the symbols  $\infty$  and  $-\infty$  to communicate the non-existence of the limits. Figures 2.7.1–2.7.3 can be used to illustrate some ways in which we communicate the **non-existence** of these types of limits.

- In Figure 2.7.1 we could (correctly) write

$$\lim_{x \rightarrow 2} k(x) = \infty, \lim_{x \rightarrow 2^-} k(x) = \infty, \text{ and } \lim_{x \rightarrow 2^+} k(x) = \infty.$$

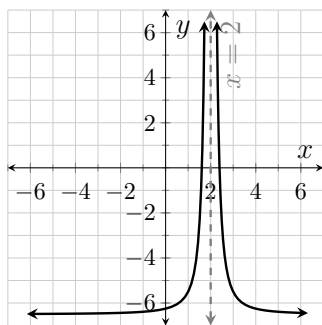
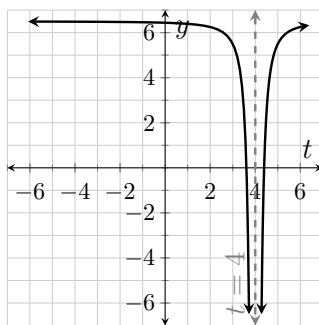
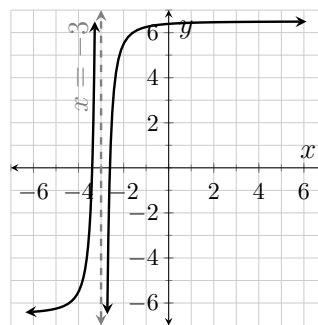
- In Figure 2.7.2 we could (correctly) write

$$\lim_{t \rightarrow 4} w(t) = -\infty, \lim_{t \rightarrow 4^-} w(t) = -\infty, \text{ and } \lim_{t \rightarrow 4^+} w(t) = -\infty.$$

- In Figure 2.7.3 we could (correctly) write

$$\lim_{x \rightarrow -3^-} T(x) = \infty \text{ and } \lim_{x \rightarrow -3^+} T(x) = -\infty.$$

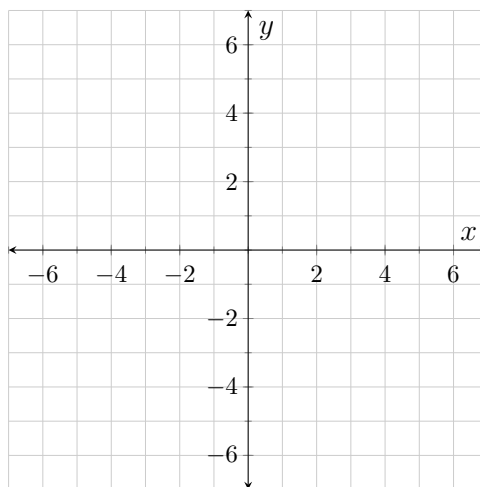
There is no shorthand way of communicating the non-existence of the two-sided limit  $\lim_{x \rightarrow -3} T(x)$ .

**Figure 2.7.1:**  $y = k(x)$ **Figure 2.7.2:**  $y = w(t)$ **Figure 2.7.3:**  $y = T(x)$ 

## Exercises

1. In the plane provided, draw the graph of a single function,  $f$ , that satisfies each of the following limit statements. Make sure that you draw the necessary asymptotes and that you label each asymptote with its equation.

$$\begin{aligned}\lim_{x \rightarrow 3^-} f(x) &= -\infty \\ \lim_{x \rightarrow \infty} f(x) &= 0 \\ \lim_{x \rightarrow 3^+} f(x) &= \infty \\ \lim_{x \rightarrow -\infty} f(x) &= 2\end{aligned}$$

**Figure 2.7.4:**  $y = f(x)$ 

## 2.8 Vertical Asymptotes

Whenever  $\lim_{x \rightarrow a} f(x) \neq 0$  but  $\lim_{x \rightarrow a} g(x) = 0$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  *does not exist* because from either side of  $a$  the value of  $\frac{f(x)}{g(x)}$  has an absolute value that will become arbitrarily large. In these situations the line  $x = a$  is a vertical asymptote for the graph of  $y = \frac{f(x)}{g(x)}$ . For example, the line  $x = 2$  is a vertical asymptote for the

function  $h$  defined by  $h(x) = \frac{x+5}{2-x}$ . We say that  $\lim_{x \rightarrow 2} \frac{x+5}{2-x}$  has the form “not-zero over zero.” (Specifically, the form of  $\lim_{x \rightarrow 2} \frac{x+5}{2-x}$  is  $\frac{7}{0}$ .) Every limit with form “not-zero over zero” *does not exist*. However, we frequently can communicate the non-existence of the limit using an infinity symbol. In the case of  $h(x) = \frac{x+5}{2-x}$  it’s pretty easy to see that  $h(1.99)$  is a positive number whereas  $h(2.01)$  is a negative number. Consequently, we can infer that  $\lim_{x \rightarrow 2^-} h(x) = \infty$  and  $\lim_{x \rightarrow 2^+} h(x) = -\infty$ . Remember, these equations are communicating that the limits *do not exist* as well as the reason for their non-existence. There is no short-hand way to communicate the non-existence of the two-sided limit  $\lim_{x \rightarrow 2} h(x)$ .

## Exercises

Suppose that  $g(t) = \frac{t+4}{t+3}$ .

1. What is the vertical asymptote on the graph of  $y = g(t)$ ?
2. Write an equality about  $\lim_{t \rightarrow -3^-} g(t)$ .
3. Write an equality about  $\lim_{t \rightarrow -3^+} g(t)$ .
4. Is it possible to write an equality about  $\lim_{t \rightarrow -3} g(t)$ ? If so, do it.
5. Which of the following limits exist?

$$\lim_{t \rightarrow -3^-} g(t)? \quad \lim_{t \rightarrow -3^+} g(t)? \quad \lim_{t \rightarrow -3} g(t)?$$

Suppose that  $z(x) = \frac{7-3x^2}{(x-2)^2}$ .

6. What is the vertical asymptote on the graph of  $y = z(x)$ ?
7. Is it possible to write an equality about  $\lim_{x \rightarrow 2} z(x)$ ? If so, do it.
8. What is the horizontal asymptote on the graph of  $y = z(x)$ ?
9. Which of the following limits exist?

$$\lim_{x \rightarrow 2^-} z(x)? \quad \lim_{x \rightarrow 2^+} z(x)? \quad \lim_{x \rightarrow 2} z(x)?$$

**10.** Consider the function  $f$  defined by  $f(x) = \frac{x+7}{x-8}$ . Complete Table 2.8.1 without the use of your calculator.

Use this as an opportunity to discuss why limits of form “not-zero over zero” are “infinite limits.” What limit equation is being illustrated in the table?

$x$	$x + 7$	$x - 8$	$f(x)$
8.1	15.1	0.1	
8.01	15.01	0.01	
8.001	15.001	0.001	
8.0001	15.0001	0.0001	

**Table 2.8.1:**  $f(x) = \frac{x+7}{x-8}$

Hear me, and hear me loud... $\infty$  *does not exist*. This, in part, is why we cannot apply Limit Law R1 to an expression like  $\lim_{x \rightarrow \infty} x$  to see that  $\lim_{x \rightarrow \infty} x = \infty$ . When we write, say,  $\lim_{x \rightarrow 7} x = 7$ , we are replacing the limit, we are replacing the limit expression *with its value*—that’s what the replacement laws are all about! When we write  $\lim_{x \rightarrow \infty} x = \infty$ , we are not replacing the limit expression with a value! We are explicitly saying that the limit has no value (i.e. does not exist) as well as saying the reason the limit does not exist. The limit laws can only be applied when all of the limits in the equation exist. With this in mind, discuss and decide whether each of the following equations are *true* or *false*.

**11.** True or False?  $\lim_{x \rightarrow 0} \left( \frac{e^x}{e^x} \right) = \frac{\lim_{x \rightarrow 0} e^x}{\lim_{x \rightarrow 0} e^x}$

**12.** True or False?  $\lim_{x \rightarrow 1} \frac{e^x}{\ln(x)} = \frac{\lim_{x \rightarrow 1} e^x}{\lim_{x \rightarrow 1} \ln(x)}$

**13.** True or False?  $\lim_{x \rightarrow 0^+} (2 \ln(x)) = 2 \lim_{x \rightarrow 0^+} \ln(x)$

**14.** True or False?  $\lim_{x \rightarrow \infty} (e^x - \ln(x)) = \lim_{x \rightarrow \infty} e^x - \lim_{x \rightarrow \infty} \ln(x)$

**15.** True or False?  $\lim_{x \rightarrow -\infty} \left( \frac{e^{-x}}{e^{-x}} \right) = \frac{\lim_{x \rightarrow -\infty} e^{-x}}{\lim_{x \rightarrow -\infty} e^{-x}}$

**16.** True or False?  $\lim_{x \rightarrow 1} \left( \frac{\ln(x)}{e^x} \right) = \frac{\lim_{x \rightarrow 1} \ln(x)}{\lim_{x \rightarrow 1} e^x}$

**17.** True or False?  $\lim_{\theta \rightarrow \infty} \frac{\sin(\theta)}{\sin(\theta)} = \frac{\lim_{\theta \rightarrow \infty} \sin(\theta)}{\lim_{\theta \rightarrow \infty} \sin(\theta)}$

**18.** True or False?  $\lim_{x \rightarrow -\infty} e^{1/x} = e^{\lim_{x \rightarrow -\infty} 1/x}$



**19.** Mindy tried to evaluate  $\lim_{x \rightarrow 6^+} \frac{4x-24}{x^2-12x+36}$  using the limit laws. Things went horribly wrong for Mindy (her work is shown below). Identify what is wrong in Mindy's work and discuss what a more reasonable approach might have been.

***This “solution” is not correct! Do not emulate Mindy’s work!***

$$\begin{aligned}
 \lim_{x \rightarrow 6^+} \frac{4x-24}{x^2-12x+36} &= \lim_{x \rightarrow 6^+} \frac{4(x-6)}{(x-6)^2} \\
 &= \lim_{x \rightarrow 6^+} \frac{4}{x-6} \\
 &= \frac{\lim_{x \rightarrow 6^+} 4}{\lim_{x \rightarrow 6^+} (x-6)} && \text{Limit Law A5} \\
 &= \frac{\lim_{x \rightarrow 6^+} 4}{\lim_{x \rightarrow 6^+} x - \lim_{x \rightarrow 6^+} 6} && \text{Limit Law A2} \\
 &= \frac{4}{6-6} && \text{Limit Law R1, Limit Law R2} \\
 &= \frac{4}{0}
 \end{aligned}$$

***This “solution” is not correct! Do not emulate Mindy’s work!***

## 2.9 Continuity

Many statements we make about functions are only true over intervals where the function is **continuous**. When we say a function is continuous over an interval, we basically mean that there are no breaks in the function over that interval; that is, there are no vertical asymptotes, holes, jumps, or gaps along that interval.

**Definition 2.9.1** (Continuity). The function  $f$  is **continuous at the number**  $a$  if and only if  $\lim_{x \rightarrow a} f(x) = f(a)$ .

There are three ways that the defining property can fail to be satisfied at a given value of  $a$ . To facilitate exploration of these three manners of failure, we can separate the defining property into three sub-properties.

1.  $f(a)$  must be defined
2.  $\lim_{x \rightarrow a} f(x)$  must exist
3.  $\lim_{x \rightarrow a} f(x)$  must equal  $f(a)$

Please note that if either Property 1 or Property 2 fails to be satisfied at a given value of  $a$ , then Property 3 also fails to be satisfied at  $a$ .

## Exercises

These questions refer to the function in Figure 2.9.2.

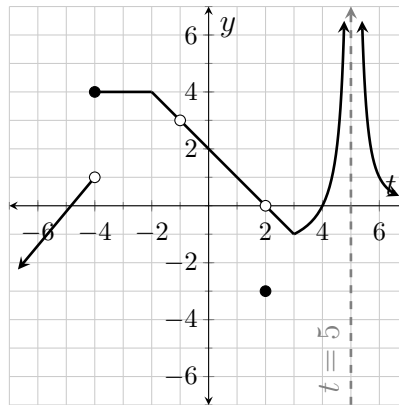


Figure 2.9.2:  $y = h(t)$

1. Complete Table 2.9.3.

Table 2.9.3: Function values and limit values for  $h$

$a$	$h(a)$	$\lim_{t \rightarrow a^-} h(t)$	$\lim_{t \rightarrow a^+} h(t)$	$\lim_{t \rightarrow a} h(t)$
-4				
-1				
2				
3				
5				

2. State the values of  $t$  at which the function  $h$  is discontinuous. For each instance of discontinuity, state (by number) all of the sub-properties in Definition 2.9.1 that fail to be satisfied.

## 2.10 Discontinuities

When a function has a discontinuity at  $a$ , the function is sometimes continuous from only the right or only the left at  $a$ . (Please note that when we say “the function is continuous at  $a$ ” we mean that the function is continuous from *both* the right and left at  $a$ .)

**Definition 2.10.1** (One-sided Continuity). The function  $f$  is continuous from the left at  $a$  if and only if

$$\lim_{x \rightarrow a^-} f(x) = f(a).$$

The function  $f$  is continuous from the right at  $a$  if and only if

$$\lim_{x \rightarrow a^+} f(x) = f(a).$$

Some discontinuities are classified as **removable discontinuities**. Discontinuities that are holes or skips (holes with a secondary point) are **removable**.

**Definition 2.10.2** (Removable Discontinuity). We say that  $f$  has a removable discontinuity at  $a$  if  $f$  is discontinuous at  $a$  but  $\lim_{x \rightarrow a} f(x)$  exists.

## Exercises

1. Referring to the function  $h$  shown in Figure 2.9.2, state the values of  $t$  where the function is continuous from the right but not the left. Then state the values of  $t$  where the function is continuous from the left but not the right.
2. Referring again to the function  $h$  shown in Figure 2.9.2, state the values of  $t$  where the function has removable discontinuities.

## 2.11 Continuity on an Interval

Now that we have a definition for continuity at a number, we can go ahead and define what we mean when we say a function is continuous over an interval.

**Definition 2.11.1** (Continuity on an Interval). The function  $f$  is **continuous** over an open interval if and only if it is continuous at each and every number on that interval.

The function  $f$  is **continuous** over a closed interval  $[a, b]$  if and only if it is continuous on  $(a, b)$ , continuous from the right at  $a$ , and continuous from the left at  $b$ . Similar definitions apply to half-open intervals.

## Exercises

1. Write a definition for continuity on the half-open interval  $(a, b]$ .

Referring to the function in Figure 2.9.2, decide whether each of the following statements are *true* or *false*.

2. True or False?  $h$  is continuous on  $[-4, -1)$ .
3. True or False?  $h$  is continuous on  $(-4, -1)$ .
4. True or False?  $h$  is continuous on  $(-4, -1]$ .
5. True or False?  $h$  is continuous on  $(-1, 2]$ .
6. True or False?  $h$  is continuous on  $(-1, 2)$ .
7. True or False?  $h$  is continuous on  $(-\infty, -4)$ .
8. True or False?  $h$  is continuous on  $(-\infty, -4]$ .

Several functions are described below. Your task is to sketch a graph for each function on its provided axis system. Do not introduce any unnecessary discontinuities or intercepts that are not directly implied by the stated properties. Make sure that you draw all implied asymptotes and label them with their equations.

9.

$$f(0) = 4 \text{ and } f(4) = 5$$

$$\lim_{x \rightarrow 4^-} f(x) = 2 \text{ and } \lim_{x \rightarrow 4^+} f(x) = 5$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 4$$

$f$  has at most one discontinuity.

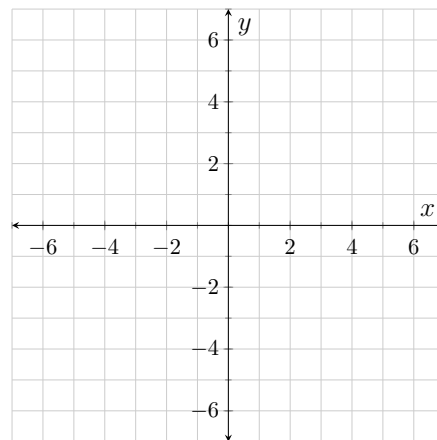


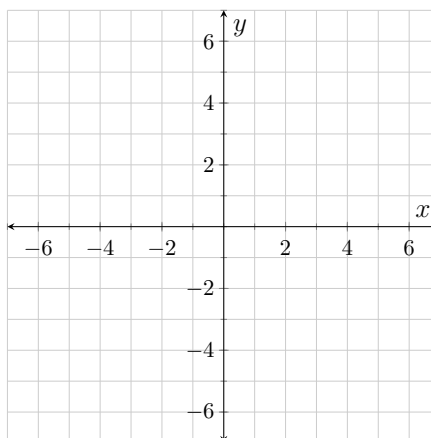
Figure 2.11.2:  $y = f(x)$

10.

$$g(0) = 4, \quad g(3) = -2, \quad \text{and} \quad g(6) = 0$$

$$\lim_{x \rightarrow -2} g(x) = \lim_{x \rightarrow -\infty} g(x) = \infty$$

$g$  is continuous and has constant slope on  $(0, \infty)$ .

Figure 2.11.3:  $y = g(x)$ 

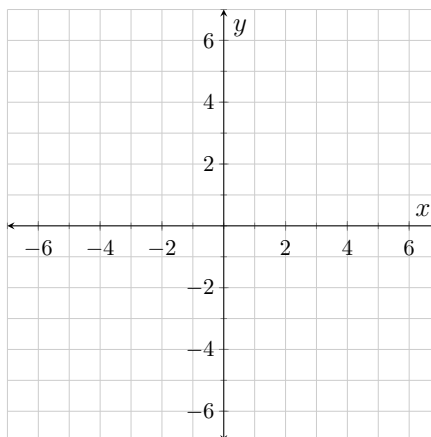
11.

$$m(-6) = 5$$

$$\lim_{x \rightarrow -4^+} m(x) = -2$$

$$\lim_{x \rightarrow 3} m(x) = \lim_{x \rightarrow \infty} m(x) = -\infty$$

$m$  only has discontinuities at  $-4$  and  $3$ .  $m$  has no zeros.  $m$  has slope  $-2$  over  $(-\infty, -4)$ .  $m$  is continuous over  $[-4, 3)$ .

Figure 2.11.4:  $y = m(x)$ 

## 2.12 Discontinuous Formulas

Discontinuities are a little more challenging to identify when working with formulas than when working with graphs. One reason for the added difficulty is

that when working with a function formula you have to dig into your memory bank and retrieve fundamental properties about certain types of functions.

## Exercises

1. What would cause a discontinuity on a rational function (a polynomial divided by another polynomial)?
2. If  $y$  is a function of  $u$ , defined by  $y = \ln(u)$ , what is always true about the **argument** of the function,  $u$ , over intervals where the function is continuous?
3. Name three values of  $\theta$  where the function  $\tan(\theta)$  is discontinuous.
4. What is the domain of the function  $k$ , where  $k(t) = \sqrt{t-4}$ ?
5. What is the domain of the function  $g$ , where  $g(t) = \sqrt[3]{t-4}$ ?

## 2.13 Piecewise-Defined Functions

Piecewise-defined functions are functions where the formula used depends upon the value of the input. When looking for discontinuities on piecewise-defined functions, you need to investigate the behavior at values where the formula changes as well as values where the issues discussed in Activity 2.12 might pop up.

## Exercises

This question is all about the function  $f$  defined by

$$f(x) = \begin{cases} \frac{4}{5-x} & x < 1 \\ \frac{x-3}{x-3} & 1 < x < 4 \\ 2x + 1 & 4 \leq x \leq 7 \\ \frac{15}{8-x} & x > 7. \end{cases}$$

1. Complete Table 2.13.1.

**Table 2.13.1:** Function values and limit values for  $f$

$a$	$f(a)$	$\lim_{x \rightarrow a^-} f(x)$	$\lim_{x \rightarrow a^+} f(x)$	$\lim_{x \rightarrow a} f(x)$
1				
3				
4				
5				
7				
8				

2. Complete Table 2.13.2.

**Table 2.13.2:** Discontinuity analysis for  $f$

Location of discontinuity	From Definition 2.9.1, sub-property (1, 2, 3) that is not met	Removable?	Continuous from one side?

Consider the function  $f$  defined by  $f(x) = \begin{cases} \frac{5}{x-10} & x \leq 5 \\ \frac{5}{5x-30} & 5 < x < 7 \\ \frac{x-2}{12-x} & x > 7. \end{cases}$

State the values of  $x$  where each of the following occur. If a stated property doesn't occur, make sure that you state that (as opposed to simply not responding to the question). No explanation necessary.

- At what values of  $x$  is  $f$  discontinuous?
- At what values of  $x$  is  $f$  continuous from the left, but not from the right?
- At what values of  $x$  is  $f$  continuous from the right, but not from the left?
- At what values of  $x$  does  $f$  have removable discontinuities?

Consider the function  $g$  defined by  $g(x) = \begin{cases} \frac{C}{x-17} & x < 10 \\ C + 3x & x = 10 \\ 2C - 4 & x > 10. \end{cases}$

The symbol  $C$  represents the same real number in each of the piecewise formulas.

7. Find the value for  $C$  that makes the function continuous on  $(-\infty, 10]$ .  
Make sure that your reasoning is clear.
  
  
  
  
  
  
  
  
  
  
8. Is it possible to find a value for  $C$  that makes the function continuous over  $(-\infty, \infty)$ ? Explain.



## 2.14 Supplement

### Exercises

For Exercises 1–5, state the limit suggested by the values in the table and state whether or not the limit exists. The correct answer for Exercise 1 has been given to help you understand the instructions.

	$t$	$g(t)$
1.	−10,000	99.97
	−100,000	999.997
	−1,000,000	9999.9997

**Answer:**  $\lim_{t \rightarrow -\infty} g(t) = \infty$ .

The limit does *not* exist.

	$x$	$f(x)$
2.	51,000	$-3.2 \times 10^{-5}$
	510,000	$-3.02 \times 10^{-7}$
	5,100,000	$-3.002 \times 10^{-9}$

	$t$	$z(t)$
3.	0.33	0.66
	0.333	0.666
	0.3333	0.6666

	$\theta$	$g(\theta)$
4.	−0.9	2,999,990
	−0.99	2,999,999
	−0.999	2,999,999.9

	$t$	$T(t)$
5.	0.778	29,990
	0.7778	299,999
	0.77778	2,999,999.9

6. Sketch onto Figure 2.14.1 a function,  $f$ , with the following properties. Your graph should include all of the features addressed in lab.

$$\begin{aligned}\lim_{x \rightarrow -4^-} f(x) &= 1 \\ \lim_{x \rightarrow -4^+} f(x) &= -2 \\ \lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow \infty} f(x) = -\infty\end{aligned}$$

The only discontinuities on  $f$  are at  $-4$  and  $3$ .  $f$  has no  $x$ -intercepts.  $f$  is continuous from the right at  $-4$ .  $f$  has constant slope  $-2$  over  $(-\infty, -4)$ .

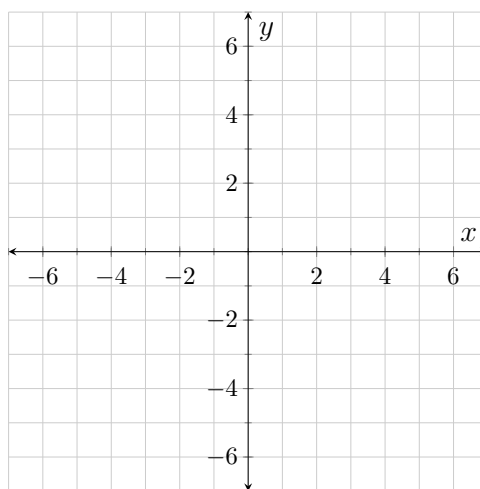


Figure 2.14.1:  $y = f(x)$

7. Sketch onto Figure 2.14.2 a function,  $f$ , with each of the properties stated below. Assume that there are no intercepts or discontinuities other than those directly implied by the given properties. Make sure that your graph includes all of the relevant features addressed in lab.

$$\begin{aligned}f(-2) &= 0 \\ f(0) &= -1 \\ f(-4) &= 5 \\ \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow \infty} f(x) = 3 \\ \lim_{x \rightarrow -4^+} f(x) &= -2 \\ \lim_{x \rightarrow -4^-} f(x) &= 5 \\ \lim_{x \rightarrow 3^-} f(x) &= -\infty \\ \lim_{x \rightarrow 3^+} f(x) &= \infty\end{aligned}$$

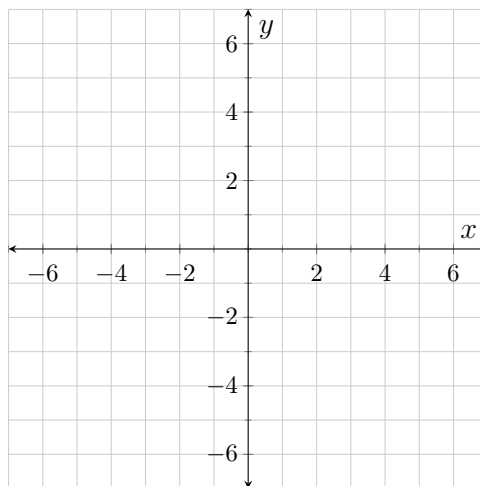


Figure 2.14.2:  $y = f(x)$

8. Determine all of the values of  $x$  where the function  $f$  (given below) has discontinuities. At each value where  $f$  has a discontinuity, determine if  $f$  is continuous from either the right or left at  $x$  and also state whether or not the

discontinuity is removable.

$$f(x) = \begin{cases} \frac{2\pi}{x} & x \leq 3 \\ \sin\left(\frac{2\pi}{x}\right) & 3 < x < 4 \\ \frac{\sin(x)}{\sin(x)} & 4 < x \leq 7 \\ 3 - \frac{2x-8}{x-4} & x > 7 \end{cases}$$

9. Determine the value(s) of  $k$  that make(s) the function  $g$  defined by  $g(t) = \begin{cases} t^2 + kt - k & t \geq 3 \\ t^2 - 4k & t < 3 \end{cases}$  continuous over  $(-\infty, \infty)$ .

Determine the appropriate symbol to write after an equal sign following each of the given limits. In each case, the appropriate symbol is either a real number,  $\infty$ , or  $-\infty$ . Also, state whether or not each limit exists and if the limit exists prove its existence (and value) by applying the appropriate limit laws. The Rational Limit Forms table in Appendix A summarizes strategies to be employed based upon the initial form of the limit.

10.  $\lim_{x \rightarrow 4^-} \left(5 - \frac{1}{x-4}\right)$

11.  $\lim_{x \rightarrow \infty} \frac{e^{2/x}}{e^{1/x}}$

12.  $\lim_{x \rightarrow 2^+} \frac{x^2 - 4}{x^2 + 4}$

13.  $\lim_{x \rightarrow 2^+} \frac{x^2 - 4}{x^2 - 4x + 4}$

14.  $\lim_{x \rightarrow \infty} \frac{\ln(x) + \ln(x^6)}{7 \ln(x^2)}$

15.  $\lim_{x \rightarrow -\infty} \frac{3x^3 + 2x}{3x - 2x^3}$

16.  $\lim_{x \rightarrow \infty} \sin\left(\frac{\pi e^{3x}}{2e^x + 4e^{3x}}\right)$

17.  $\lim_{x \rightarrow \infty} \frac{\ln(1/x)}{\ln(x/x)}$

18.  $\lim_{x \rightarrow 5} \sqrt{\frac{x^2 - 12x + 35}{5 - x}}$

19.  $\lim_{h \rightarrow 0} \frac{4(3+h)^2 - 5(3+h) - 21}{h}$

20.  $\lim_{h \rightarrow 0} \frac{5h^2 + 3}{2 - 3h^2}$

21.  $\lim_{h \rightarrow 0} \frac{\sqrt{9-h} - 3}{h}$

22.  $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin(\theta + \frac{\pi}{2})}{\sin(2\theta + \pi)}$

23.  $\lim_{x \rightarrow 0^+} \frac{\ln(x^e)}{\ln(e^x)}$

Draw sketches of the curves  $y = e^x$ ,  $y = e^{-x}$ ,  $y = \ln(x)$ , and  $y = \frac{1}{x}$ . Note the coordinates of any and all intercepts. This is something you should be able to do from memory/intuition. If you cannot already do so, spend some time reviewing whatever you need to review so that you can do so. Once you have the graphs drawn, fill in each of the blanks below and decide whether or not each limit exists. The appropriate symbol for some of the blanks is either  $\infty$  or  $-\infty$ .

24.  $\lim_{x \rightarrow \infty} e^x = \underline{\hspace{1cm}}$  (exists?)

26.  $\lim_{x \rightarrow 0} e^x = \underline{\hspace{1cm}}$  (exists?)

28.  $\lim_{x \rightarrow -\infty} e^{-x} = \underline{\hspace{1cm}}$  (exists?)

30.  $\lim_{x \rightarrow \infty} \ln(x) = \underline{\hspace{1cm}}$  (exists?)

32.  $\lim_{x \rightarrow 0^+} \ln(x) = \underline{\hspace{1cm}}$  (exists?)

34.  $\lim_{x \rightarrow -\infty} \frac{1}{x} = \underline{\hspace{1cm}}$  (exists?)

36.  $\lim_{x \rightarrow 0^-} \frac{1}{x} = \underline{\hspace{1cm}}$  (exists?)

38.  $\lim_{x \rightarrow \infty} \frac{1}{e^x} = \underline{\hspace{1cm}}$  (exists?)

40.  $\lim_{x \rightarrow \infty} \frac{1}{e^{-x}} = \underline{\hspace{1cm}}$  (exists?)

25.  $\lim_{x \rightarrow -\infty} e^x = \underline{\hspace{1cm}}$  (exists?)

27.  $\lim_{x \rightarrow \infty} e^{-x} = \underline{\hspace{1cm}}$  (exists?)

29.  $\lim_{x \rightarrow 0} e^{-x} = \underline{\hspace{1cm}}$  (exists?)

31.  $\lim_{x \rightarrow 1} \ln(x) = \underline{\hspace{1cm}}$  (exists?)

33.  $\lim_{x \rightarrow \infty} \frac{1}{x} = \underline{\hspace{1cm}}$  (exists?)

35.  $\lim_{x \rightarrow 0^+} \frac{1}{x} = \underline{\hspace{1cm}}$  (exists?)

37.  $\lim_{x \rightarrow \infty} e^{1/x} = \underline{\hspace{1cm}}$  (exists?)

39.  $\lim_{x \rightarrow -\infty} \frac{1}{e^x} = \underline{\hspace{1cm}}$  (exists?)

41.  $\lim_{x \rightarrow -\infty} \frac{1}{e^{-x}} = \underline{\hspace{1cm}}$  (exists?)

# Lab 3

## Introduction to the First Derivative

### 3.1 Instantaneous Velocity

Most of the focus in Lab 1 was on average rates of change. While the idea of rates of change at one specific instant was alluded to, we couldn't explore that idea formally because we hadn't yet talked about limits. Now that we have covered average rates of change and limits we can put those two ideas together to discuss rates of change at specific instances in time.

Suppose that an object is tossed into the air in such a way that the elevation of the object (measured in ft) is given by the function  $s$  defined by  $s(t) = 40 + 40t - 16t^2$  where  $t$  is the amount of time that has passed since the object was tossed (measured in s). Let's determine the velocity of the object 2 seconds into this flight.

Recall that the difference quotient  $\frac{s(2+h)-s(2)}{h}$  gives us the average velocity for the object between the times  $t = 2$  and  $t = 2 + h$ . So long as  $h$  is positive, we can think of  $h$  as the length of the time interval. To infer the velocity exactly 2 seconds into the flight we need the time interval as close to 0 as possible; this is done using the appropriate limit in Example 3.1.1.

**Example 3.1.1.**

$$\begin{aligned} v(2) &= \lim_{h \rightarrow 0} \frac{s(2+h) - s(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[40 + 40(2+h) - 16(2+h)^2] - [40 + 40(2) - 16(2)^2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{40 + 80 + 40h - 64 - 64h - 16h^2 - 56}{h} \\ &= \lim_{h \rightarrow 0} \frac{-24h - 16h^2}{h} \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{h(-24 - 16h)}{h} \\
&= \lim_{h \rightarrow 0} (-24 - 16h) \\
&= -24 - 16 \cdot 0 \\
&= -24
\end{aligned}$$

From this we can infer that the velocity of the object 2 seconds into its flight is  $-24 \frac{\text{ft}}{\text{s}}$ . From that we know that the object is falling at a speed of  $24 \frac{\text{ft}}{\text{s}}$ .

## Exercises

Suppose that an object is tossed into the air so that its elevation  $p$  (measured in m) is given by  $p(t) = 300 + 10t - 4.9t^2$ , where  $t$  is the amount of time that has passed since the object was tossed (measured in s).

$t_1$	$\frac{p(t_1) - p(4)}{t_1 - 4}$
3.9	
3.99	
3.999	
4.001	
4.01	
4.1	

**Table 3.1.2:** Average Velocities

1. Evaluate  $\lim_{h \rightarrow 0} \frac{p(4+h) - p(4)}{h}$  showing each step in the simplification process (as illustrated in Example 3.1.1).
2. What is the unit for the value calculated in Exercise 1 and what does the value (including unit) tell you about the motion of the object?
3. Copy Table 3.1.2 onto your paper and compute and record the missing values. Do these values support your answer to Exercise 2?

## 3.2 Tangent Lines

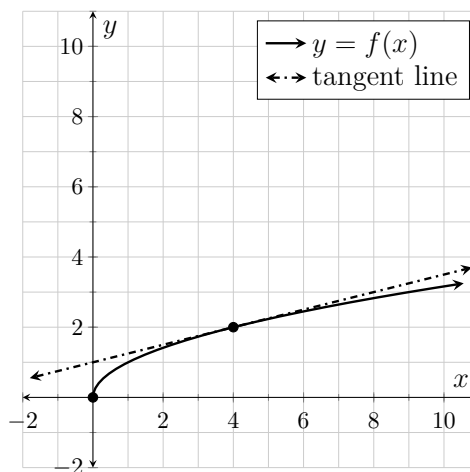
In previous activities we saw that if  $p$  is a position function, then the difference quotient for  $p$  can be used to calculate average velocities and the expression  $\frac{p(t_0+h) - p(t_0)}{h}$  calculates the instantaneous velocity at time  $t_0$ .

Graphically, the difference quotient of a function  $f$  can be used to calculate the slope of secant lines to  $f$ . What happens when we take the run of the secant

line to zero? Basically, we are connecting two points on the line that are really, really, (*really*), close to one another. As mentioned above, sending  $h$  to zero turns an average velocity into an instantaneous velocity. Graphically, sending  $h$  to zero turns a secant line into a **tangent line**.

**Example 3.2.1.** The tangent line at  $x = 4$  to the function  $f$  defined by  $f(x) = \sqrt{x}$  is shown in Figure 3.2.2. Here is a calculation of the slope of this line.

$$\begin{aligned}
 m_{\text{tan}} &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\
 &= \lim_{h \rightarrow 0} \left( \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \right) \\
 &= \lim_{h \rightarrow 0} \frac{4 + h - 4}{h(\sqrt{4+h} + 2)} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{4+h} + 2)} \\
 &= \lim_{h \rightarrow 0} \frac{1}{1(\sqrt{4+h} + 2)} \\
 &= \frac{1}{\sqrt{4+0} + 2} \\
 &= \frac{1}{4}
 \end{aligned}$$

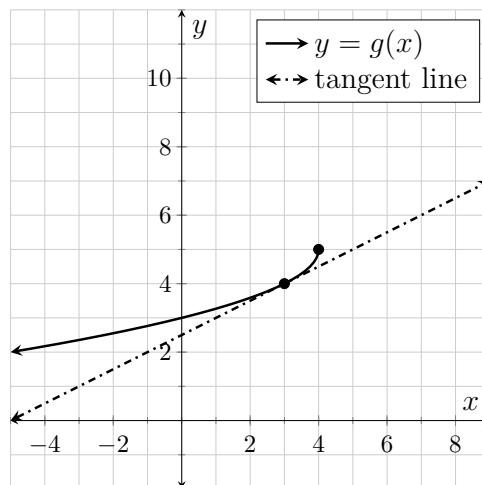


**Figure 3.2.2:**  $y = f(x)$  and its tangent line at  $x = 4$

You should verify that the slope of the tangent line shown in Figure 3.2.2 is indeed  $\frac{1}{4}$ . You should also verify that the equation of the tangent line is  $y = \frac{1}{4}x + 1$ .

## Exercises

Consider the function  $g$  given by  $g(x) = 5 - \sqrt{4 - x}$  graphed in Figure 3.2.3.



**Figure 3.2.3:**  $y = g(x)$  and its tangent line at  $x = 3$

1. Find the slope of the tangent line shown in Figure 3.2.3 using

$$m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{g(3+h) - g(3)}{h}.$$

Show work consistent with that illustrated in Example 3.2.1.

2. Use the line in Figure 3.2.3 to verify your answer to Exercise 1.
3. State the equation of the tangent line to  $g$  at  $x = 3$ .

## 3.3 The First Derivative

We have seen two applications of expressions of the form  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ . It turns out that this expression is so important in mathematics, the sciences, economics, and many other fields that it deserves a name in and of its own right. We call the expression **the first derivative of  $f$  at  $a$** .

So far we've always fixed the value of  $a$  before making the calculation. There's no reason why we couldn't use a variable for  $a$ , make the calculation, and then replace the variable with specific values; in fact, it seems like this might be a



better plan all around. This leads us to a definition of **the first derivative function**.

**Definition 3.3.1** (The First Derivative Function). If  $f$  is a function of  $x$ , then we define **the first derivative function**,  $f'$ , as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The symbols  $f'(x)$  are read aloud as “ $f$  prime of  $x$ ” or “ $f$  prime at  $x$ .”

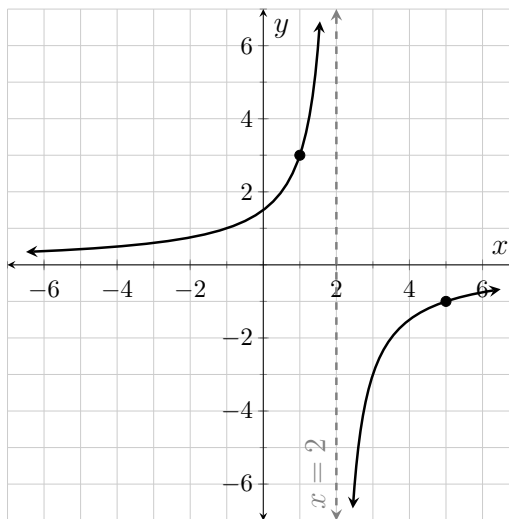
As we’ve already seen,  $f'(a)$  gives us the slope of the tangent line to  $f$  at  $a$ .

We’ve also seen that if  $s$  is a position function, then  $s'(a)$  gives us the instantaneous velocity at  $a$ . It’s not too much of a stretch to infer that the velocity function for  $s$  would be  $v(t) = s'(t)$ .

Let’s find a first derivative.

**Example 3.3.2.** Let  $f(x) = \frac{3}{2-x}$ . We can find  $f'(x)$  as follows.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3}{2-(x+h)} - \frac{3}{2-x}}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{\frac{3}{2-x-h} - \frac{3}{2-x}}{h} \cdot \frac{(2-x-h)(2-x)}{(2-x-h)(2-x)} \right) \\ &= \lim_{h \rightarrow 0} \frac{3(2-x) - 3(2-x-h)}{h(2-x-h)(2-x)} \\ &= \lim_{h \rightarrow 0} \frac{6 - 3x - 6 + 3x + 3h}{h(2-x-h)(2-x)} \\ &= \lim_{h \rightarrow 0} \frac{3h}{h(2-x-h)(2-x)} \\ &= \lim_{h \rightarrow 0} \frac{3}{(2-x-h)(2-x)} \\ &= \frac{3}{(2-x-0)(2-x)} \\ &= \frac{3}{(2-x)^2} \end{aligned}$$



**Figure 3.3.3:**  $y = f(x) = \frac{3}{2-x}$

## Exercises

A graph of the function  $f$  defined by  $f(x) = \frac{3}{2-x}$  is shown in Figure 3.3.3 and the formula for  $f'(x)$  is derived in Example 3.3.2.

1. Use the formula  $f'(x) = \frac{3}{(2-x)^2}$  to calculate  $f'(1)$  and  $f'(5)$ .
2. Draw onto Figure 3.3.3 a line through the point  $(1, 3)$  with a slope of  $f'(1)$ . Also draw a line through the point  $(5, -1)$  with a slope of  $f'(5)$ . What are the names for the two lines you just drew? What are their equations?
3. Showing work consistent with that shown in Example 3.3.2, find the formula for  $g'(x)$  where  $g(x) = \frac{5}{2x+1}$ .

Suppose that the elevation of an object (measured in ft) is given by  $s(t) = -16t^2 + 112t + 5$  where  $t$  is the amount of time that has passed since the object was launched into the air (measured in s).

4. Use the equations below to find the formula for the velocity function associated with this motion. The first two lines of your presentation should be an exact copy of the two lines below.

$$\begin{aligned} v(t) &= s'(t) \\ &= \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h} \end{aligned}$$

5. Find the values of  $v(2)$  and  $v(5)$ . What is the unit on each of these values? What do the values tell you about the motion of the object? Don't just say "the velocity"—describe what is actually happening to the object 2 seconds and 5 seconds into its travel.
6. Use the velocity function to determine when the object reaches its maximum elevation. (Think about what must be true about the velocity at that instant.) Also, what is the common mathematical term for the point on the parabola  $y = s(t)$  that occurs at that value of  $t$ ?
7. Use (3.3.1) to find the formula for  $v'(t)$ . The first line of your presentation should be an exact copy of (3.3.1).

$$v'(t) = \lim_{h \rightarrow 0} \frac{v(t+h) - v(t)}{h} \tag{3.3.1}$$

8. What is the common name for the function  $v'(t)$ ? Is its formula consistent with what you know about objects in free fall on Earth?
9. What is the constant slope of the function  $w$  defined by  $w(x) = 12$ ? Verify this by using Definition 3.3.1 to find the formula for  $w'(x)$ .

### 3.4 Derivative Units

We can think about the instantaneous velocity as being *the instantaneous rate of change* in position. In general, whenever you see the phrase “rate of change” you can assume that the rate of change at one instant is being discussed. When we want to discuss average rates of change over a time interval we always say “average rate of change.”

In general, if  $f$  is any function, then  $f'(a)$  *tells us the rate of change in  $f$  at  $a$* . Additionally, if  $f$  is an applied function with an input unit of  $i_{\text{unit}}$  and an output unit of  $f_{\text{unit}}$ , then the unit on  $f'(a)$  is  $\frac{f_{\text{unit}}}{i_{\text{unit}}}$ . Please note that this unit loses all meaning if it is simplified in any way. Consequently, *we do not simplify derivative units in any way, shape, or form*.

For example, if  $v(t)$  is the velocity of your car (measured in  $\frac{\text{mi}}{\text{h}}$ ) where  $t$  is the amount of time that has passed since you hit the road (measured in minutes), then the unit on  $v'(t)$  is  $\frac{\text{mi/h}}{\text{min}}$ .

### Exercises

Determine the unit for the first derivative function for each of the following functions. Remember, *we do not simplify derivative units in any way, shape, or form*.

1.  $V(r)$  is the volume of a sphere (measured in mL) with radius  $r$  (measured in mm).
2.  $A(x)$  is the area of a square (measured in  $\text{ft}^2$ ) with sides of length  $x$  (measured in ft).
3.  $V(t)$  is the volume of water in a bathtub (measured in gal) where  $t$  is the amount of time that has elapsed since the bathtub began to drain (measured in minutes).
4.  $R(t)$  is the rate at which a bathtub is draining (measured in  $\frac{\text{gal}}{\text{min}}$ ) where  $t$  is the amount of time that has elapsed since the bathtub began to drain (measured in minutes).

5. Akbar was given a formula for the function described in Exercise 3. Akbar did some calculations and decided that the value of  $V'(2)$  was (without unit) 1.5. Nguyen took one look at Akbar's value and said "that's wrong" What is it about Akbar's value that caused Nguyen to dismiss it as wrong?

6. After a while Nguyen convinced Akbar that he was wrong, so Akbar set about doing the calculation over again. This time Akbar came up with a value of  $-12528$ . Nguyen took one look at Akbar's value and declared "still wrong." What's the problem now?

Consider the function described in Exercise 4.

7. What would it mean if the value of  $R(t)$  was zero for all  $t > 0$ ?

8. What would it mean if the value of  $R'(t)$  was zero for all  $t$  in  $(0, 2.25)$ ?

## 3.5 Supplement

### Exercises

Find the first derivative formula for each of the following functions twice: first by evaluating  $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$  and then by evaluating  $\lim_{t \rightarrow x} \frac{f(t)-f(x)}{t-x}$ .

1.  $f(x) = x^2$

2.  $f(x) = \sqrt{x}$

3.  $f(x) = 7$

4. It can be shown that  $\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$  and  $\lim_{h \rightarrow 0} \frac{\cos(h)-1}{h} = 0$ . Use these limits to help you to establish the first derivative formula for  $\sin(x)$ .

Suppose that an object is tossed into the air in such a way that the elevation of the object (measured in ft)  $t$  seconds after the object was tossed is given by the function  $s(t) = 150 + 60t - 16t^2$ .

5. Find the velocity function for this motion and use it to determine the velocity of the object 4.1 s into its motion.

6. Find the acceleration function for this motion and use that function to determine the acceleration of the object 4.1 s into its motion.

Determine the unit on the first derivative function for each of the following functions. Remember, we do not simplify derivative units in any way, shape, or form.

7.  $R(p)$  is Carl's heart rate (beats per min) when he jogs at a rate of  $p$  (measured in  $\frac{\text{ft}}{\text{min}}$ ).

8.  $F(v)$  is the fuel consumption rate  $\frac{\text{gal}}{\text{mi}}$  of Hanh's pick-up when she drives it on level ground at a constant speed of  $v$  (measured in  $\frac{\text{mi}}{\text{h}}$ ).

9.  $v(t)$  is the velocity of the space shuttle  $\frac{\text{mi}}{\text{h}}$  where  $t$  is the amount of time that has passed since liftoff (measured in s).

10.  $h(t)$  is the elevation of the space shuttle (mi) where  $t$  is the amount of time that has passed since lift-off (measured in s).

Referring to the functions in Exercises 7–10, write sentences that explain the meaning of each of the following function values.

11.  $R(300) = 84$

12.  $R'(300) = 0.02$

13.  $F(50) = 0.03$

14.  $F'(50) = -0.0006$

15.  $v(20) = 266$

16.  $v'(20) = 18.9$

17.  $h(20) = 0.7$

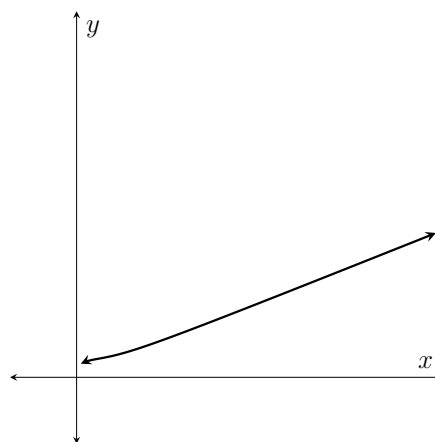
18.  $h'(20) = 0.074$

It can be shown that the derivative formula for the function  $f$  defined by  $f(x) = \frac{2x^3+2x+1}{3+5x^2}$  is  $f'(x) = \frac{10x^4+8x^2-10x+6}{25x^4+30x^2+9}$ .

19. Determine the equation of the tangent line to  $f$  at 1.

20. A graph of  $f$  is shown in Figure 3.5.1; axis scales have deliberately been omitted from the graph.

The graph shows that  $f$  quickly resembles a line. In a detailed sketch of  $f$  we would reflect this apparent linear behavior by adding a skew asymptote. What is the slope of this skew asymptote?



**Figure 3.5.1:**

$$y = f(x) = \frac{2x^3+2x+1}{3+5x^2}$$

# Lab 4

## Functions, Derivatives, and Antiderivatives

### 4.1 Graph Features

Functions, derivatives, and antiderivatives have many entangled properties. For example, over intervals where the first derivative of a function is always positive, we know that the function itself is always increasing. (Do you understand why?) Many of these relationships can be expressed graphically. Consequently, it is imperative that you fully understand the meaning of some commonly used graphical expressions. These expressions are loosely defined in Definition 4.1.1.

**Definition 4.1.1** (Some Common Graphical Phrases). These are loose, informal definitions. Technical, precise definitions can be found elsewhere.

**Positive function.** The vertical coordinates of points on the function are positive. A function is positive whenever its graph lies above the horizontal axis.

**Negative function.** The vertical coordinates of points on the function are negative. A function is negative whenever its graph lies below the horizontal axis.

**Increasing function.** The vertical coordinates of the function increase as you move along the curve from left to right. Linear functions with positive slope are always increasing.

**Decreasing function.** The vertical coordinates of the function decrease as you move along the curve from left to right. Linear functions with negative slope are always decreasing.

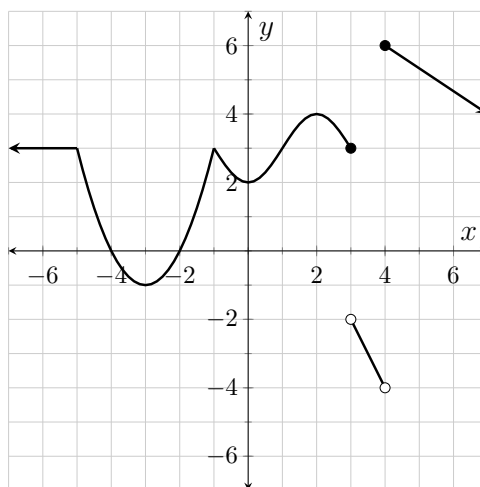
**Concave up function.** A function is concave up at  $a$  if the tangent line to the function at  $a$  lies below the curve. The bottom half of a circle is concave up.

**Concave down function.** A function is concave down at  $a$  if the tangent line to the function at  $a$  lies above the curve. The top half of a circle is concave down.

## Exercises

Answer each of the following questions in reference to the function shown in Figure 4.1.2. Each answer is an interval (or intervals) along the  $x$ -axis.

Use interval notation when expressing your answers. Make each interval as wide as possible; that is, do not break an interval into pieces if the interval does not need to be broken. Assume that the slope of the function is constant on  $(-\infty, -5)$ ,  $(3, 4)$ , and  $(4, \infty)$ .



**Figure 4.1.2:**  $y = f(x)$

1. Over what intervals is the function positive?
2. Over what intervals is the function negative?
3. Over what intervals is the function increasing?
4. Over what intervals is the function decreasing?
5. Over what intervals is the function concave up?
6. Over what intervals is the function concave down?
7. Over what intervals is the function linear?
8. Over what intervals is the function constant?



## 4.2 Graphical Derivatives

Much information about a function’s first derivative can be gleaned simply by looking at a graph of the function. In fact, a person with good visual skills can “see” the graph of the derivative while looking at the graph of the function. This activity focuses on helping you develop that skill.

### Exercises

A parabolic function is shown in Figure 4.2.1. The next several questions are in reference to that function.

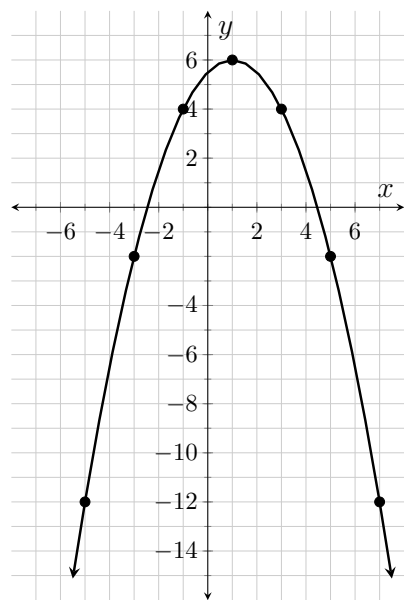


Figure 4.2.1:  $y = g(x)$

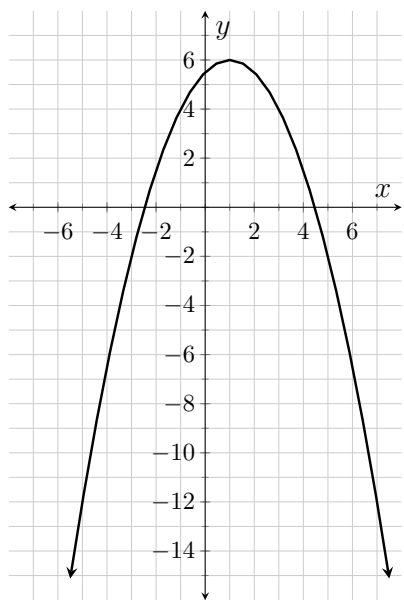


Figure 4.2.2:  $y = g(x)$

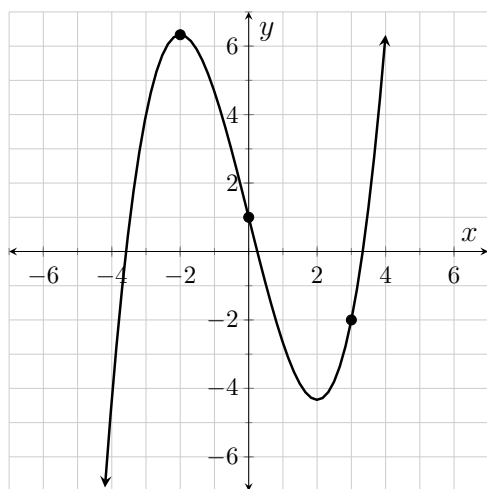
$x$	$y$
-5	6
-3	
-1	2
1	
3	-2
5	-4
7	

Table 4.2.3:  
 $y = g'(x)$

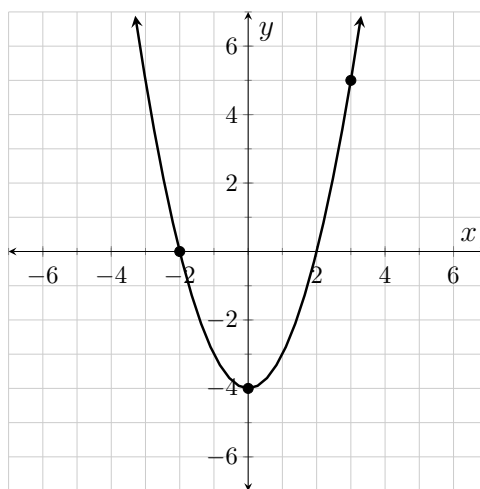
1. Several values of  $g'$  are given in Table 4.2.3. For each given value of  $x$  draw onto Figure 4.2.1 a nice long line segment at the corresponding point on  $g$  whose slope is equal to the value of  $g'$ . If we think of these line segments as actual lines, what do we call the lines?
2. What is the value of  $g'(1)$ ? How do you know? Enter that value into Table 4.2.3.

3. The function  $g$  is symmetric across the line  $x = 1$ ; that is, if we move equal distance to the left and right from this line the corresponding  $y$ -coordinates on  $g$  are always equal. Notice that the slopes of the tangent lines are “equal but opposite” at points that are equally removed from the axis of symmetry; this is reflected in the values of  $g'(-1)$  and  $g'(3)$ . Use the idea of “equal but opposite slope equidistant from the axis of symmetry” to complete Table 4.2.3.
4. Plot the points from Table 4.2.3 onto Figure 4.2.2 and connect the dots. Determine the formula for the resultant linear function.
5. The formula for  $g(x)$  is  $-0.5x^2 + x + 5.5$ . Use Definition 3.3.1 to determine the formula for  $g'(x)$ .
6. The line you drew onto Figure 4.2.2 is *not* a tangent line to  $g$ . Just what exactly is this line?

A function  $f$  is shown in Figure 4.2.4 and the corresponding first derivative function  $f'$  is shown in Figure 4.2.5. Answer each of the following questions referencing these two functions.



**Figure 4.2.4:**  $y = f(x)$



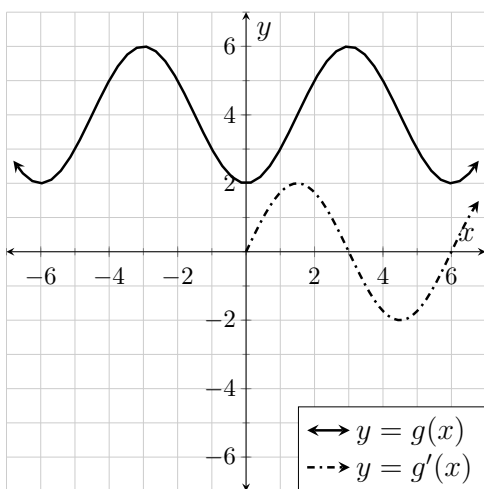
**Figure 4.2.5:**  $y = f'(x)$

7. Draw the tangent line to  $f$  at the three points indicated in Figure 4.2.4 after first using the graph of  $f'$  to determine the exact slope of the respective tangent lines.
8. Write a sentence relating the slope of the tangent line to  $f$  with the corresponding  $y$ -coordinate on  $f'$ .

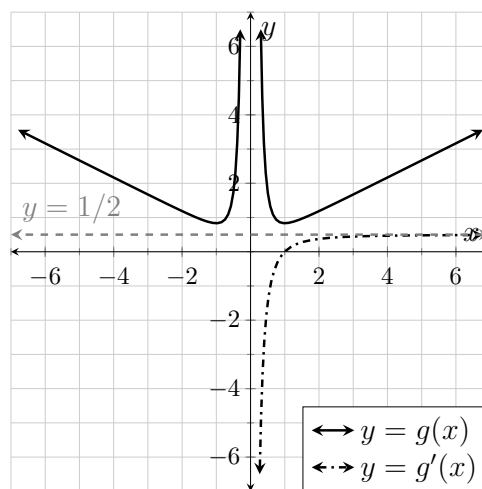
9. Copy each of the following sentences onto your paper and supply the words or phrases that correctly complete each sentence.

- Over the interval where  $f'$  is negative,  $f$  is ...
- Over the interval where  $f'$  is positive,  $f$  is ...
- Over the interval where  $f'$  is increasing,  $f$  is ...
- Over the interval where  $f'$  is decreasing,  $f$  is ...

10. In each of Figures 4.2.6 and 4.2.7 a function (the solid curve) is given; both of these functions are symmetric about the  $y$ -axis. The first derivative of each function (the dash-dotted curves) have been drawn over the interval  $(0, 7)$ . Use the given portion of the first derivative together with the symmetry of the function to help you draw each first derivative over the interval  $(-7, 0)$ . Please note that a function that is symmetric across a vertical line has a first derivative that is symmetric about a point. Similarly, a function that is symmetric about a point has a first derivative that is symmetric across a vertical line.

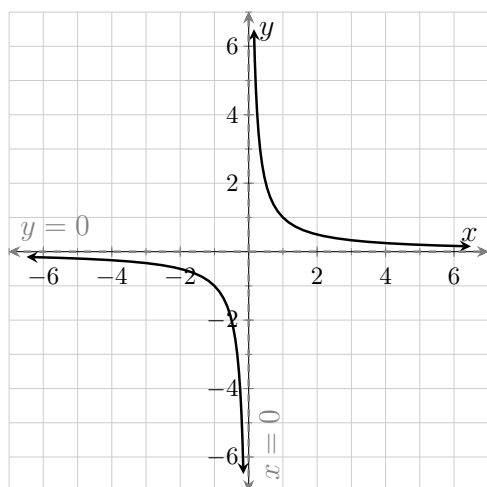


**Figure 4.2.6:**  $y = f(x)$  and part of the curve  $y = f'(x)$

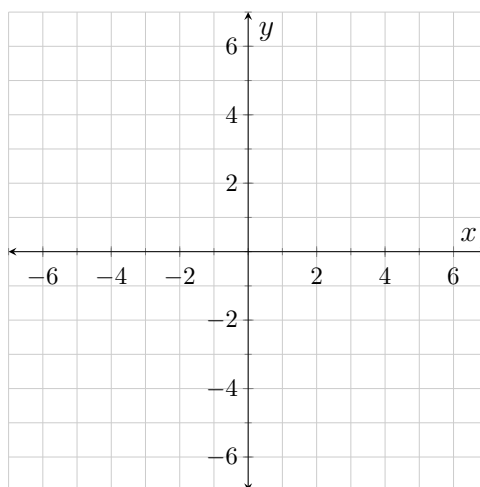


**Figure 4.2.7:**  $y = f(x)$  and part of the curve  $y = f'(x)$

A graph of the function  $h$  given by  $h(x) = \frac{1}{x}$  is shown in Figure 4.2.8.

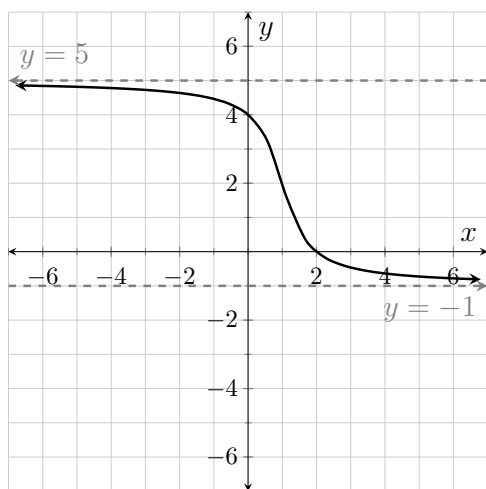
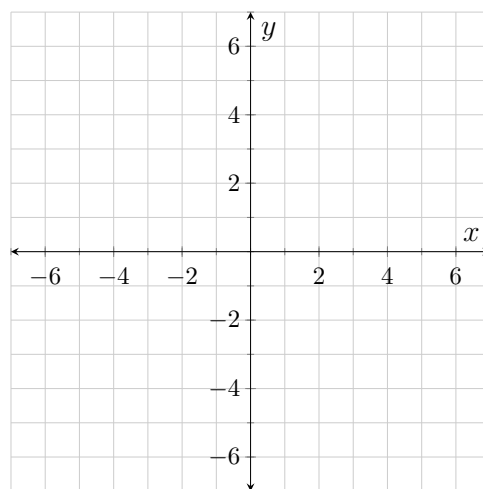


**Figure 4.2.8:**  $y = h(x) = \frac{1}{x}$

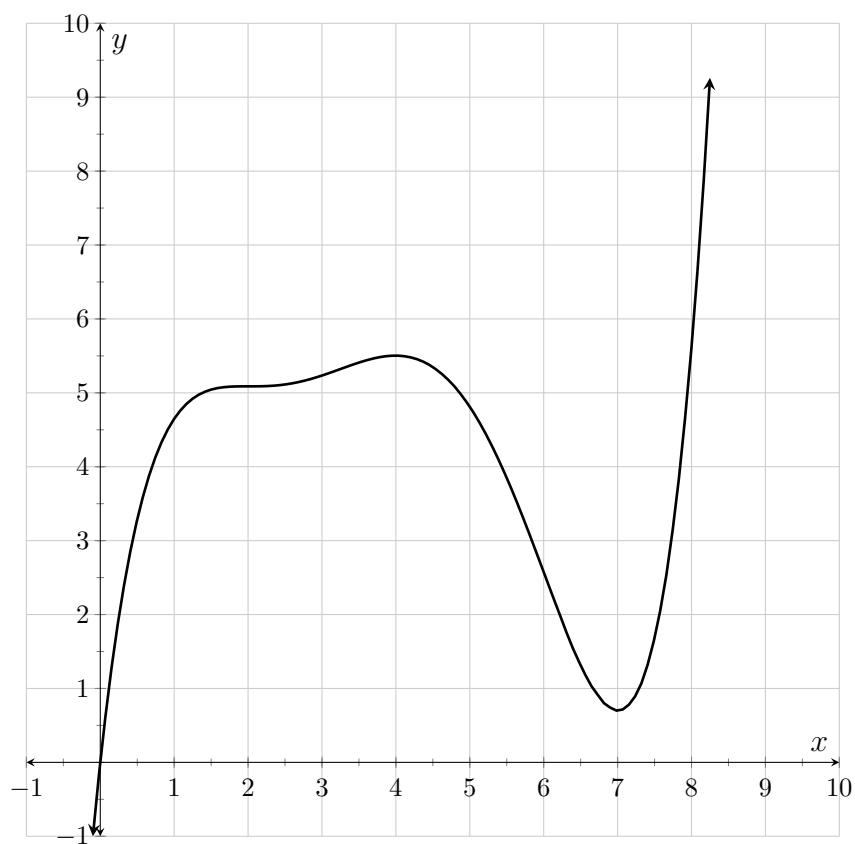


**Figure 4.2.9:**  $y = h'(x)$

11. Except at 0, there is something that is always true about the value of  $h'$ ; what is the common trait?
12. Use Definition 3.3.1 to find the formula for  $h'(x)$ .
13. Does the formula for  $h'(x)$  support your answer to Exercise 11?
14. Use the formula for  $h'(x)$  to determine the horizontal and vertical asymptotes to the graph of  $y = h'(x)$ .
15. Keeping it simple, draw onto Figure 4.2.9 a curve with the asymptotes found in Exercise 14 and the property determined in Exercise 11. Does the curve you drew have the properties you would expect to see in the first derivative of  $h$ ? For example,  $h$  is concave down over  $(-\infty, 0)$ , and concave up over  $(0, \infty)$ ; what are the corresponding differences in the behavior of  $h'$  over those two intervals?
16. A graph of the function  $g$  is shown in Figure 4.2.10. The absolute minimum value ever obtained by  $g'$  is  $-3$ . With that in mind, draw  $g'$  onto Figure 4.2.11. Make sure that you draw and label any and all necessary asymptotes. Please note that  $g$  is symmetric about the point  $(1, 2)$ .

**Figure 4.2.10:**  $y = g(x)$ **Figure 4.2.11:**  $y = g'(x)$ 

Answer the following questions in reference to the function  $w$ , shown in Figure 4.2.12.

**Figure 4.2.12:**  $y = w(x)$

17. An **inflection point** is a point where the function is continuous and the concavity of the function changes.

The inflection points on  $w$  occur at 2, 3.25, and 6. With that in mind, over each interval stated in Table 4.2.13 exactly two of the properties from its caption apply to  $w'$ . Complete Table 4.2.13 with the appropriate pairs of properties.

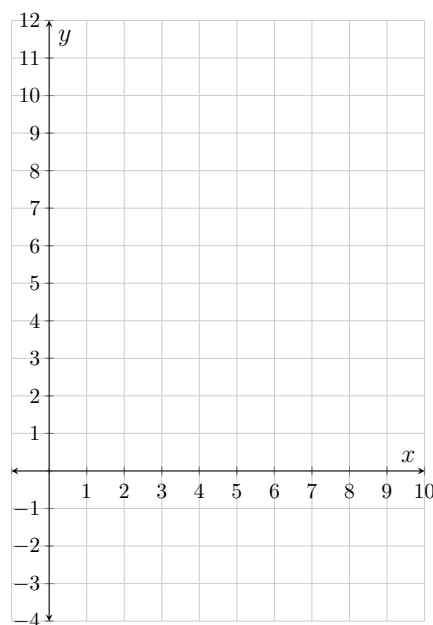
Interval	Properties of $w'$
$(-\infty, 2)$	
$(2, 3.25)$	
$(3.25, 4)$	
$(4, 6)$	
$(6, 7)$	
$(7, \infty)$	

**Table 4.2.13:** Properties of  $w'$  (positive, negative, increasing, or decreasing)

18. In Table 4.2.14, three possible values are given for  $w'$  at several values of  $x$ . In each case, one of the values is correct. Use tangent lines to  $w$  to determine each of the correct values.

$x$	Proposed values
0	$2/3$ or $8/3$ or $28/3$
1	$1/2$ or $3/2$ or $5/2$
3	$1/3$ or 1 or 3
5	$-1/2$ or $-1$ or $-3/2$
6	$-4/3$ or $-8/3$ or $-4$
8	1 or 6 or 12

**Table 4.2.14:** Choose correct values for  $w'$



**Figure 4.2.15:**  $y = w'(x)$

19. The value of  $w'$  is the same at 2, 4, and 7. What is this common value?
20. Put it all together and draw  $w'$  onto Figure 4.2.15.

## 4.3 Nondifferentiability

A function is said to be nondifferentiable at any value where its first derivative is undefined. There are three graphical behaviors that lead to non-differentiability.

- $f$  is nondifferentiable at  $a$  if  $f$  is discontinuous at  $a$ .
- $f$  is nondifferentiable at  $a$  if the slope of  $f$  is different from the left and right at  $a$ .
- $f$  is nondifferentiable at  $a$  if  $f$  has a vertical tangent line at  $a$ .

### Exercises

Consider the function  $k$  shown in Figure 4.3.1. Please note that  $k$  has a vertical tangent line at  $-4$ .

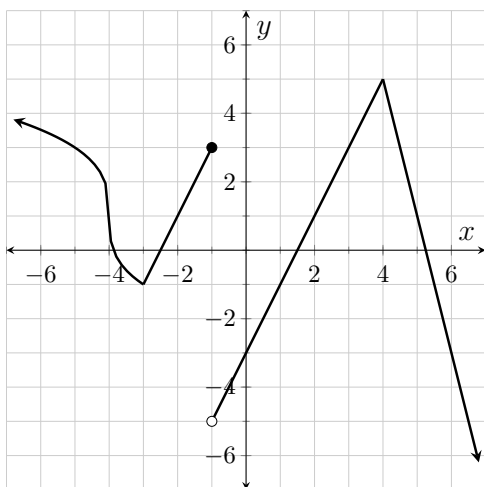


Figure 4.3.1:  $y = k(x)$

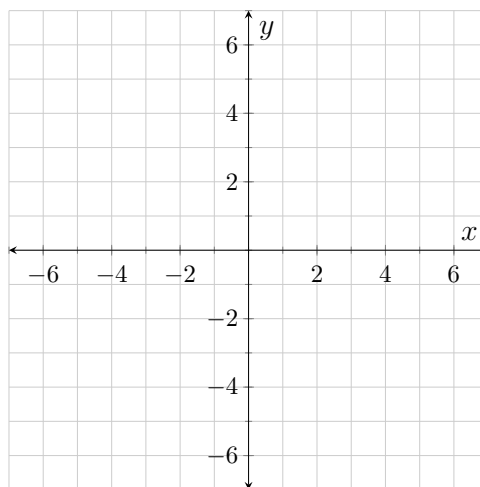
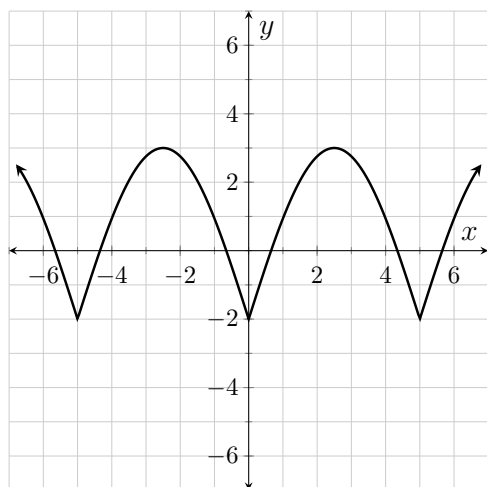


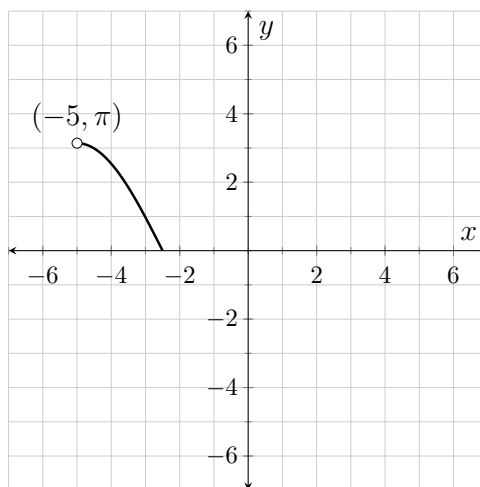
Figure 4.3.2:  $y = k'(x)$

1. There are four values where  $k$  is nondifferentiable; what are these values?
2. Draw  $k'$  onto Figure 4.3.2.

Consider the function  $g$  shown in Figure 4.3.3.



**Figure 4.3.3:**  $y = g(x)$



**Figure 4.3.4:**  $y = g'(x)$

3.  $g'$  has been drawn onto Figure 4.3.4 over the interval  $(-5, -2.5)$ . Use the piecewise symmetry and periodic behavior of  $g$  to help you draw the remainder of  $g'$  over  $(-7, 7)$ .
4. What six-syllable word applies to  $g$  at  $-5$ ,  $0$ , and  $5$ ?
5. What five-syllable and six-syllable words apply to  $g'$  at  $-5$ ,  $0$ , and  $5$ ?

## 4.4 Higher Order Derivatives

Seeing as the first derivative of  $f$  is a function in its own right,  $f'$  must have its own first derivative. The first derivative of  $f'$  is the **second derivative** of  $f$  and is symbolized as  $f''$  ( $f$  double-prime). Likewise,  $f'''$  ( $f$  triple-prime) is the first derivative of  $f''$ , the second derivative of  $f'$ , and the **third derivative** of  $f$ .



All of the graphical relationships you’ve established between  $f$  and  $f'$  work their way down the derivative chain; this is illustrated in Table 4.4.1.

When $f'$ is ...	$f$ is ...
When $f''$ is ...	$f'$ is ...
When $f'''$ is ...	$f''$ is ...
Positive	Increasing
Negative	Decreasing
Constantly Zero	Constant
Increasing	Concave Up
Decreasing	Concave Down
Constant	Linear

**Table 4.4.1:** Relationships between  $f$ ,  $f'$ ,  $f''$  and  $f'''$

Exercises

- 1. Extrapolating from Table 4.4.1, what must be true about  $f$  over intervals where  $f''$  is, respectively: positive, negative, or constantly zero?
- 2. A function,  $g$ , and its first three derivatives are shown in Figures 4.4.2–4.4.5, although not in that order. Determine which curve is which function ( $g, g', g'', g'''$ ).

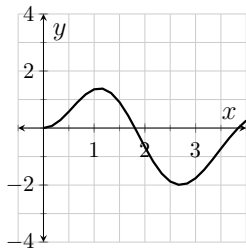


Figure 4.4.2:

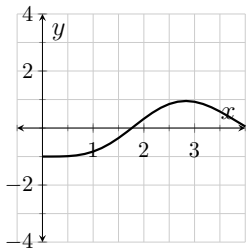


Figure 4.4.3:

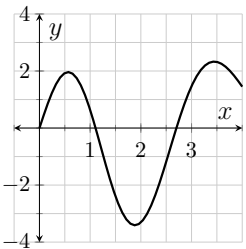


Figure 4.4.4:

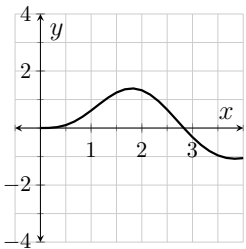


Figure 4.4.5:

Three containers are shown in Figures 4.4.6–4.4.8. Each of the following questions are in reference to these containers.

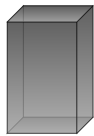


Figure 4.4.6:

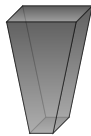


Figure 4.4.7:

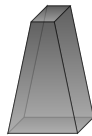


Figure 4.4.8:

3. Suppose that water is being poured into each of the containers at a constant rate. Let  $h_a$ ,  $h_b$ , and  $h_c$  be the heights (measured in cm) of the liquid in containers 4.4.6–4.4.8 respectively,  $t$  seconds after the water began to fill the containers. What would you expect the sign to be on the second derivative functions  $h_a''$ ,  $h_b''$ , and  $h_c''$  while the containers are being filled? (Hint: Think about the shape of the curves  $y = h_a(t)$ ,  $y = h_b(t)$ , and  $y = h_c(t)$ .)
4. Suppose that water is being drained from each of the containers at a constant rate. Let  $h_a$ ,  $h_b$ , and  $h_c$  be the heights (measured in cm) of the liquid remaining in the containers  $t$  seconds after the water began to drain. What would you expect the sign to be on the second derivative functions  $h_a''$ ,  $h_b''$ , and  $h_c''$  while the containers are being drained?
5. During the recession of 2008–2009, the total number of employed Americans decreased every month. One month a talking head on the television made the observation that “at least the second derivative was positive this month.” Why was it a good thing that the second derivative was positive?
6. During the early 1980s the problem was inflation. Every month the average price for a gallon of milk was higher than the month before. Was it a good thing when the second derivative of this function was positive? Explain.

## 4.5 Antiderivatives

The derivative continuum can be expressed backwards as well as forwards. When you move from function to function in the reverse direction the resultant functions are called **antiderivatives** and the process is called **antidifferentiation**. These relationships are shown in Figures 4.5.1 and 4.5.2.

$$f \xrightarrow{\text{differentiate}} f' \xrightarrow{\text{differentiate}} f'' \xrightarrow{\text{differentiate}} f''' \xrightarrow{\text{differentiate}} \dots$$

**Figure 4.5.1:** Differentiating

$$\dots f''' \xrightarrow{\text{antidifferentiate}} f'' \xrightarrow{\text{antidifferentiate}} f' \xrightarrow{\text{antidifferentiate}} f \xrightarrow{\text{antidifferentiate}} F$$

**Figure 4.5.2:** Antidifferentiating

There are (at least) two important differences between the differentiation chain and the antidifferentiation chain (besides their reversed order).

- When you differentiate, the resultant function is unique. When you antidifferentiate, you do not get a unique function—you get a family of functions; specifically, you get a set of parallel curves.
- We introduce a new function in the antidifferentiation chain. We say that  $F$  is an antiderivative of  $f$ . This is where we stop in that direction; we do not have a variable name for an antiderivative of  $F$ .

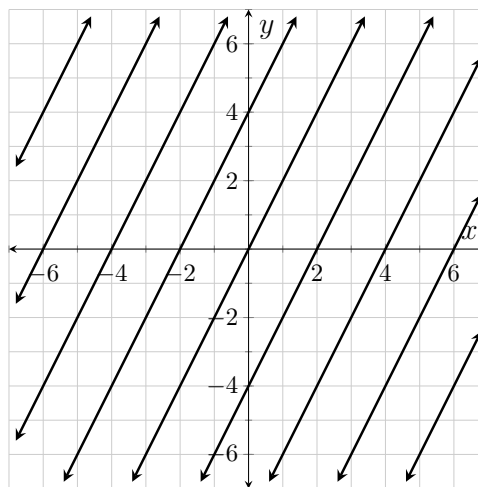
Since  $F$  is considered an antiderivative of  $f$ , it must be the case that  $f$  is the first derivative of  $F$ . Hence we can add  $F$  to our derivative chain resulting in Figure 4.5.3.

$$F \xrightarrow{\text{differentiate}} f \xrightarrow{\text{differentiate}} f' \xrightarrow{\text{differentiate}} f'' \xrightarrow{\text{differentiate}} f''' \xrightarrow{\text{differentiate}} \dots$$

**Figure 4.5.3:** Differentiating

## Exercises

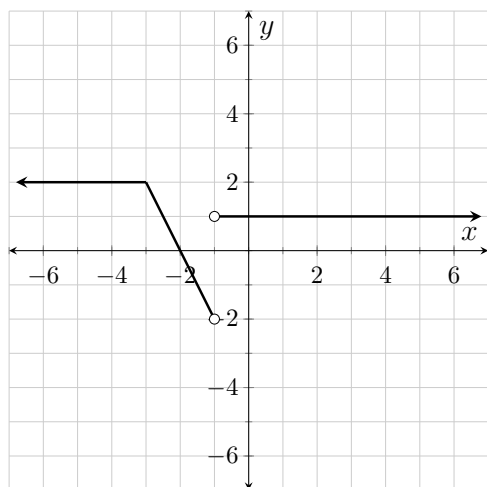
Each of the linear functions in Figure 4.5.4 has the same first derivative function.



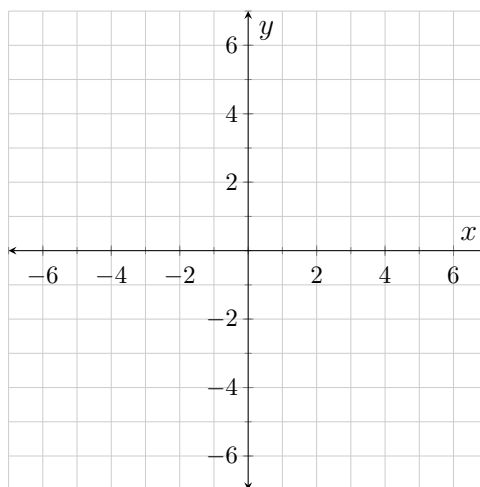
**Figure 4.5.4:**

1. Draw this common first derivative function onto Figure 4.5.4 and label it  $g$ .
2. Each of the given lines Figure 4.5.4 is called what in relation to  $g$ ?

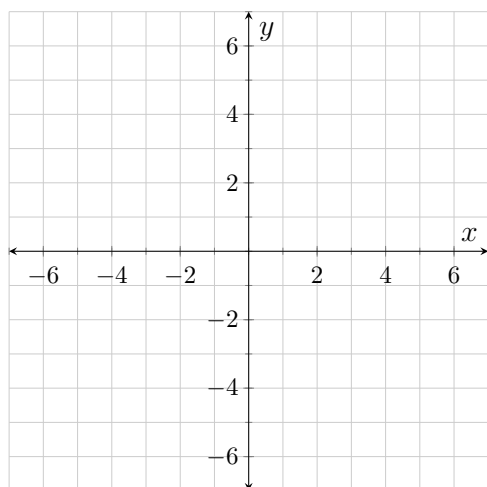
The function  $f$  is shown in Figure 4.5.5. Reference this function in the following questions.



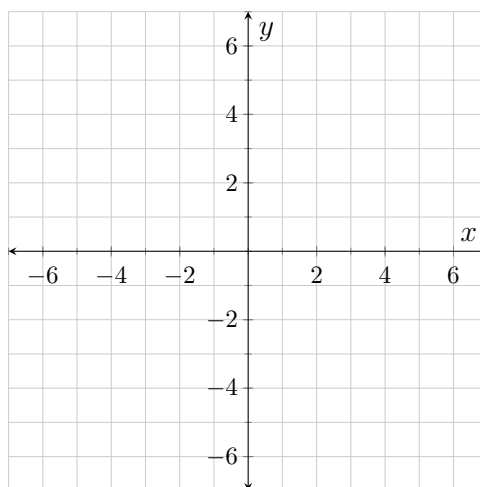
**Figure 4.5.5:**  $y = f(x)$



**Figure 4.5.6:**



**Figure 4.5.7:**



**Figure 4.5.8:**

3. At what values of  $x$  is  $f$  nondifferentiable?
4. At what values of  $x$  are antiderivatives of  $f$  nondifferentiable?
5. Draw onto Figure 4.5.6 the continuous antiderivative of  $f$  that passes through the point  $(-3, 1)$ . Note that every antiderivative of  $f$  increases exactly one unit over the interval  $(-3, -2)$ .

6. Because  $f$  is not continuous, there are other antiderivatives of  $f$  that pass through the point  $(-3, 1)$ . Specifically, antiderivatives of  $f$  may or may not be continuous at  $-1$ . Draw onto Figures 4.5.7 and 4.5.8 different antiderivatives of  $f$  that pass through the point  $(-3, 1)$ .

7. Recall that a function,  $f$ , is periodic if there exists a constant,  $C$ , with the property that  $f(x + C) = f(x)$  for all values of  $x$  in the domain of  $f$ . The function  $y = \sin(x)$  is an example of a periodic function. Specifically, the function has a period of  $2\pi$  because over any interval of length  $2\pi$  the behavior of the function is exactly the same as it was the previous interval of length  $2\pi$ . A little more precisely,  $\sin(x + 2\pi) = \sin(x)$  regardless of the value of  $x$ .

Jasmine was thinking and told her lab assistant that derivatives and antiderivatives of periodic functions must also be periodic. Jasmine's lab assistant told her that she was half right. Which half did Jasmine have correct—the part about the derivatives necessarily being periodic or the part about antiderivatives necessarily being periodic? Also, draw a function that illustrates that the other half of Jasmine's statement is not correct.

8. Consider the function  $g$  shown in Figure 4.5.9.

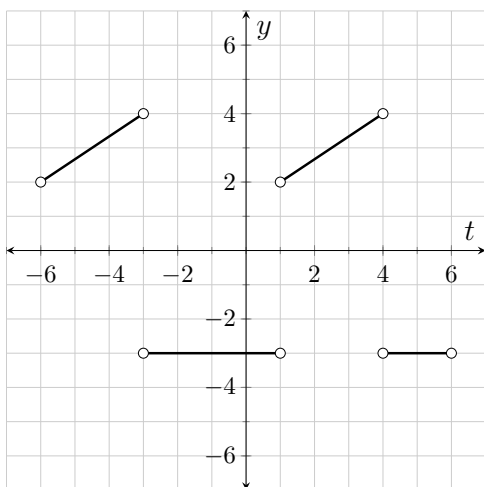


Figure 4.5.9:  $y = g(t)$

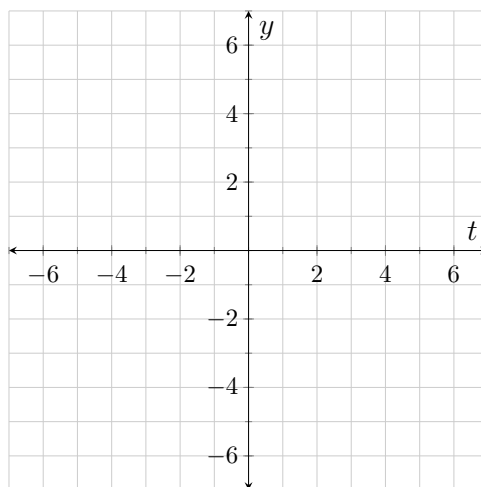
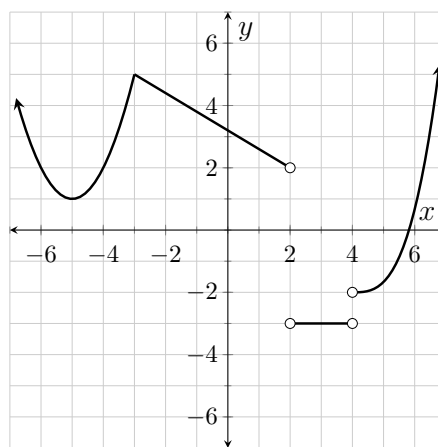


Figure 4.5.10:  $y = G(t)$

Let  $G$  be an antiderivative of  $g$ . Suppose that  $G$  is continuous on  $[-6, 6]$ ,  $G(-6) = -3$ , and that the greatest value  $G$  ever achieves is 6. Draw  $G$  onto Figure 4.5.10.

Answer the following questions in reference to a continuous function  $g$  whose first derivative is shown in Figure 4.5.11. You do not need to state how you made your determination; just state the interval(s) or values of  $x$  that satisfy the stated property. Note: the correct answer to one or more of these questions may be “There is no way of knowing.”



**Figure 4.5.11:**  $y = g'(x)$

9. Over what interval(s) is  $g''$  positive and increasing?
10. At what value(s) of  $x$  is  $g$  nondifferentiable?
11. Over what interval(s) is  $g$  never negative?
12. At what value(s) of  $x$  is every antiderivative of  $g$  nondifferentiable?
13. Over what interval(s) does  $g''$  have a constant value?
14. Over what interval(s) is  $g$  linear?
15. Over what interval(s) are antiderivatives of  $g$  linear?
16. Over what interval(s) is  $g'''$  never negative?

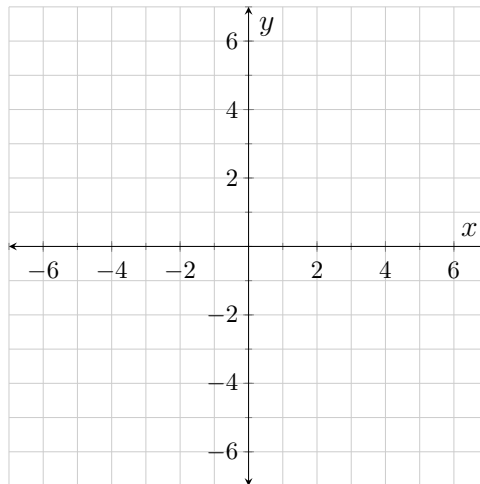
## 4.6 Graphical Features from Derivatives

When given a function formula, we often find the first and second derivative formulas to determine behaviors of the given function. In a later lab we will use the first and second derivative formulas to help us graph a function given the formula for the function. One thing we do with the derivative formulas is determine where they are positive, negative, zero, and undefined. This helps us determine where the given function is increasing, decreasing, concave up, concave down, and linear.

## Exercises

In the next few exercises, you will construct a function and then answer questions about it.

Interval	$f'$	$f''$
$(-\infty, -1)$	Positive	Negative
$(-1, \infty)$	Positive	Positive



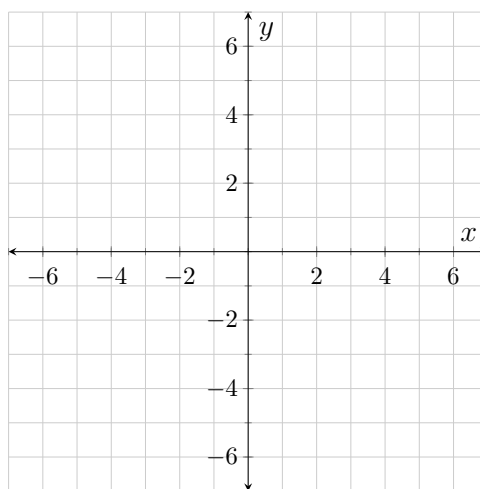
**Table 4.6.1:** Signs on  $f'$  and  $f''$

**Figure 4.6.2:**  $y = f(x)$

1. Draw onto Figure 4.6.2 a *continuous* function  $f$  that has a horizontal tangent line at the point  $(-1, 2)$  along with the properties stated in Table 4.6.1.
2. Given the conditions stated in Exercise 1, does it have to be the case that  $f''(-1) = 0$ ?
3. Is  $f$  increasing at  $-1$ ? How do you know?
4. Can a continuous, everywhere differentiable function satisfy the properties stated in Table 4.6.1 and not have a slope of zero at  $-1$ ? Draw a picture that supports your answer.

In the next few exercises, you will construct a function and then answer questions about it.

Interval	$g'$	$g''$
$(-\infty, -1)$	Positive	Positive
$(-1, \infty)$	Positive	Negative

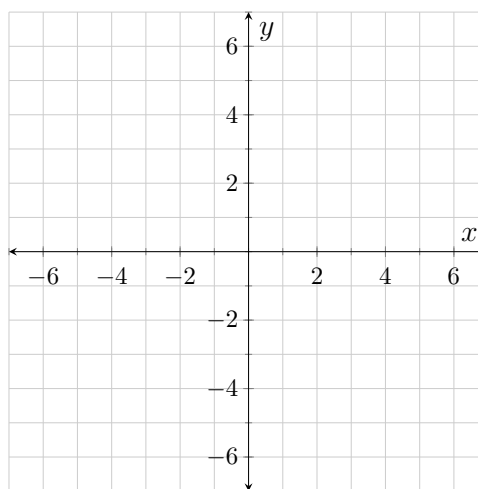
**Table 4.6.3:** Signs on  $g'$  and  $g''$ **Figure 4.6.4:**  $y = g(x)$ 

5. Draw onto Figure 4.6.4 a continuous function  $g$  that has a vertical tangent line at the point  $(-1, 2)$  along with the properties stated in Table 4.6.3.
6. Given the conditions stated in Exercise 5, does it have to be the case that  $g''(-1)$  is undefined?
7. Is  $g$  increasing at  $-1$ ? How do you know?
8. Can a continuous, everywhere differentiable function satisfy the properties stated in Table 4.6.3 and not have a vertical tangent line at  $1$ ? Draw a picture that supports your answer.

In the next few exercises, you will construct a function and then answer questions about it.



Interval	$k'$	$k''$
$(-\infty, -1)$	Negative	Positive
$(-1, \infty)$	Positive	Negative

**Table 4.6.5:** Signs on  $k'$  and  $k''$ **Figure 4.6.6:**  $y = k(x)$ 

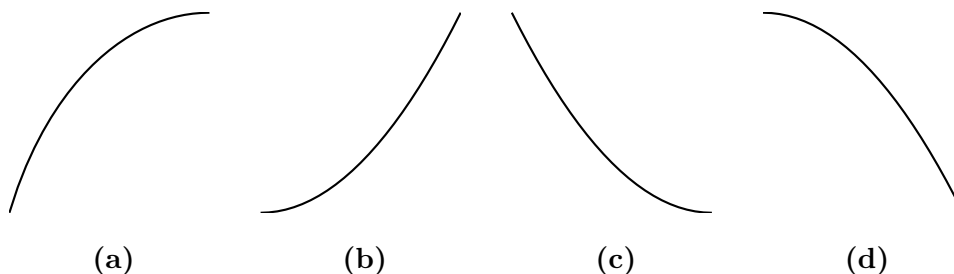
9. Draw onto Figure 4.6.6 the graph of a continuous function  $k$  that passes through the point  $(-1, 2)$  and also satisfies the properties stated in Table 4.6.5.

10. At what values of  $x$  is  $k$  nondifferentiable?

You should know by now that the first derivative continually increases over intervals where the second derivative is constantly positive and that the first derivative continually decreases over intervals where the second derivative is constantly negative. A person might infer from this that a function changes more and more quickly over intervals where the second derivative is constantly positive and that a function changes more and more slowly over intervals where the second derivative is constantly negative. We are going to explore that idea in this problem.

11. Suppose that  $V(t)$  is the volume of water in an ice cube (mL) where  $t$  is the amount of time that has passed since noon (measure in minutes). Suppose that  $V'(6) = 0 \frac{\text{mL}}{\text{min}}$  and that  $V''(t)$  has a constant value of  $-0.3 \frac{\text{mL}}{\text{min}}$  over the interval  $[6, 11]$ . What is the value of  $V'(11)$ ? When is  $V$  changing more quickly, at 12:06 pm or at 12:11 pm?

12. Referring to the function  $V$  in Exercise 11, what would the shape of  $V$  be over the interval  $[6, 11]$ ? (Choose from options 4.6.7(a)–4.6.7(d).)



**Figure 4.6.7:**

13. Again referring to options 4.6.7(a)–4.6.7(d), which functions are changing more and more rapidly from left to right and which functions are changing more and more slowly from left to right? Which functions have positive second derivatives and which functions have negative second derivatives? Do the functions with positive second derivative values both change more and more quickly from left to right?
14. Consider the signs on both the first and second derivatives in options 4.6.7(a)–4.6.7(d). Is there something that the two functions that change more and more quickly have in common that is different in the functions that change more and more slowly?

Resolve each of the following disputes.

15. One day Sara and Jermaine were working on an assignment. One question asked them to draw a function over the domain  $(-2, \infty)$  with the properties that the function is always increasing and always concave down. Sara insisted that the curve must have a vertical asymptote at  $-2$  and Jermaine insisted that the function must have a horizontal asymptote somewhere. Were either of these students correct?
16. The next question Sara and Jermaine encountered described the same function with the added condition that the function is never positive. Sara and Jermaine made the same contentions about asymptotes. Is one of them now correct?

17. At another table Pedro and Yoshi were asked to draw a continuous curve that, among other properties, was never concave up. Pedro said “OK, so the curve is always concave down” to which Yoshi replied “Pedro, you need to open your mind to other possibilities.” Who’s right?
18. In the next problem Pedro and Yoshi were asked to draw a function that is everywhere continuous and that is concave down at every value of  $x$  *except* 3. Yoshi declared “impossible” and Pedro responded “have some faith, Yosh-man.” Pedro then began to draw. Is it possible that Pedro came up with such a function?

Determine the correct answer to each of the following questions. Pictures of the situation may help you determine the correct answers.

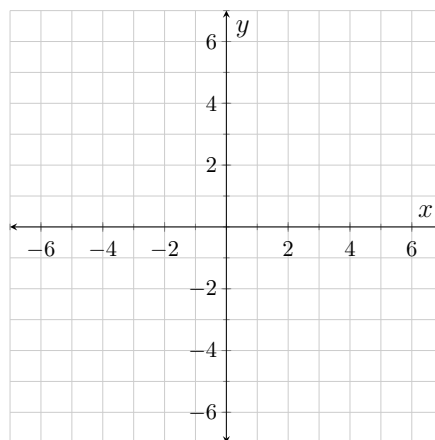
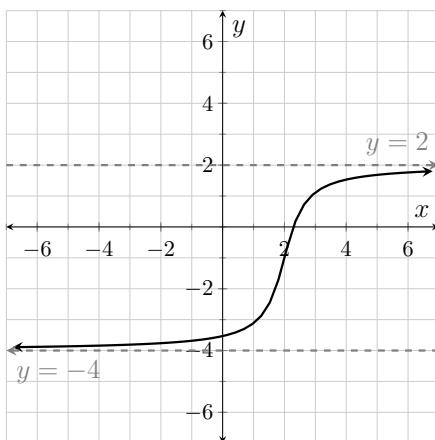
19. Which of the following propositions is true? If a given proposition is not true, draw a graph that illustrates its untruth.
- |   |   |
|---|---|
| (a) If the graph of $f$ has a vertical asymptote, then the graph of $f'$ must also have a vertical asymptote. | (b) If the graph of $f'$ has a vertical asymptote, then the graph of $f$ must also have a vertical asymptote. |
|---|---|
20. Suppose that the function  $f$  is continuous, concave down, and is such that  $f(7) = 5$  and  $f'(7) = 3$ . Which of the following is true?
- |                 |  |
|-----------------|--|
| (a) $f(9) < 11$ | (d) There is not enough information to determine the relationship between $f(9)$ and 11. |
| (b) $f(9) = 11$ |  |
| (c) $f(9) > 11$ |  |

## 4.7 Supplement

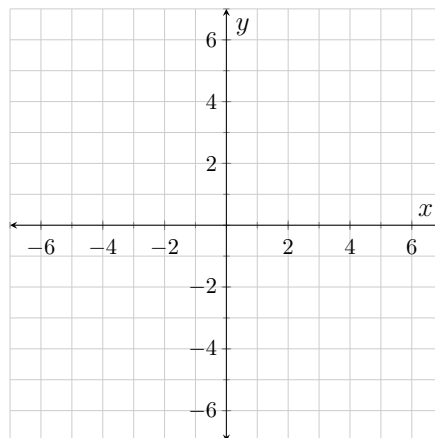
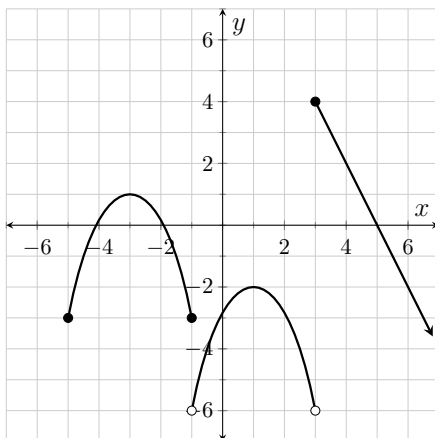
### Exercises

Sketch first derivatives of the functions shown in Exercises 1–3.

1.



2.



3.

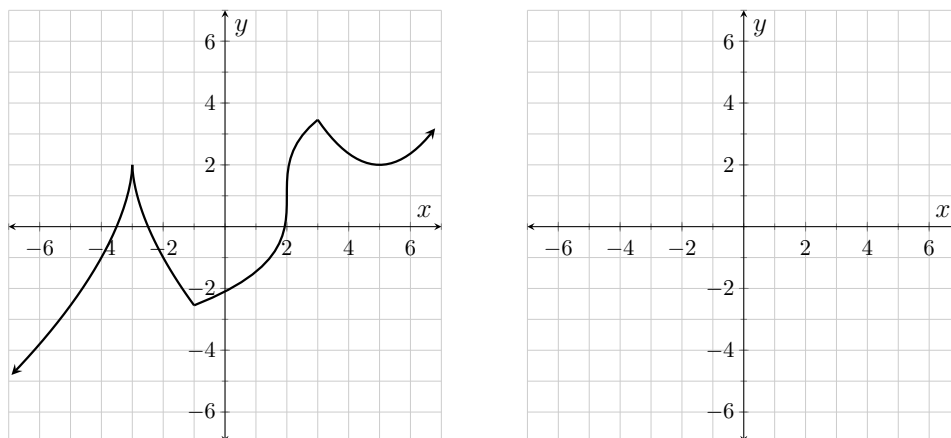
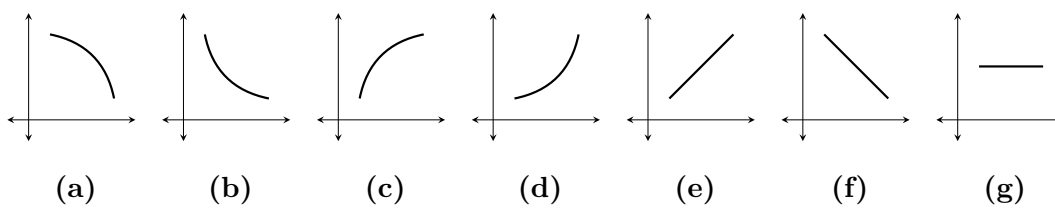


Figure 4.7.1 gives seven curves. For each statement below, identify the figure letters that go along with the given statement. Assume each statement is being made only in reference to the portion of the curves shown in each of the figures. Assume in all cases that the axes have the traditional positive/negative orientation. The first question has been answered for you to help you get started.



**Figure 4.7.1:** Seven curves

4. Each of these functions is always increasing: 4.7.1(c), 4.7.1(d), 4.7.1(e)
5. The first derivative of each of these functions is always increasing.
6. The second derivative of each of these functions is always negative.
7. The first derivative of each of these functions is always positive.
8. Any antiderivative of each of these functions is always concave up.
9. The second derivative of each of these functions is always equal to zero.
10. Any antiderivative of each of these functions graphs to a line.

11. The first derivative of each of these functions is a constant function.
12. The second derivative of each of these functions is never positive.
13. The first derivative of each of these functions *must* have a graph that is a straight line.

As water drains from a tank the drainage rate continually decreases over time. Suppose that a tank that initially holds 360 gal of water drains in exactly 6 minutes. Answer each of the following questions about this problem situation.

14. Suppose that  $V(t)$  is the amount of water (gal) left in the tank where  $t$  is the amount of time that has elapsed (s) since the drainage began. What are the units on  $V'(45)$  and  $V''(45)$ ? Which of the following is the most realistic value for  $V'(45)$ ?

(a) 0.8                      (b) 1.2                      (c)  $-0.8$                       (d)  $-1.2$

15. Suppose that  $R(t)$  is the water's flow rate ( $\frac{\text{gal}}{\text{s}}$ ) where  $t$  is the amount of time that has elapsed (s) since the drainage began. What are the units on  $R'(45)$  and  $R''(45)$ ? Which of the following is the most realistic value for  $R'(45)$ ?

(a) 1.0                      (b) 0.001                      (c)  $-1.0$                       (d)  $-0.001$

16. Suppose that  $V(t)$  is the amount of water (gal) left in the tank where  $t$  is the amount of time that has elapsed (s) since the drainage began. Which of the following is the most realistic value for  $V''(45)$ ?

(a) 1.0                      (b) 0.001                      (c)  $-1.0$                       (d)  $-0.001$

For each statement in Exercises 17–21, decide which of the following is true.

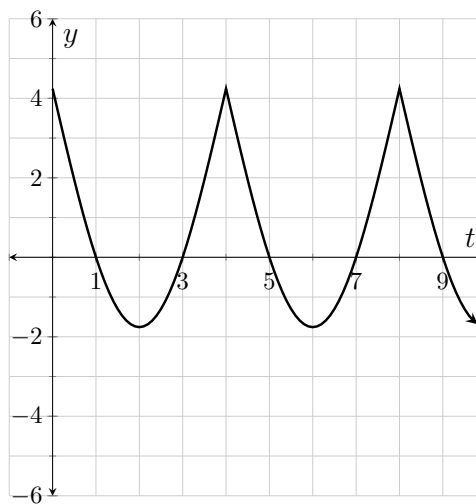
- (a) The statement is true regardless of the specific function  $f$ .
- (b) The statement is true for some functions and false for other functions.
- (c) The statement is false regardless of the specific function  $f$ .

17. If a function  $f$  is increasing over the entire interval  $(-3, 7)$  and  $f'(0)$  exists, then  $f'(0) > 0$ .
18. If a function  $f$  is increasing over the entire interval  $(-3, 7)$  and  $f'(4)$  exists, then  $f'(4) = 0$ .
19. If a function  $f$  is concave down over the entire interval  $(-3, 7)$  and  $f'(-2)$  exists, then  $f'(-2) < 0$ .
20. If the slope of  $f$  is increasing over the entire interval  $(-3, 7)$  and  $f'(0)$  exists, then  $f'(0) > 0$ .
21. If a function  $f$  has a local maximum at 3, then  $f'(3) = 0$ .

For a certain function  $g$ ,  $g'(t) = -8$  at all values of  $t$  on  $(-\infty, -3)$  and  $g'(t) = 2$  at all values of  $t$  on  $(-3, \infty)$ .

22. Janice thinks that  $g$  must be discontinuous at  $-3$  but Lisa disagrees. Who's right?
23. Lisa thinks that the graph of  $g''$  is the line  $y = 0$  but Janice disagrees. Who's right?

Each statement below is in reference to the function shown in Figure 4.7.2. Decide whether each statement is *True* or *False*. A caption has been omitted from Figure 4.7.2 because the identity of the curve changes from question to question.



**Figure 4.7.2:**

24. True or False? If the given function is  $f'$ , then  $f(3) > f(2)$ .

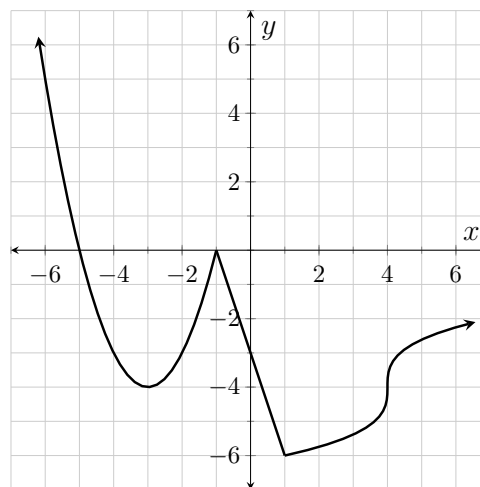
25. True or False? If the given function is  $f$ , then  $f'(3) > f'(2)$ .
26. True or False? If the given function is  $f$ , then  $f'(1) > f''(1)$ .
27. True or False? If the given function is  $f'$ , then the tangent line to  $f$  at 6 is horizontal.
28. True or False? If the given function is  $f'$ , then  $f''$  is always increasing.
29. True or False? If the given function is  $f'$ , then  $f$  is not differentiable at 4.
30. True or False? If the given function is  $f'$ , then  $f'$  is not differentiable at 4.
31. True or False? The first derivative of the given function is periodic (starting at 0).
32. True or False? Antiderivatives of the given function are periodic (starting at 0).
33. True or False? If the given function was measuring the rate at which the volume of air in your lungs was changing  $t$  seconds after you were frightened, then there was the same amount of air in your lungs 9 seconds after the fright as was there 7 seconds after the fright.
34. True or False? If the given function is measuring the position of a weight attached to a spring relative to a table edge (with positive and negative positions corresponding to the weight being above and below the edge of the table), then the weight is in the same position 9 seconds after it begins to bob as it is 7 seconds after the bobbing commenced (assuming that  $t$  represents the number of seconds that pass after the weight begins to bob).
35. True or False? If the given function is measuring the velocity of a weight attached to a spring relative to a table edge (with positive and negative positions corresponding, respectively, to the weight being above and below the edge of the table), then the spring was moving downward over the interval  $(1, 3)$  (assuming that  $t$  represents the number of seconds that pass after the weight begins to bob).
36. Complete as many cells in Table 4.7.3 as possible so that when read left to right the relationships between the derivatives, function, and antiderivative are always true. Please note that several cells will remain blank.



**Table 4.7.3:**

$f''$	$f'$	$f$	$F$
		Positive	
		Negative	
		Constant Zero	
		Increasing	
		Decreasing	
		Constant	
		Concave Up	
		Concave Down	
		Linear	

Answer each question about the function  $f$  whose first derivative is shown in Figure 4.7.4. The correct answer to one or more of the questions may be “there’s no way of knowing.”

**Figure 4.7.4:**  $y = f'(x)$ 

37. Where, over  $(-6, 6)$ , is  $f'$  nondifferentiable?
38. Where, over  $(-6, 6)$ , are antiderivatives of  $f$  nondifferentiable?
39. Where, over  $(-6, 6)$ , is  $f$  decreasing?
40. Where, over  $(-6, 6)$ , is  $f$  concave down?
41. Where, over  $(-6, 6)$ , is  $f''$  decreasing?
42. Where, over  $(-6, 6)$ , is  $f''$  positive?
43. Where, over  $(-6, 6)$ , are antiderivatives of  $f$  concave up?
44. Where, over  $(-6, 6)$ , are antiderivatives of  $f$  concave up?
45. Where, over  $(-6, 6)$ , does  $f$  have its maximum value?

46. Suppose  $f(-3) = 14$ . What is the equation of the tangent line to  $f$  at  $-3$ ?

47. A certain antiderivative,  $F$ , of the function  $f$  shown in Figure 4.7.5 passes through the points  $(3, -2)$  and about  $(-3, 5.6)$ . Draw this antiderivative (over the interval  $(-6, 6)$ ) onto the provided graph.

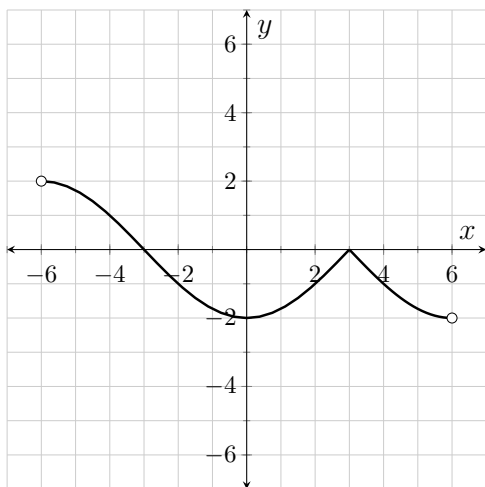


Figure 4.7.5:  $y = f(x)$

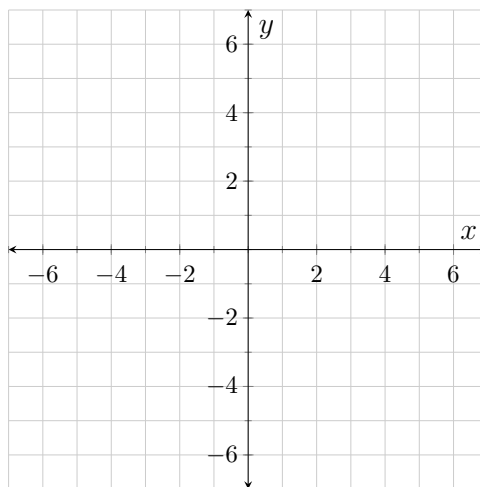


Figure 4.7.6:  $y = F(x)$

48. Draw onto Figure 4.7.7 the function  $f$  given the following properties of  $f$ .

The only discontinuity on  $f$  occurs at the vertical asymptote  $x = 2$ .

$$f(0) = f(4) = 3$$

$$f'(-1) = 0$$

$$f'(x) > 0 \text{ on } (-\infty, -1) \cup (-1, 2) \cup (4, \infty)$$

$$f'(x) < 0 \text{ on } (2, 4)$$

$$f''(x) > 0 \text{ on } (-1, 2) \cup (2, 4)$$

$$f''(x) < 0 \text{ on } (-\infty, -1)$$

$$f''(x) = 0 \text{ on } (4, \infty)$$

$$\lim_{x \rightarrow 4^-} f'(x) = -1$$

$$\lim_{x \rightarrow 4^+} f'(x) = 1$$

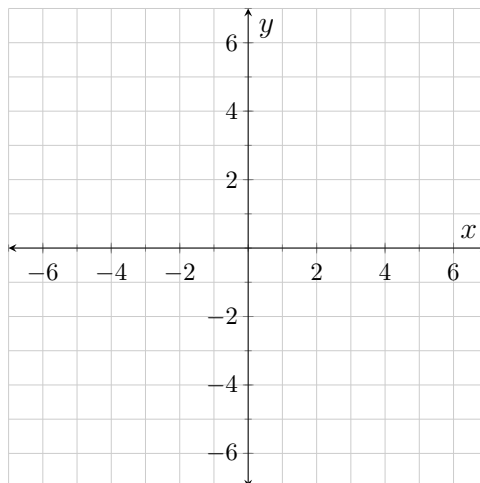


Figure 4.7.7:  $y = f(x)$

# Lab 5

## Derivative Formulas

### 5.1 Leibniz Notation

While the primary focus of this lab is to help you develop shortcut skills for finding derivative formulas, there are inevitable notational issues that must be addressed. It turns out that the latter issue is the one we are going to address first.

**Definition 5.1.1** (Leibniz Notation). If  $y = f(x)$ , we say that the derivative of  $y$  with respect to  $x$  is equal to  $f'(x)$ . Symbolically, we write  $\frac{dy}{dx} = f'(x)$ .

While the symbol  $\frac{dy}{dx}$  certainly looks like a fraction, it is *not* a fraction. The symbol is **Leibniz notation** for the first derivative of  $y$  with respect to  $x$ . The short way of reading the symbol aloud is “ $d y d x$ .”

If  $z = g(t)$ , we say that the derivative of  $z$  with respect to  $t$  is equal to  $g'(t)$ . Symbolically, we write  $\frac{dz}{dt} = g'(t)$ . (Read aloud as “ $d z d t$  equals  $g$  prime of  $t$ .”)

### Exercises

Take the derivative of both sides of each equation with respect to the independent variable as indicated in the function notation. Write and say the derivative using Leibniz notation on the left side of the equal sign and function notation on the right side of the equal sign. Make sure that every one in your group says at least one of the derivative equations aloud using both the formal reading and informal reading of the Leibniz notation.

1.  $y = k(t)$

2.  $V = f(r)$

3.  $T = g(P)$

## 5.2 The Derivative Operator

$\frac{dy}{dx}$  is the name of a derivative in the same way that  $f'(x)$  is the name of a derivative. We need a different symbol that tells us to *take* the derivative of a given expression (in the same way that we have symbols that tell us to take a square root, sine, or logarithm of an expression).

The symbol  $\frac{d}{dx}$  is used to tell us to *take the derivative with respect to  $x$  of something*. The symbol itself is an incomplete phrase in the same way that the symbol  $\sqrt{\quad}$  is an incomplete phrase; in both cases we need to indicate the object to be manipulated—what number or formula are we taking the square root of? What number or formula are we differentiating?

One thing you can do to help you remember the difference between the symbols  $\frac{dy}{dx}$  and  $\frac{d}{dx}$  is to get in the habit of always writing grouping symbols after  $\frac{d}{dx}$ . In this way the symbols  $\frac{d}{dx}(\sin(x))$  mean “the derivative with respect to  $x$  of the sine of  $x$ .” Similarly, the symbols  $\frac{d}{dt}(t^2)$  mean “the derivative with respect to  $t$  of  $t$ -squared.”

### Exercises

Write the Leibniz notation for each of the following expressions.

1. The derivative with respect to  $\beta$  of  $\cos(\beta)$ .
2. The derivative with respect to  $x$  of  $\frac{dy}{dx}$ .
3. The derivative with respect to  $t$  of  $\ln(x)$ . (Yes, we will do such things.)
4. The derivative of  $z$  with respect to  $x$ .
5. The derivative with respect to  $t$  of  $g(8)$ . (Yes, we will also do such things.)

## 5.3 The Power Rule

The first differentiation rule we are going to explore is called **the power rule of differentiation**.

$$\frac{d}{dx}(x^n) = n x^{n-1} \quad (5.3.1)$$

When  $n$  is a positive integer, it is fairly easy to establish this rule using Definition 3.3.1. The proof of the rule gets a little more complicated when  $n$  is

negative, fractional, or irrational. For purposes of this lab, we are going to just accept the rule as valid.

This rule is one you just “do in your head” and then write down the result. Three examples of what you would be expected to write when differentiating power functions are shown in Table 5.3.1.

**Table 5.3.1:** Examples of the Power Rule

Given function	$y = x^7$	$f(t) = \sqrt[3]{t^7}$	$z = \frac{1}{y^5}$
You should “see”		$f(t) = t^{7/3}$	$z = y^{-5}$
You should write	$\frac{dy}{dx} = 7x^6$	$f'(t) = \frac{7}{3}t^{4/3}$ $= \frac{7}{3}\sqrt[3]{t^4}$	$\frac{dz}{dy} = -5y^{-6}$ $= -\frac{5}{y^6}$

Notice that the type of notation used when naming the derivative is dictated by the manner in which the original function is expressed. For example,  $y = x^7$  is telling us the relationship between two variables; in this situation we name the derivative using the notation  $\frac{dy}{dx}$ . On the other hand, function notation is being used to name the rule in  $f(t) = \sqrt[3]{t^7}$ ; in this situation we name the derivative using the function notation  $f'(t)$ .

## Exercises

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure to use the appropriate name for each derivative.

$$1. f(x) = x^{43} \qquad 2. z = \frac{1}{t^7} \qquad 3. P = \sqrt[5]{t^2} \qquad 4. h(x) = \frac{1}{\sqrt{x}}$$

## 5.4 The Constant Factor Rule

The next rule you are going to practice is the **constant factor rule** of differentiation. This is another rule you do in your head.

$$\frac{d}{dx}(k f(x)) = k \cdot \frac{d}{dx}(f(x)) \quad \text{for } k \in \mathbb{R}. \quad (5.4.1)$$

In the following examples and problems several derivative rules are used that are shown in Appendix B. While working this lab you should refer to those rules; you may want to cover up the chain rule and implicit derivative columns this week! You should make a goal of having all of the basic formulas memorized within a week.

**Table 5.4.1:** Examples of the Constant Factor Rule

Given function	$f(x) = 6x^9$	$f(\theta) = 2 \cos(\theta)$	$z = 2 \ln(x)$
You should write	$f'(x) = 54x^8$	$f'(\theta) = -2 \sin(\theta)$	$\frac{dz}{dx} = \frac{2}{x}$

## Exercises

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure to use the appropriate name for each derivative.

1.  $z = 7t^4$
2.  $P(x) = -7 \sin(x)$
3.  $h(t) = \frac{1}{3} \ln(t)$
4.  $z(x) = \pi \tan(x)$
5.  $P = \frac{-8}{t^4}$
6.  $T = 4\sqrt{t}$

## 5.5 The Constant Divisor Rule

When an expression is divided by the constant  $k$ , we can think of the expression as being multiplied by the fraction  $\frac{1}{k}$ . In this way, the constant factor rule of differentiation can be applied when a formula is multiplied *or divided* by a constant.

**Table 5.5.1:** Examples of the Constant Divisor Rule

Given function	$f(x) = \frac{x^4}{8}$	$y = \frac{3 \tan(\alpha)}{2}$	$z = \frac{\ln(y)}{3}$
You should “see”	$f(x) = \frac{1}{8}x^4$	$y = \frac{3}{2} \tan(\alpha)$	$z = \frac{1}{3} \ln(y)$
You should write	$f'(x) = \frac{1}{2}x^3$	$\frac{dy}{d\alpha} = \frac{3}{2} \sec^2(\alpha)$	$\frac{dz}{dy} = \frac{1}{3y}$

## Exercises

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by

the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative. In the last problem,  $G$ ,  $m_1$ , and  $m_2$  are constants.

$$1. z(t) = \frac{\sin^{-1}(t)}{6} \qquad 2. V(r) = \frac{\pi r^3}{3} \qquad 3. f(r) = \frac{Gm_1m_2}{r^2}$$

## 5.6 The Sum and Difference Rules

When taking the derivative of two or more terms, you can take the derivatives term by term and insert plus or minus signs as appropriate. Collectively we call this the **sum and difference rules** of differentiation.

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x)) \quad (5.6.1)$$

In the following examples and problems we introduce linear and constant terms into the functions being differentiated.

$$\frac{d}{dx}(k) = 0 \quad \text{for } k \in \mathbb{R} \quad (5.6.2)$$

$$\frac{d}{dx}(kx) = k \quad \text{for } k \in \mathbb{R} \quad (5.6.3)$$

**Table 5.6.1:** Examples of the Sum and Difference Rules

Given function	$y = 4\sqrt[5]{t^6} - \frac{1}{6\sqrt{t}} + 8t$	$P(\gamma) = \frac{\sin(\gamma) - \cos(\gamma)}{2} + 4$
You should “see”	$y = 4t^{\frac{6}{5}} - \frac{1}{6}t^{-\frac{1}{2}} + 8t$	$P(\gamma) = \frac{1}{2}\sin(\gamma) - \frac{1}{2}\cos(\gamma) + 4$
You should write	$\frac{dy}{dt} = \frac{24}{5}t^{\frac{1}{5}} + \frac{1}{12}t^{-\frac{3}{2}} + 8$ $= \frac{24\sqrt[5]{t}}{5} + \frac{1}{12\sqrt{t^3}} + 8$	$P'(\gamma) = \frac{1}{2}\cos(\gamma) + \frac{1}{2}\sin(\gamma)$

### Exercises

1. Explain why both the constant rule (Equation (5.6.2)) and linear rule (Equation (5.6.3)) are “obvious.” Hint: think about the graphs  $y = k$  and  $y = kx$ . What does a first derivative tell you about a graph?

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

2.  $T = \sin(t) - 2 \cos(t) + 3$

3.  $k(\theta) = \frac{4 \sec(\theta) - 3 \csc(\theta)}{4}$

4.  $r(x) = \frac{x}{5} + 7$

5.  $r = \frac{x}{3\sqrt[3]{x}} - \frac{\ln(x)}{9} + \ln(2)$

## 5.7 The Product Rule

The next rule we are going to explore is called the **product rule** of differentiation. We use this rule when there are two or more variable factors in the expression we are differentiating. (Remember, we already have the constant factor rule to deal with two factors when one of the two factors is a constant.)

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x)) \quad (5.7.1)$$

Intuitively, what is happening in this rule is that we are alternately treating one factor as a constant (the one not being differentiated) and the other factor as a variable function (the one that is being differentiated). We then add these two rates of change together.

Ultimately, you want to perform this rule in your head just like all of the other rules. Your instructor, however, may initially want you to show steps; under that presumption, steps are going to be shown in each and every example of this lab when the product rule is applied. Two simple examples of the product rule are shown in Table 5.7.1.

**Table 5.7.1:** Examples of the Product Rule

Given function	Derivative
$y(x) = x^2 \sin(x)$	$y'(x) = \frac{d}{dx}(x^2) \cdot \sin(x) + x^2 \cdot \frac{d}{dx}(\sin(x))$ $= 2x \sin(x) + x^2 \cos(x)$
$P = e^t \cos(t)$	$\frac{dP}{dt} = \frac{d}{dt}(e^t) \cdot \cos(t) + e^t \cdot \frac{d}{dt}(\cos(t))$ $= e^t \cos(t) - e^t \sin(t)$

A decision you'll need to make is how to handle a constant factor in a term that requires the product rule. Two options for taking the derivative of  $f(x) = 5x^2 \ln(x)$  are shown in Figure 5.7.2.



Option A	Option B
$f'(x) = 5 \left[ \frac{d}{dx}(x^2) \ln(x) + x^2 \frac{d}{dx}(\ln(x)) \right]$ $= 5 \left[ 2x \ln(x) + x^2 \cdot \frac{1}{x} \right]$ $= 10x \ln(x) + 5x$	$f'(x) = \frac{d}{dx}(5x^2) \ln(x) + 5x^2 \frac{d}{dx}(\ln(x))$ $= 10x \ln(x) + 5x^2 \cdot \frac{1}{x}$ $= 10x \ln(x) + 5x$

**Figure 5.7.2:**

In Option B we are treating the factor of 5 as a part of the first variable factor. In doing so, the factor of 5 distributes itself. This is the preferred treatment of the author, so this is what you will see illustrated in this lab.

## Exercises

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

- |                                |                                 |
|--------------------------------|---------------------------------|
| 1. $T(t) = 2 \sec(t) \tan(t)$  | 2. $k = \frac{e^t \sqrt{t}}{2}$ |
| 3. $y = 4x \ln(x) + 3^x - x^3$ | 4. $f(x) = \cot(x) \cot(x) - 1$ |

Find each of the following derivatives *without first simplifying the formula*; that is, go ahead and use the product rule on the expression as written. Simplify each resultant derivative formula. For each derivative, *check* your answer by simplifying the original expression and then taking the derivative of that simplified expression.

- |                            |                                   |   |
|----------------------------|-----------------------------------|---|
| 5. $\frac{d}{dx}(x^4 x^7)$ | 6. $\frac{d}{dx}(x \cdot x^{10})$ | 7. $\frac{d}{dx}(\sqrt{x} \sqrt{x^{21}})$ |
|----------------------------|-----------------------------------|---|

## 5.8 The Quotient Rule

The next rule we are going to explore is called the **quotient rule** of differentiation. We never use this rule unless there is a variable factor in the denominator of the expression we are differentiating. (Remember, we already have the constant factor/divisor rule to deal with constant factors in the denominator.)

$$\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - f(x) \cdot \frac{d}{dx}(g(x))}{[g(x)]^2} \quad (5.8.1)$$

Like all of the other rules, you ultimately want to perform the quotient rule in your head. Your instructor, however, may initially want you to show steps when applying the quotient rule; under that presumption, steps are going to be shown in each and every example of this lab when the quotient rule is applied. Two simple examples of the quotient rule are shown in Table 5.8.1.

**Table 5.8.1:** Examples of the Quotient Rule

Given function	Derivative
$y = \frac{4x^3}{\ln(x)}$	$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{dx}(4x^3) \cdot \ln(x) - 4x^3 \cdot \frac{d}{dx}(\ln(x))}{[\ln(x)]^2} \\ &= \frac{12x^2 \cdot \ln(x) - 4x^3 \cdot \frac{1}{x}}{[\ln(x)]^2} \\ &= \frac{12x^2 \ln(x) - 4x^2}{[\ln(x)]^2} \end{aligned}$
$V = \frac{\csc(t)}{\tan(t)}$	$\begin{aligned} \frac{dV}{dt} &= \frac{\frac{d}{dt}(\csc(t)) \cdot \tan(t) - \csc(t) \cdot \frac{d}{dt}(\tan(t))}{\tan^2(t)} \\ &= \frac{-\csc(t) \cot(t) \tan(t) - \csc(t) \sec^2(t)}{\tan^2(t)} \\ &= \frac{-\csc(t) (1 + \sec^2(t))}{\tan^2(t)} \end{aligned}$

### Exercises

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

1.  $g(x) = \frac{4 \ln(x)}{x}$

2.  $j(y) = \frac{\sqrt[3]{y^5}}{\cos(y)}$

$$3. y = \frac{\sin(x)}{4 \sec(x)}$$

$$4. f(t) = \frac{t^2}{e^t}$$

Find each of the following derivatives *without first simplifying the formula*; that is, go ahead and use the quotient rule on the expression as written. For each derivative, *check* your answer by simplifying the original expression and then taking the derivative of that simplified expression.

$$5. \frac{d}{dt} \left( \frac{\sin(t)}{\sin(t)} \right)$$

$$6. \frac{d}{dx} \left( \frac{x^6}{x^2} \right)$$

$$7. \frac{d}{dx} \left( \frac{10}{2x} \right)$$

## 5.9 Simplification

In Activity 5.7 Exercises 5–7 and Activity 5.8 Exercises 5–7 you applied the product and quotient rules to expressions where the derivative could have been found much more quickly had you simplified the expression before taking the derivative. For example, while you can find the correct derivative formula using the quotient rule when working Exercise 7, the derivative can be found much more quickly if you simplify the expression before applying the rules of differentiation. This is illustrated in Example 5.9.1.

**Example 5.9.1.**

$$\begin{aligned} \frac{d}{dx} \left( \frac{10}{2x} \right) &= \frac{d}{dx} (5x^{-1}) \\ &= -5x^{-2} \\ &= -\frac{5}{x^2} \end{aligned}$$

Part of learning to take derivatives is learning to make good choices about the methodology to employ when taking derivatives. In Examples 5.9.2 and 5.9.3, the need to use the product rule or quotient rule is obviated by first simplifying the expression being differentiated.

**Example 5.9.2.**

Find  $\frac{dy}{dx}$  if  $y = \sec(x) \cos(x)$ .

$$\begin{aligned} y &= \sec(x) \cos(x) \\ &= \frac{1}{\cos(x)} \cos(x) \\ &= 1 \end{aligned}$$

$$\frac{dy}{dx} = 0$$

Note that since  $y$  is undefined for  $x = \frac{\pi}{2} + k\pi$  with  $k \in \mathbb{Z}$ ,  $\frac{dy}{dx}$  is also undefined at such  $x$ -values.

**Example 5.9.3.**

$$\begin{aligned} \text{Find } f'(t) \text{ if } f(t) &= \frac{4t^5 - 3t^3}{2t^2}. & f(t) &= \frac{4t^5 - 3t^3}{2t^2} \\ & & &= \frac{4t^5}{2t^2} - \frac{3t^3}{2t^2} \\ & & &= 2t^3 - \frac{3}{2}t \\ f'(t) &= 6t^2 - \frac{3}{2} \\ &= \frac{12t^2 - 3}{2} \end{aligned}$$

Note that since  $f$  is undefined at 0,  $f'$  is also undefined at 0.

## Exercises

Find the derivative with respect to  $x$  for each of the following functions after first completely simplifying the formula being differentiated. In each case you should *not* use either the product rule or the quotient rule while finding the derivative formula.

$$1. y = \frac{4x^{12} - 5x^4 + 3x^2}{x^4}$$

$$2. g(x) = \frac{-4 \sin(x)}{\cos(x)}$$

$$3. h(x) = \frac{4 - x^6}{3x^{-2}}$$

$$4. z(x) = \sin^2(x) + \cos^2(x)$$

$$5. z = (x + 4)(x - 4)$$

$$6. T(x) = \frac{\ln(x)}{\ln(x^2)}$$

## 5.10 Product and Quotient Rules Together

Sometimes both the product rule and quotient rule need to be applied when finding a derivative formula.

## Exercises

Consider the functions defined by

$$f(x) = x^2 \frac{\sin(x)}{e^x} \quad g(x) = \frac{x^2 \sin(x)}{e^x}.$$

1. Discuss why  $f$  and  $g$  are in fact two representations of the same function.
2. Find  $f'(x)$  by first applying the product rule and then applying the quotient rule (where necessary).
3. Find  $g'(x)$  by first applying the quotient rule and then applying the product rule (where necessary).
4. Rigorously establish that the formulas for  $f'(x)$  and  $g'(x)$  are indeed the same.

## 5.11 Derivative Formulas and Function Behavior

Derivative formulas can give us much information about the behavior of a function. For example, the derivative formula for  $f(x) = x^2$  is  $f'(x) = 2x$ . Clearly  $f'$  is negative when  $x$  is negative and  $f'$  is positive when  $x$  is positive. This tells us that  $f$  is decreasing when  $x$  is negative and that  $f$  is increasing when  $x$  is positive. This matches the behavior of the parabola  $y = x^2$ .

## Exercises

The amount of time (seconds),  $T$ , required for a pendulum to complete one period is a function of the pendulum's length (meters),  $L$ . Specifically,  $T = 2\pi\sqrt{\frac{L}{g}}$  where  $g$  is the gravitational acceleration constant for Earth (roughly  $9.8 \frac{\text{m}}{\text{s}^2}$ ).

1. Find  $\frac{dT}{dL}$  after first rewriting the formula for  $T$  as a constant times  $\sqrt{L}$ .
2. The sign on  $\frac{dT}{dL}$  is the same regardless of the value of  $L$ . What is this sign and what does it tell you about the relative periods of two pendulums with different lengths?

The gravitational force (Newtons) between two objects of masses  $m_1$  and  $m_2$  (kg) is a function of the distance (meters) between the objects' centers of mass,  $r$ . Specifically,  $F(r) = \frac{Gm_1m_2}{r^2}$  where  $G$  is the universal gravitational constant (which is approximately  $6.7 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}$ .)

3. Leaving  $G$ ,  $m_1$ , and  $m_2$  as constants, find  $F'(r)$  after first rewriting the formula for  $F$  as a constant times a power of  $r$ .
4. The sign on  $F'(r)$  is the same regardless of the value of  $r$ . What is this sign and what does it tell you about the effect on the gravitational force between two objects when the distance between the objects is changed?
5. Find each of  $F'(1.00 \times 10^{12})$ ,  $F'(1.01 \times 10^{12})$ , and  $F'(1.02 \times 10^{12})$ , leaving  $G$ ,  $m_1$ , and  $m_2$  as constants.
6. Calculate  $\frac{F(1.02 \times 10^{12}) - F(1.00 \times 10^{12})}{1.02 \times 10^{12} - 1.00 \times 10^{12}}$ . Which of the quantities found in Exercise 5 comes closest to this value? Draw a sketch of  $F$  and discuss why this result makes sense.

## 5.12 Supplement

### Exercises

Apply the constant factor and power rules to find the first derivative of each of the following functions. In each case take the derivative with respect to the variable suggested by the expression on the right side of the equation. Make sure that you assign the proper name to the derivative function. Please note that there is no need to employ either the quotient rule or the product rule to find any of these derivative formulas.

$$1. f(x) = \frac{\sqrt[11]{x^6}}{6}$$

$$2. y = -\frac{5}{t^7}$$

$$3. y(u) = 12\sqrt[3]{u}$$

$$4. z(\alpha) = e\alpha^\pi$$

$$5. z = \frac{8}{\sqrt{t^7}}$$

$$6. T = \frac{4\sqrt[3]{t^7}}{t^2}$$

Apply the constant factor and product rules to find the first derivative of each of the following functions. Write out the Leibniz notation for the product rule as applied to the given function. In each case take the derivative with respect to the variable suggested by the expression on the right side of the equation. Make sure that you assign the proper name to the derivative function.

$$7. y = 5 \sin(x) \cos(x)$$

$$8. y = \frac{5t^2 e^t}{7}$$

$$9. F(x) = 4x \ln(x)$$

$$10. z = x^2 \sin^{-1}(x)$$

$$11. T(t) = (t^2 + 1) \tan^{-1}(t)$$

$$12. T = \frac{x^7 7^x}{3}$$

Apply the constant factor and quotient rules to find the first derivative of each of the following functions. Write out the Leibniz notation for the quotient rule as applied to the given function. In each case take the derivative with respect to the variable suggested by the expression on the right side of the equation. Make sure that you assign the proper name to the derivative function.

$$13. q(\theta) = \frac{4e^\theta}{e^\theta + 1}$$

$$14. u(x) = 2 \frac{\ln(x)}{x^4}$$

$$15. F = \frac{\sqrt{t}}{3t^2 - 5\sqrt{t^3}}$$

$$16. F(x) = \frac{\tan(x)}{\tan^{-1}(x)}$$

$$17. p = \frac{\tan^{-1}(t)}{1 + t^2}$$

$$18. y = \frac{4}{\sin(\beta) - 2 \cos(\beta)}$$

Multiple applications of the product rule are required when finding the derivative formula of the product of three or more functions. After simplification, however, the predictable pattern shown below emerges.

$$\begin{aligned}\frac{d}{dx}(fg) &= \frac{d}{dx}(f)g + f\frac{d}{dx}(g) \\ \frac{d}{dx}(fgh) &= \frac{d}{dx}(f)gh + f\frac{d}{dx}(g)h + fg\frac{d}{dx}(h) \\ \frac{d}{dx}(fghk) &= \frac{d}{dx}(f)ghk + f\frac{d}{dx}(g)hk + fg\frac{d}{dx}(h)k + fgh\frac{d}{dx}(k)\end{aligned}$$

- 19.** Apply the product rule twice to the expression  $\frac{d}{dx}(f(x)g(x)h(x))$  to verify the formula stated above for the product of three functions. The first application of the product rule is shown below to help you get you started.

$$\frac{d}{dx}(f(x)g(x)h(x)) = \frac{d}{dx}(f(x) \cdot [g(x)h(x)])$$

- 20.** Apply the formula shown for the product of four functions to determine the derivative with respect to  $x$  of the function  $p$  defined by  $p(x) = x^2e^x \sin(x) \cos(x)$ .

Find the first derivative with respect to  $x$  for each of the following functions and for each function find the equation of the tangent line at the stated value of  $x$ . Make sure that you use appropriate techniques of differentiation.

**21.**  $f(x) = \frac{x^3 + x^2}{x}$ ; tangent line at 5

**22.**  $h(x) = \frac{x}{1+x}$ ; tangent line at  $-2$

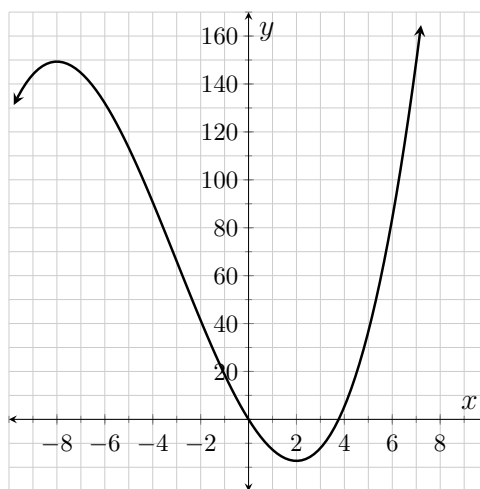
**23.**  $K(x) = \frac{1+x}{2x+2}$ ; tangent line at 8

**24.**  $r(x) = 3\sqrt[3]{x}e^x$ ; tangent line at 0

- 25.** Find the second derivative of  $y$  with respect to  $t$  if  $y = 4\sqrt{t}t^5$ . Make sure that you use appropriate techniques of differentiation and that you use proper notation on both sides of the equal sign.



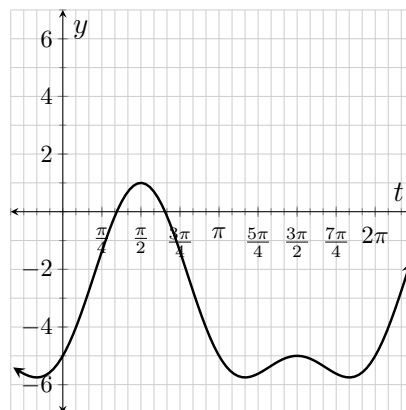
A function  $f$  is shown in Figure 5.12.1. Each question below is in reference to this function. Please note that the numeric endpoints of each interval answer are all integers.



**Figure 5.12.1:**  $y = f(x)$

26. Over what intervals is  $f'$  positive? Over what intervals is  $f'$  negative?
27. Over what intervals is  $f'$  increasing? Over what intervals is  $f'$  decreasing?
28. Over what intervals is  $f''$  positive? Over what intervals is  $f''$  negative?
29. The formula for  $f$  is  $f(x) = \frac{x^3}{3} + 3x^2 - 16x$ . Find the formulas for  $f'$  and  $f''$ , and then graph the derivatives and verify your answers to Exercises 26–29.

A function  $g$  is shown in Figure 5.12.2. Each question below is in reference to this function.



**Figure 5.12.2:**  $y = g(t)$

30. Where over the interval  $[0, 2\pi]$  does  $g'(t) = 0$ ? Please note that the  $t$ -scale in Figure 5.12.2 is  $\frac{\pi}{12}$ .
31. The formula for  $g$  is  $g(t) = 3\sin(t) + 3\sin^2(t) - 5$ . Find the formula for  $g'$  and use that formula to solve the equation  $g'(t) = 0$  over the interval  $[0, 2\pi]$ . Compare your answers to those found in Exercise 30.

**32.** Find the equation of the tangent line to  $f'$  at  $\pi$  if  $f(x) = \sin(2x)$ .

In Table 5.12.3 several function and derivative values are given for the functions  $f$  and  $g$ . The exercises that follow are based upon these two functions.

**Table 5.12.3:** Various values for derivatives of  $f$  and  $g$

$x$	$f(x)$	$f'(x)$	$f''(x)$	$g(x)$	$g'(x)$	$g''(x)$
0	2	-1	-2	-3	-3	-2
1	1	0	4	-5	2	16
2	4	7	10	11	37	58
3	15	20	16	87	126	124
4	46	39	22	289	293	214

Suppose that you were asked to find the value of  $h'(2)$  where  $h(x) = f(x)g(x)$ . The first thing you would need to do is find a formula for  $h'(x)$ . You would then replace  $x$  with 2, substitute the appropriate function and derivative values, and simplify. This is illustrated below.

$$\begin{aligned}
 h'(x) &= \frac{d}{dx}(f(x))g(x) + f(x)\frac{d}{dx}(g(x)) \\
 &= f'(x)g(x) + f(x)g'(x) \\
 h'(2) &= f'(2)g(2) + f(2)g'(2) \\
 &= 7(11) + 4(37) \\
 &= 225
 \end{aligned}$$

Following the process outlined in this example, find each of the following.

**33.** Find  $h'(4)$ , where  $h(x) = f(x)g(x)$ .

**34.** Find  $h''(2)$ , where  $h(x) = f(x)g(x)$ .

**35.** Find  $k'(3)$ , where  $k(x) = \frac{g(x)}{f(x)}$ .

**36.** Find  $p'(4)$ , where  $p(x) = 6\sqrt{x}f(x)$ .

**37.** Find  $r'(1)$ , where  $r(x) = (g(x))^2$ .

**38.** Find  $s'(2)$ , where  $s(x) = xf(x)g(x)$ .

**39.** Find  $F'(4)$ , where  $F(x) = \sqrt{x}g(4)$ .

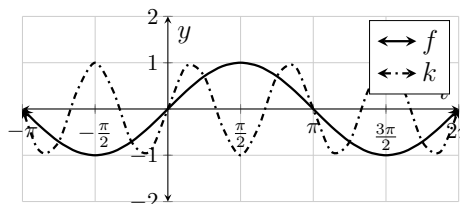
**40.** Find  $T''(0)$ , where  $T(x) = \frac{f(x)}{e^x}$ .

# Lab 6

## The Chain Rule

### 6.1 Introduction to the Chain Rule

The functions  $f(t) = \sin(t)$  and  $k(t) = \sin(3t)$  are shown in Figure 6.1.1. Since  $f'(t) = \cos(t)$ , it is reasonable to speculate that  $k'(t) = \cos(3t)$ . But this would imply that  $k'(0) = f'(0) = 1$ , and a quick glance of the two functions at 0 should convince you that this is not true; clearly  $k'(0) > f'(0)$ .



**Figure 6.1.1:**  $y = f(t)$  and  $y = k(t)$

The function  $k$  moves through three periods for every one period generated by the function  $f$ . Since the amplitudes of the two functions are the same, the only way  $k$  can generate periods at a rate of 3 : 1 (compared to  $f$ ) is if its rate of change is three times that of  $f$ . In fact,  $k'(t) = 3\cos(3t)$ ; please note that 3 is the first derivative of  $3t$ . This means that the formula for  $k'(t)$  is the product of the rates of change of the outside function ( $\frac{d}{du}(\sin(u)) = \cos(u)$ ) and the inside function ( $\frac{d}{dt}(3t) = 3$ ).

$k$  is an example of a composite function (as illustrated in Equation (6.1.1)). If we define  $g$  by the rule  $g(t) = 3t$ , then  $k(t) = f(g(t))$ .

$$t \xrightarrow{g} 3t \xrightarrow{f} \sin(3t) \quad (6.1.1)$$

$$\begin{aligned} g(t) &= 3t \\ f(u) &= \sin(u) \\ k(t) &= f(g(t)) \end{aligned}$$

Taking the output from  $g$  and processing it through a second function,  $f$ , is the action that characterizes  $k$  as a composite function.

Note that  $k'(0) = g'(0)f'(g(0))$ . This last equation is an example of what we call the *chain rule for differentiation*. Loosely, the chain rule tells us that when finding the rate of change for a composite function (at 0), we need to multiply the rate of change of the outside function,  $f'(g(0))$ , with the rate of change of the inside function,  $g'(0)$ . This is symbolized for general values of  $x$  in Equation (6.1.2) where  $u$  represents a function of  $x$  (e.g.  $u = g(x)$ ).

$$\frac{d}{dx}(f(u)) = f'(u) \frac{d}{dx}(u) \quad (6.1.2)$$

This rule is used to find derivative formulas in Examples 6.1.2 and 6.1.3.

### Example 6.1.2.

Problem	Solution
Find $\frac{dy}{dx}$ if $y = \sin(x^2)$ .	$\begin{aligned} \frac{dy}{dx} &= \cos(x^2) \cdot \frac{d}{dx}(x^2) \\ &= \cos(x^2) \cdot 2x \\ &= 2x \cos(x^2) \end{aligned}$

The factor of  $\frac{d}{dx}(x^2)$  is called a **chain rule factor**.

**Example 6.1.3.**

Problem	Solution
Find $f'(t)$ if $f(t) = \sec^9(t)$ .	$f(t) = [\sec(t)]^9$
	$f'(t) = 9 [\sec(t)]^8 \cdot \frac{d}{dt}(\sec(t))$
	$= 9 \sec^8(t) \cdot \sec(t) \tan(t)$
	$= 9 \sec^9(t) \tan(t)$

The factor of  $\frac{d}{dt}(\sec(t))$  is called a **chain rule factor**.

**Example 6.1.4.**

Problem	Solution
Find $\frac{dy}{dx}$ if $y = 4^x$ .	$\frac{dy}{dx} = \ln(4) \cdot 4^x$

Please note that *the chain rule was not applied* here because the function being differentiated was not a composite function. Here we just applied the formula for the derivative of an exponential function with base 4.

**Example 6.1.5.**

Problem	Solution
Find $\frac{dy}{dx}$ if $y = 4^x$	$y = e^{\ln(4)x}$
$= (e^{\ln(4)})^x$	$\frac{dy}{dx} = e^{\ln(4)x} \cdot \frac{d}{dx}(\ln(4)x)$
$= e^{\ln(4)x}$ .	$= e^{\ln(4)x} \cdot \ln(4).$
	$= \ln(4) \cdot 4^x$

While this is the same task as in Example 6.1.4, here we view the function as a composite function, and apply the chain rule.

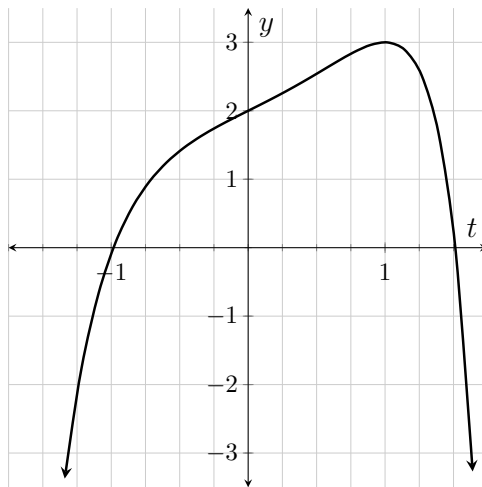
While you ultimately want to perform the chain rule step in your head, your instructor may want you to illustrate the step while you are first practicing the rule. For this reason, the step will be explicitly shown in every example given in this lab.

**Exercises**

Find the first derivative formula for each function. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative (e.g.  $h'(t)$ ).

- |                              |                                 |   |
|------------------------------|---------------------------------|---|
| 1. $h(t) = \cos(\sqrt{t})$   | 2. $P = \sin(\theta^4)$         | 3. $w(\alpha) = \cot(\sqrt[3]{\alpha})$ |
| 4. $z = 7[\ln(t)]^3$         | 5. $z(\theta) = \sin^4(\theta)$ | 6. $P(\beta) = \tan^{-1}(\beta)$        |
| 7. $y = [\sin^{-1}(t)]^{17}$ | 8. $T = 2^{\ln(x)}$             | 9. $y(x) = \sec^{-1}(e^x)$              |

A function,  $f$ , is shown in Figure 6.1.6. Answer the following questions in reference to this function.



**Figure 6.1.6:**  $y = f(t)$

10. Use the graph to rank the following in decreasing order:  $f(1)$ ,  $f'(1)$ ,  $f''(1)$ ,  $f'(0)$ , and  $f'(-1)$ .
11. The formula for  $f$  is  $f(t) = t e^t - e^{t^2} + 3$ . Find the formulas for  $f'$  and  $f''$  and use the formulas to verify your answer to Exercise 10.
12. Find the equation to the tangent line to  $f$  at 0.
13. Find the equation to the tangent line to  $f'$  at 0.
14. There is an antiderivative of  $f$  that passes through the point  $(0, 7)$ . Find the equation of the tangent line to this antiderivative at 0.

## 6.2 Order to Apply Rules

When finding derivatives of complex formulas you need to apply the rules for differentiation in the reverse of order of operations. For example, when finding  $\frac{d}{dx}(\sin(x e^x))$  the first rule you need to apply is the derivative formula for  $\sin(u)$  but when finding  $\frac{d}{dx}(x \sin(e^x))$  the first rule that needs to be applied is the product rule.

## Exercises

Find the first derivative formula for each function. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative (e.g.  $f'(x)$ ).

1.  $f(x) = \sin(x e^x)$

2.  $g(x) = x \sin(e^x)$

3.  $y = \frac{\tan(\ln(x))}{x}$

4.  $z = 5t + \frac{\cos^2(t^2)}{3}$

5.  $f(y) = \sin\left(\frac{\ln(y)}{y}\right)$

6.  $G = x \sin^{-1}(x \ln(x))$

## 6.3 Not Simplifying First

As always, you want to simplify an expression before jumping in to take its derivative. Nevertheless, it can build confidence to see that the rules work even if you don't simplify first.

## Exercises

Consider the functions  $f(x) = \sqrt{x^2}$  and  $g(x) = (\sqrt{x})^2$ .

1. Assuming that  $x$  is not negative, how does each of these formulas simplify? Use the simplified formula to find the formulas for  $f'(x)$  and  $g'(x)$ .
2. Use the chain rule (without first simplifying) to find the formulas for  $f'(x)$  and  $g'(x)$ ; simplify each result (assuming that  $x$  is positive).
3. Are  $f$  and  $g$  the same function? Explain why or why not.
4. So long as  $x$  falls on the interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$ ,  $\tan^{-1}(\tan(x)) = x$ . Use the chain rule to find  $\frac{d}{dx}(\tan^{-1}(\tan(x)))$  and show that it simplifies as it should.
5. How does the formula  $g(t) = \ln(e^{5t})$  simplify and what does this tell you about the formula for  $g'(t)$ ? After answering those questions, use the chain rule to find the formula for  $g'(t)$  and show that it simplifies as it should.

Consider the function  $g(t) = \ln\left(\frac{5}{t^3 \sec(t)}\right)$ .

6. Find the formula for  $g'(t)$  without first simplifying the formula for  $g(t)$ .
7. Use the quotient, product, and power rules *of logarithms* to expand the formula for  $g(t)$  into three logarithmic terms. Then find  $g'(t)$  by taking the derivative of the expanded version of  $g$ .
8. Show the two resultant formulas are in fact the same. Also, reflect upon which process of differentiation was less work and easier to “clean up.”

## 6.4 Chain Rule with Leibniz Notation

So far we have worked with the chain rule as expressed using function notation. In some applications it is easier to think of the chain rule using Leibniz notation. Consider the following example.

**Example 6.4.1.** During the 1990s, the amount of electricity used per day in Etown increased as a function of population at the rate of 18 kW/person. On July 1, 1997, the population of Etown was 100,000 and the population was decreasing at a rate of 6 people/day. In Equation (6.4.1) we use these values to determine the rate at which electrical usage was changing (with respect to time) in Etown on 7/1/1997. Please note that in this extremely simplified example we are ignoring all factors that contribute to citywide electrical usage other than population (such as temperature).

$$\left(18 \frac{\text{kW}}{\text{person}}\right) \left(-6 \frac{\text{people}}{\text{day}}\right) = -108 \frac{\text{kW}}{\text{day}} \quad (6.4.1)$$

Let's define  $g(t)$  as the population of Etown  $t$  years after January 1, 1990 and  $f(u)$  as the daily amount of electricity used in Etown when the population was  $u$ . From the given information,  $g(7.5) = 100,000$ ,  $g'(7.5) = -6$ , and  $f'(u) = 18$  for all values of  $u$ . If we let  $y = f(u)$  where  $u = g(t)$ , then we have (from Equation (6.4.1)):

$$\left(\frac{dy}{du}\bigg|_{u=100,000}\right) \left(\frac{du}{dt}\bigg|_{t=7.5}\right) = \frac{dy}{dt}\bigg|_{t=7.5}. \quad (6.4.2)$$

You should note that Equation (6.4.2) is an application of the chain rule expressed in Leibniz notation; specifically, the expression on the left side of the equal sign represents  $f'(g(7.5))g'(7.5)$ .

In general, we could express the chain rule as shown in Equation (6.4.3).

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (6.4.3)$$



## Exercises

Suppose that Carla is jogging in her sweet new running shoes. Suppose further that  $r = f(t)$  is Carla's pace (mph)  $t$  hours after 1:00 pm and  $y = h(r)$  is Carla's heart rate (bpm) when she jogs at a rate of  $r$  mph. In this context we can assume that all of Carla's motion was in one direction, so the words 'speed' and 'velocity' are completely interchangeable.

1. What is the meaning of  $f(0.75) = 7$ ?
2. What is the meaning of  $h(7) = 125$ ?
3. What is the meaning of  $(h \circ f)(0.75) = 125$ ?
4. What is the meaning of  $\left. \frac{dr}{dt} \right|_{t=0.75} = -0.00003$ ?
5. What is the meaning of  $\left. \frac{dy}{dr} \right|_{r=7} = 8$ ?
6. Assuming that all of the previous values are accurate, what is the value of  $\left. \frac{dy}{dt} \right|_{t=0.75}$  and what does this value tell you about Carla?

Portions of SW 35th Avenue are extremely hilly. Suppose that you are riding your bike along SW 35th Ave from Vermont Street to Capitol Highway. Let  $u = d(t)$  be the distance you have traveled (ft) where  $t$  is the number of seconds that have passed since you began your journey. Suppose that  $y = e(u)$  is the elevation (m) of SW 35th Ave where  $u$  is the distance (ft) from Vermont St headed towards Capitol Highway.

7. What, including units, would be the meanings of  $d(25) = 300$ ,  $e(300) = 140$ , and  $(e \circ d)(25) = 140$ ?
8. What, including units, would be the meanings of  $\left. \frac{du}{dt} \right|_{t=25} = 14$  and  $\left. \frac{dy}{du} \right|_{u=300} = -0.1$ ?
9. Suppose that the values stated in Exercise 8 are accurate. What, including unit, is the value of  $\left. \frac{dy}{dt} \right|_{t=25}$ ? What does this value tell you in the context of this problem?

According to Hooke's Law, the force (lb),  $F$ , required to hold a spring in place when its displacement from the natural length of the spring is  $x$  (ft), is given by the formula  $F = kx$  where  $k$  is called the **spring constant**. The value of  $k$  varies from spring to spring.

Suppose that it requires 120 lb of force to hold a given spring 1.5 ft beyond its natural length.

10. Find the spring constant for this spring. Include units when substituting the values for  $F$  and  $x$  into Hooke's Law so that you know the unit on  $k$ .
11. What, including unit, is the constant value of  $\frac{dF}{dx}$ ?
12. Suppose that the spring is stretched at a constant rate of  $0.032 \frac{\text{ft}}{\text{s}}$ . If we define  $t$  to be the amount of time (s) that passes since the stretching begins, what, including unit, is the constant value of  $\frac{dx}{dt}$ ?
13. Use the chain rule to find the constant value (including unit) of  $\frac{dF}{dt}$ . What is the contextual significance of this value?

## 6.5 Supplement

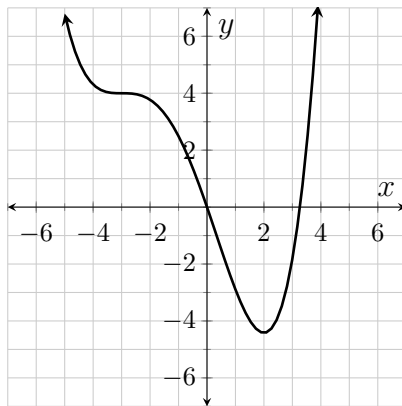
### Exercises

1. The functions  $f(x) = e^{3x}$ ,  $g(x) = (e^x)^3$ , and  $h(x) = (e^3)^x$  are equivalent. Find the first derivative formulas for each of the functions (without altering their given forms) and then explicitly establish that the derivative formulas are the same.

Consider the function  $k(\theta) = \sin^{-1}(\sin(\theta))$ .

2. Over the interval  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ ,  $k(\theta) = \theta$ . What does this tell you about the formula for  $k'(\theta)$  over  $(-\frac{\pi}{2}, \frac{\pi}{2})$ ?
3. Use the chain rule and an appropriate trigonometric identity to verify your answer to Exercise 2. Please note that  $\cos(\theta) > 0$  over  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .
4. What is the constant value of  $k'(\theta)$  over  $(\frac{\pi}{2}, \frac{3\pi}{2})$ ?
5. What is the value of  $k'(\frac{\pi}{2})$ ?

Consider the function  $f$  shown in Figure 6.5.1.



**Figure 6.5.1:**  $y = f(x)$

6. Suppose that  $g(x) = [f(x)]^4$ . Over what intervals is  $g'$  positive?
7. Suppose that  $r(x) = e^{f(x)}$ . Over what intervals is  $r'$  positive?
8. Suppose that  $w(x) = e^{f(-x)}$ . Over what intervals is  $w'$  positive?
9. Suppose that  $h(x) = \frac{1}{f(x)}$ . Is  $h$  differentiable at  $-3$ ? Please note that  $f$  has a horizontal tangent line at  $-3$ .

Decide whether or not it is necessary to use the chain rule when finding the derivative with respect to  $x$  of each of the following functions.

10.  $f(x) = \ln(x + 1)$

11.  $f(x) = \frac{2}{x^5}$

12.  $f(x) = \cos(\pi)$

13.  $f(x) = \cos(x)$

14.  $f(x) = \cos(\pi x)$

15.  $f(x) = \frac{\cos(\pi x)}{\pi}$

Find the first derivative with respect to  $x$  of each of the following functions. In all cases, look for appropriate simplifications before taking the derivative. Please note that some of the functions will be simpler to differentiate if you first use the rules of logarithms to expand and simplify the logarithmic expression.

16.  $f(x) = \tan^{-1}(\sqrt{x})$

17.  $f(x) = e^{e^{\sin(x)}}$

18.  $f(x) = \sin^{-1}(\cos(x))$

19.  $f(x) = \tan(x \sec(x))$

20.  $f(x) = \tan(x) \sec(\sec(x))$

21.  $f(x) = \sqrt[3]{(\sin(x^2))^2}$

22.  $f(x) = 4x \sin^2(x)$

23.  $f(x) = \ln(x \ln(x))$

24.  $f(x) = \ln\left(\frac{5}{xe^x}\right)$

25.  $f(x) = 2 \ln\left(\sqrt[3]{x \tan^2(x)}\right)$

26.  $f(x) = \ln\left(\frac{e^{x+2}}{\sqrt{x+2}}\right)$

27.  $f(x) = \ln(x^e + e)$

28.  $f(x) = \sec^4(e^x)$

29.  $f(x) = \sec^{-1}(e^x)$

30.  $f(x) = \csc\left(\frac{1}{\sqrt{x}}\right)$

31.  $f(x) = \frac{1}{\csc(\sqrt{x})}$

32.  $f(x) = \frac{\tan^{-1}(2x)}{2}$

33.  $f(x) = x^3 \sin\left(\frac{x}{3}\right)$

34.  $f(x) = \frac{4}{\sqrt{\frac{3}{x^7}}}$

35.  $f(x) = \frac{e^{xe^x}}{x}$

36.  $f(x) = xe^{xe^2}$

37.  $f(x) = \frac{\sin^5(x) - \sqrt{\sin(x)}}{\sin(x)}$

38.  $f(x) = 4x \sin(x) \cos(x^2)$

39.  $f(x) = \sin(x \cos^2(x))$

# Lab 7

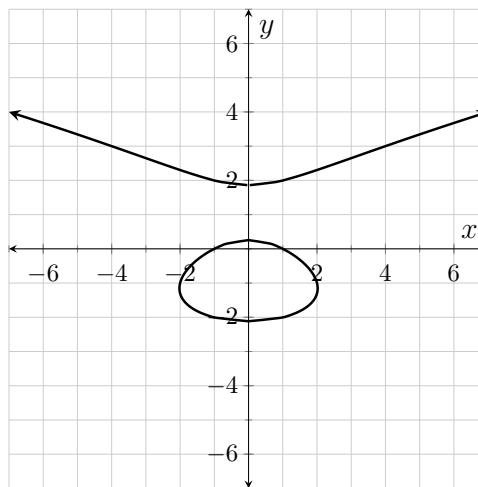
## Implicit Differentiation

### 7.1 General Implicit Differentiation

**Read the next lab early!** Before coming to the *next* lab session, Lab 8, read the special note at the beginning of that lab. You are asked to do some reading outside of class before you begin working on the lab in class. More explanation is given in that note.

Some points that satisfy the equation  $y^3 - 4y = x^2 - 1$  are graphed in Figure 7.1.1. Clearly this set of points does not constitute a function where  $y$  is a function of  $x$ ; for example, there are three points that have an  $x$ -coordinate of 1.

Nevertheless, there is a unique tangent line to the curve at each point on the curve and so long as the tangent line is not vertical it has a unique slope. We still identify the value of this slope using the symbol  $\frac{dy}{dx}$ , so it would be helpful if we had a formula for  $\frac{dy}{dx}$ . If we could solve  $y^3 - 4y = x^2 - 1$  for  $y$ , finding the formula for  $\frac{dy}{dx}$  would be a snap. Hopefully you quickly see that such an approach is just not reasonable.



**Figure 7.1.1:**  $y^3 - 4y = x^2 - 1$

To get around this problem we are going to employ a technique called **implicit differentiation**. We use this technique to find the formula for  $\frac{dy}{dx}$  whenever the

equation relating  $x$  and  $y$  is not explicitly solved for  $y$ . What we are going to do is treat  $y$  as if it were a function of  $x$  and set the derivatives of the two sides of the equation equal to one another. This is actually a reasonable thing to do because so long as we are at a point on the curve where the tangent line is not vertical, we could make  $y$  a function of  $x$  using appropriate restrictions on the domain and range.

Since we are treating  $y$  as a function of  $x$ , we need to make sure that we use the chain rule when differentiating terms like  $y^3$ . When  $u$  is a function of  $x$ , we know that  $\frac{d}{dx}(u^3) = 3u^2 \frac{d}{dx}(u)$ . Since the name we give  $\frac{d}{dx}(y)$  is  $\frac{dy}{dx}$ , it follows that when  $y$  is a function of  $x$ ,  $\frac{d}{dx}(y^3) = 3y^2 \frac{dy}{dx}$ . The derivation of  $\frac{dy}{dx}$  for the equation  $y^3 - 4y = x^2 - 1$  is shown in Example 7.1.2.

### Example 7.1.2.

$$\begin{aligned}
 y^3 - 4y &= x^2 - 1 \\
 \frac{d}{dx}(y^3 - 4y) &= \frac{d}{dx}(x^2 - 1) && \text{Differentiate each side with respect to } x. \\
 3y^2 \frac{dy}{dx} - 4 \frac{dy}{dx} &= 2x && \text{Respect the chain rule with the } y\text{-terms.} \\
 (3y^2 - 4) \frac{dy}{dx} &= 2x && \text{Begin to isolate } \frac{dy}{dx} \\
 \frac{dy}{dx} &= \frac{2x}{3y^2 - 4}.
 \end{aligned}$$

At first it might be unsettling that the formula for  $\frac{dy}{dx}$  contains both the variables  $x$  and  $y$ . However, if you think it through you should conclude that the formula *must* include the variable  $y$ ; otherwise, how could the formula generate three different slopes at the points  $(1, 2)$ ,  $(1, 0)$ , and  $(1, -2)$ ? These slopes are given below. The reader should verify their values by drawing lines onto Figure 7.1.1 with the indicated slopes at the indicated points.

$$\begin{aligned}
 \left. \frac{dy}{dx} \right|_{(1,2)} &= \frac{2(1)}{3(2)^2 - 4} && \left. \frac{dy}{dx} \right|_{(1,0)} = \frac{2(1)}{3(0)^2 - 4} && \left. \frac{dy}{dx} \right|_{(1,-2)} = \frac{2(1)}{3(-2)^2 - 4} \\
 &= \frac{1}{4} && && = -\frac{1}{2} && = \frac{1}{4}
 \end{aligned}$$

## Exercises

Use the process of implicit differentiation to find a formula for  $\frac{dy}{dx}$  for the curves generated by each of the following equations. Do not simplify the equations before taking the derivatives. You will need to use the product rule for differentiation in Exercises 4–6.

1.  $3x^4 = -6y^5$

2.  $\sin(x) = \sin(y)$

3.  $4y^2 - 2y = 4x^2 - 2x$

4.  $x = ye^y$

5.  $y = xe^y$

6.  $xy = e^{xy-1}$

7. Several points that satisfy the equation  $y = xe^y$  are graphed in Figure 7.1.3.

Find the slope and equation of the tangent line to this curve at the origin. (Please note that you already found the formula for  $\frac{dy}{dx}$  in Exercise 5.)

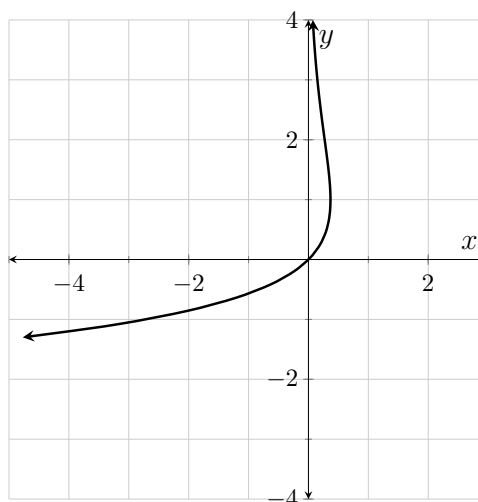


Figure 7.1.3:  $y = xe^y$

Consider the set of points that satisfy the equation  $xy = 4$ .

8. Use implicit differentiation to find a formula for  $\frac{dy}{dx}$ .

9. Find a formula for  $\frac{dy}{dx}$  after first solving the equation  $xy = 4$  for  $y$ .

10. Show that the two formulas are equivalent so long as  $xy = 4$ .

11. A set of points that satisfy the equation  $x \cos(xy) = 4 - y$  is graphed in Figure 7.1.4.

Find the slope and equation of the tangent line to this curve at the point  $(0, 4)$ .

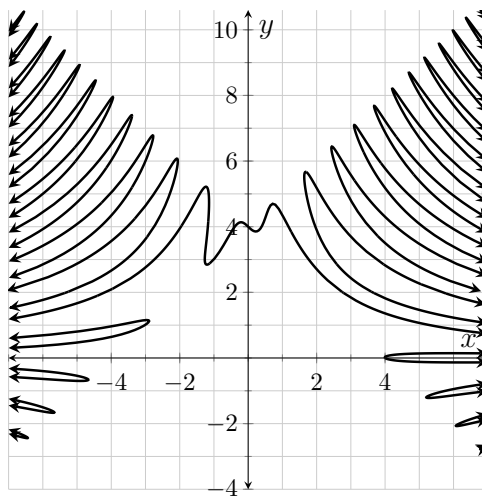


Figure 7.1.4:  $x \cos(xy) = 4 - y$

## 7.2 Derivatives of Inverse Functions

Using Definition 3.3.1, it is easy to establish that  $\frac{d}{dx}(x^2) = 2x$ . We can use this formula and implicit differentiation to find the formula for  $\frac{d}{dx}(\sqrt{x})$ . If  $y = \sqrt{x}$ , then  $y^2 = x$  (and  $y \geq 0$ ). Using implicit differentiation we have:

$$\begin{aligned} y^2 &= x \\ \frac{d}{dx}(y^2) &= \frac{d}{dx}(x) \\ 2y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{2y}. \end{aligned}$$

But  $y = \sqrt{x}$ , so we have:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2y} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

In a similar manner, we can use the fact that  $\frac{d}{dx}(\sin(x)) = \cos(x)$  to come up with a formula for  $\frac{d}{dx}(\sin^{-1}(x))$ . If  $y = \sin^{-1}(x)$ , then  $\sin(y) = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ . Note that because of the angle range that  $y$  is confined to, that  $\cos(y)$  is positive. So the Pythagorean identity  $\cos^2(y) = 1 - \sin^2(y)$  implies that  $\cos(y) = \sqrt{1 - \sin^2(y)}$ . This gives us:

$$\sin(y) = x$$



$$\begin{aligned}
\frac{d}{dx}(\sin(y)) &= \frac{d}{dx}(x) \\
\cos(y) \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos(y)} \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1 - \sin^2(y)}} \\
\frac{dy}{dx} &= \frac{1}{\sqrt{1 - x^2}}.
\end{aligned}$$

## Exercises

1. Use the fact that  $\frac{d}{dx}(e^x) = e^x$  together with implicit differentiation to show that  $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ . Begin by using the fact that  $y = \ln(x)$  implies that  $e^y = x$  (and  $x > 0$ ). Your first step is to differentiate both sides of the equation  $e^y = x$  with respect to  $x$ .
2. Use the fact that  $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$  together with implicit differentiation to show that  $\frac{d}{dx}(e^x) = e^x$ . Begin by using the fact that  $y = e^x$  implies that  $\ln(y) = x$ . Your first step is to differentiate both sides of the equation  $\ln(y) = x$  with respect to  $x$ .
3. Use the fact that  $\frac{d}{dx}(\tan(x)) = \sec^2(x)$  together with implicit differentiation to show that  $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$ . Begin by using the fact that  $y = \tan^{-1}(x)$  implies that  $\tan(y) = x$  (and  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ ). Your first step is to differentiate both sides of the equation  $\tan(y) = x$  with respect to  $x$ . Please note that you will need to use the Pythagorean identity that relates the tangent and secant functions while working this problem.

## 7.3 Logarithmic Differentiation

You have formulas that allow you to differentiate  $x^2$ ,  $2^x$ , and  $2^2$ . You don't, however, have a formula to differentiate  $x^x$ . In this section you are going to use a process called **logarithmic differentiation** to determine the derivative formula for the function  $y = x^x$ . Example 7.3.1 shows this process for a different function.

With logarithmic differentiation, when  $y$  is written as a function of  $x$ , especially one involving only products, quotients, and powers, begin by taking the logarithm of both sides of the equation relating  $y$  to  $x$ . This technique will usually only be helpful if the logarithm rules allow you to break up the right side some. Proceed to differentiate each side with respect to  $x$ . On the left side, respect the

rules for implicit differentiation. Once this is done, algebraically solve for  $\frac{dy}{dx}$ . If there are any instances of  $y$  left, replace them with the original expression in  $x$ .

**Example 7.3.1** (Logarithmic Differentiation).

$$\begin{aligned}
 y &= \frac{x^{e^x}}{4x+1} \\
 \ln(y) &= \ln\left(\frac{x^{e^x}}{4x+1}\right) \\
 \ln(y) &= \ln(x^{e^x}) - \ln(4x+1) \\
 \ln(y) &= e^x \ln(x) - \ln(4x+1) \\
 \frac{d}{dx}(\ln(y)) &= \frac{d}{dx}(e^x \ln(x) - \ln(4x+1)) \\
 \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx}(e^x) \cdot \ln(x) + e^x \cdot \frac{d}{dx}(\ln(x)) - \frac{1}{4x+1} \cdot \frac{d}{dx}(4x+1) \\
 \frac{1}{y} \frac{dy}{dx} &= e^x \cdot \ln(x) + e^x \cdot \frac{1}{x} - \frac{1}{4x+1} \cdot 4 \\
 \frac{1}{y} \frac{dy}{dx} &= e^x \ln(x) + \frac{e^x}{x} - \frac{4}{4x+1} \\
 \frac{dy}{dx} &= y \left( e^x \ln(x) + \frac{e^x}{x} - \frac{4}{4x+1} \right) \\
 \frac{dy}{dx} &= \frac{x^{e^x}}{4x+1} \left( e^x \ln(x) + \frac{e^x}{x} - \frac{4}{4x+1} \right)
 \end{aligned}$$

In the olden days (before symbolic calculators) we would use the process of logarithmic differentiation to find derivative formulas for complicated functions. The reason this process is “simpler” than straight forward differentiation is that we can obviate the need for the product and quotient rules if we completely expand the logarithmic expression before taking the derivative.

## Exercises

1. The function  $y = x^x$  is only defined for positive values of  $x$  (which in turn means  $y$  is also positive), so we can say that  $\ln(y) = \ln(x^x)$ . Use implicit differentiation to find a formula for  $\frac{dy}{dx}$  *after first applying the power rule of logarithms to the logarithmic expression on the right side of the equal sign*. Once you have your formula for  $\frac{dy}{dx}$ , substitute  $x^x$  for  $y$ . Voila! You will have the derivative formula for  $x^x$ . So go ahead and do it.

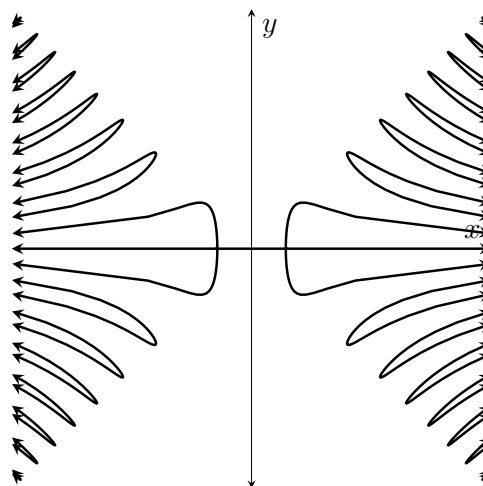
## 7.4 Supplement

**Read the next lab early!** Before coming to the *next* lab session, Lab 8, read the special note at the beginning of that lab. You are asked to do some reading outside of class before you begin working on the lab in class. More explanation is given in that note.

### Exercises

1. The curve  $x \sin(xy) = y$  is shown in Figure 7.4.1.

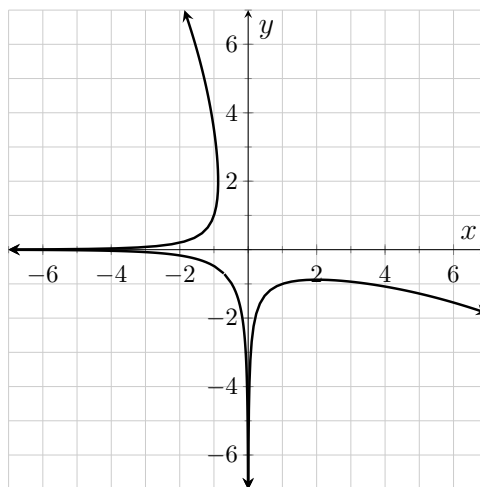
Find a formula for  $\frac{dy}{dx}$  and use that formula to determine the  $x$ -coordinate at each of the two points the curve crosses the  $x$ -axis. (Note: The tangent line to the curve is vertical at each of these points.) Scales have deliberately been omitted in Figure 7.4.1.



**Figure 7.4.1:**  $x \sin(xy) = y$

2. For the curve  $x \sin(xy) = y$  plotted in Figure 7.4.1, find the slope of the tangent line at  $\left(\sqrt{\pi/3}, \sqrt{\pi/12}\right)$ .
3. Solutions to the equation  $\ln(x^2y^2) = x + y$  are graphed in Figure 7.4.2.

Determine the equation of the tangent line to this curve at the point  $(1, -1)$ .



**Figure 7.4.2:**  $\ln(x^2 y^2) = x + y$

Use the process of logarithmic differentiation to find a first derivative formula for each of the following functions.

4.  $y = \frac{x \sin(x)}{\sqrt{x-1}}$

5.  $y = \frac{e^{2x}}{\sin^4(x) \sqrt[4]{x^5}}$

6.  $y = \frac{\ln(4x^3)}{x^5 \ln(x)}$

# Lab 8

## Related Rates

**Read this lab early!** The author of the lab manual uses a specific approach to finding solutions to related rates problems. Because the author's approach may vary in its details from that shown by your instructor, it is important that you read a couple of examples before you come to lab. You don't need to fully understand the examples, you just need to have a general sense of the author's approach so that you can follow the directions in the actual activities. Make sure that you read the examples from Activity 8.1 and Activity 8.2 before you come to lab so that you are prepared to work on the actual activities as soon as lab begins.

### 8.1 Introduction to Related Rates

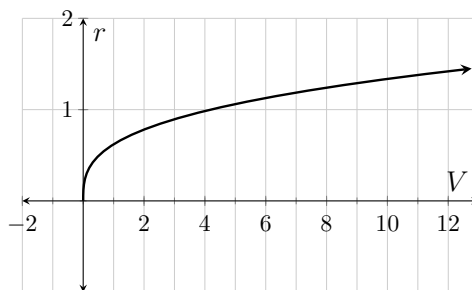
Consider the following problem.

A perfectly spherical snowball melts at a constant rate of  $0.3 \frac{\text{cm}^3}{\text{min}}$ . At what rate is the radius decreasing when the radius of the snowball is exactly 1.4 cm?

Before we actually solve the problem, let's take a moment to think about the situation. If the snowball were really large, there would be a lot of surface area to account for the loss of water. When the snowball gets small, however, there's less surface area to contribute water and, consequently, the radius will decrease at a faster rate than it does when the snowball is large. Mathematically, this increase in the rate at which the radius changes over time is a result of the manner in which the radius changes with respect to volume.

Figure 8.1.1 shows a graph of  $r = f(V)$  where  $r$  is the radius of a sphere of volume  $V$ . Note that the tangent lines to the curve have greater slope the closer  $V$  gets to zero. This tells us that a small change in volume has a greater effect on the radius the closer  $V$  is to zero.

Once the value of  $V$  gets large, the slope of the tangent lines is almost zero. This tells us that when the volume is large, a small change in volume has almost no effect on the radius.



**Figure 8.1.1:**  $r = f(V)$

Since the volume is decreasing at a constant rate, and a change in volume has a greater and greater effect on the radius as the volume decreases, the rate at which the radius decreases must increase as time goes by. This relationship between the rates at which the volume and radius change is an example of what is called **related rates**.

An acronym sometimes associated with the solution process for related rates problems is DREDS. The first four letters of the acronym stand for Diagram, Rates, Equation, and Differentiate. The final letter stands for Substitute. The reason the substitution steps comes last is that you need to have a rates equation before you have a place to substitute your values.

The first thing we need to do is clearly identify our variables. In related rates problems there are three broad types of variables.

1. Variables that measure physical quantities. These include, among other things, length variables, angle measurement variables, area variables, and volume variables. We will call these **quantity-variables**.
2. Variables that measure rates (with respect to time). These variables always *emerge* by taking the derivatives (with respect to time) of the quantity-variables. We will call these **rate-variables**.
3. A **time variable**. Depending on context, time may be measured in seconds, minutes, hours, days, weeks, years, etc.

The first three letters of DREDS (Diagram, Rates, Equation) are all interrelated and are not always to be taken literally. Basically they are meant to suggest that you need to identify the rates in the problem (either a given rate or the unknown rate you seek to find), identify quantity-variables that differentiate to each rate, and find an equation that relates those quantity-variables. A diagram is frequently useful during this part of the problem solving process.

Let's make a variable table for the snowball problem. Please note that we will need a volume variable. We know this because of the presence of  $\text{cm}^3$  in the given rate unit; cubic-centimeters always measure volume (just as square-centimeters

always measure area and centimeters always measure length). We will also need a radius variable (as the rate of change in the radius is stated in the problem).

**Table 8.1.2:** Variable table for the snowball problem

Variable	Variable Description	Unit
$V$	The volume of the snowball $t$ minutes after the snowball begins to melt.	$\text{cm}^3$
$r$	The radius of the snowball $t$ minutes after the snowball begins to melt.	$\text{cm}$
$\frac{dV}{dt}$	The rate of change in the volume of the snowball $t$ minutes after the snowball begins to melt.	$\frac{\text{cm}^3}{\text{min}}$
$\frac{dr}{dt}$	The rate of change in the radius of the snowball $t$ minutes after the snowball begins to melt.	$\frac{\text{cm}}{\text{min}}$
$t$	The amount of time that elapses after the snowball begins to melt.	$\text{min}$

Since we are asked to determine a rate at which the radius changes, we are ultimately going to need to determine a value of  $\frac{dr}{dt}$ . We cannot do that until we have an equation that includes the symbol  $\frac{dr}{dt}$ . There is a two-step process that we follow to find that equation.

1. Find an equation that relates the quantity variables. This step is one of the places where a diagram sometimes comes into play. This step corresponds to E in the DREDS acronym.
2. Differentiate both sides of the first equation with respect to time and simplify each side. The resultant equation is called the *related rates equation*. This step corresponds to the second D in the DREDS acronym.

The quantity variables in the snowball problem are  $V$  and  $r$ . We don't need a diagram in this problem as it is "standard knowledge" that the equation that relates the volume and radius of a sphere is  $V = \frac{4}{3}\pi r^3$ . Now that we have our (E)quation, we need to (D)ifferentiate both sides of this equation with respect to  $t$ , keeping in mind that both  $V$  and  $r$  are variables dependent upon  $t$ .

$$\begin{aligned}
 V &= \frac{4}{3}\pi r^3 \\
 \frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \\
 \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt}
 \end{aligned}$$

This is our related rates equation.

Now that we have our related rates equation (and only now), we can finally (s)ubstitute the given values into the appropriate variables. Before doing so, we

need to alert the reader with an introductory sentence or phrase. We also need to clearly state the values of each of the variables for which we know the value. In most math texts this part of the problem is worked without units. This is because some problems contain constants with “hidden” units, and it can be confusing when the units do not work out in the end. Getting back to the actual solution...

When the radius of the snowball is 1.4 cm and the snowball is melting at the rate of  $0.3 \frac{\text{cm}^3}{\text{L}}$ :

**Table 8.1.3:** What we know about the variables at the described moment

Variable	What we know
$V$	This variable is not in the related rates equation so it is no longer relevant in the problem solving process.
$r$	The value of this variable is 1.4.
$\frac{dV}{dt}$	The value of this variable is $-0.3$ . The value is negative because the volume decreases over time.
$\frac{dr}{dt}$	This is the variable for which we need to solve.

From our related rates equation and the values in Table 8.1.3:

$$\begin{aligned} \frac{dr}{dt} &= \frac{\frac{dV}{dt}}{4\pi r^2} \\ \left. \frac{dr}{dt} \right|_{r=1.4, \frac{dV}{dt}=-0.3} &= \frac{-0.3}{4\pi(1.4)^2} \\ &\approx -0.01218 \dots \end{aligned}$$

To finish up we need to write a conclusion.

At the moment the radius of the snowball is 1.4 cm, the radius is decreasing at the rate of about  $0.01218 \frac{\text{cm}}{\text{min}}$ .

## Exercises

Consider the following problem.

A square is getting larger. The sides are each growing at a constant rate of  $3 \frac{\text{in}}{\text{min}}$ . At what rate is the area of the square growing the moment its area is  $25 \text{ in}^2$ ?



1. Create your variable table. Use  $x$  for the side length. (DR)

**Table 8.1.4:** Variable table for the square problem

Variable	Variable Description	Unit

2. Write an equation that relates the two quantity-variables. (E)
3. Differentiate both sides of this equation with respect to  $t$ , and simplify. This equation is called the **related rates equation**. (D)
4. Write an orientation sentence or phrase and create a summary table for your quantity and rate variables. (preparing for S)

**Table 8.1.5:** What we know about the variables at the described moment

Variable	What we know

5. Substitute the known values into the related rates equation and solve for the unknown rate. (S)
6. Write a conclusion.

7. The radius of a circle is decreasing at the rate of  $0.2 \frac{\text{cm}}{\text{min}}$ . At what rate is the area of the circle decreasing when the area of the circle is  $4\pi \text{ cm}^2$ ?

Use the problem solving strategy:

- (a) Create your variable table. (DR)
- (b) Write an equation that relates the two quantity-variables. (E)

- (c) Differentiate both sides of the equation you just wrote with respect to  $t$ . Simplify the result. This is the **related rates equation**. (D)
- (d) Write an orientation sentence or phrase and create a summary table for your quantity and rate variables. (preparing for S)
- (e) Substitute the known values into the related rates equation and solve for the unknown rate. (S)
- (f) Write a conclusion.

8. As salt water boils, the only thing that evaporates is the water—the amount of salt in the mixture remains constant. As a consequence, the concentration of salt in the mixture increases over time.

Suppose that the concentration of salt in a pot of boiling water is given by  $C = \frac{200}{V}$  where  $C$  is the concentration of salt in the mixture ( $\frac{\text{mg}}{\text{L}}$ ) and  $V$  is the total amount of mixture in the pot (L). Determine the rate at which the salt concentration is changing at the moment there are 5.2 L of liquid in the pot if at that moment water is evaporating at the rate of  $0.13 \frac{\text{L}}{\text{min}}$ .

## 8.2 Diagrams for Related Rates

A particular challenge when working most related rates problems is identifying the quantity-variables. For example, when working a problem that involves a right triangle, there are six potential quantity-variables (the lengths of the three sides, the measurements of the two acute angles, and the area). In each related rates problem involving a right triangle, at least two of these measurements must be quantity-variables, or you will not end up with two (or more) rates to relate!

**Example 8.2.1.** A small drone is flying straight upward at the rate of  $0.88 \frac{\text{ft}}{\text{s}}$ . The drone is tethered to the ground (with a lot of slack) at a point that lies 400 ft west of the launch point of the drone. Assuming that the ground is perfectly level, determine the rate at which the angle of elevation from the tether point to the drone increases at the moment the drone is 300 ft above ground.

When searching for your quantity-variables, you need to identify sentences or phrases in the problem statement that contain the word “rate” or something similar (hence the R in DREDS). There are two such phrases in this problem:

- “A small drone is flying straight upward at the rate of...”
- “...determine the rate at which the angle of elevation from the tether point to the drone increase...”

The pieces of the right triangle that correspond to those phrases have been marked in Figure 8.2.2 as, respectively,  $h$  and  $\theta$ . The distance between the tether

point and launch point remains constant, hence the side labeled as 400. Please note that there should be no other annotations of any kind on this diagram. If writing other numbers down helps you think through the problem, draw a second triangle on your scratch paper and annotate that in any way the helps. That said, you need a clean variable diagram in your solution, so leave Figure 8.2.2 as it is.

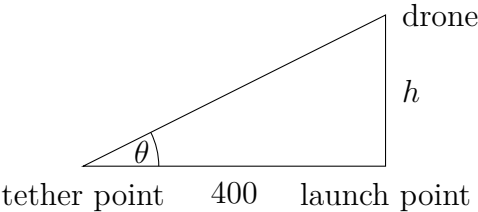
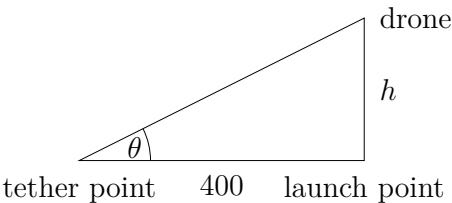


Figure 8.2.2: Variable Diagram

Now we can finish using the rest of the DREDS process.

Table 8.2.3: Variable table for the drone problem

Variable	Variable Description	Unit
$h$	The elevation of the drone $t$ seconds after the drone begins to rise.	ft
$\theta$	The angle of elevation from the tether point to the drone $t$ seconds after the drone begins to rise.	rad
$\frac{dh}{dt}$	The rate of change in elevation of the drone $t$ seconds after the drone begins to rise.	$\frac{\text{ft}}{\text{s}}$
$\frac{d\theta}{dt}$	The rate of change in the angle of elevation from the tether point to the drone $t$ seconds after the drone begins to rise.	$\frac{\text{rad}}{\text{s}}$
$t$	The amount of time that elapses after the drone begins to rise.	s



$$\tan(\theta) = \frac{h}{400}$$
$$\frac{d}{dt}(\tan(\theta)) = \frac{d}{dt}\left(\frac{h}{400}\right)$$
$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{400}\frac{dh}{dt}$$

This is our related rates equation.

Figure 8.2.4: Variable Diagram

At the moment the drone is 300 ft above the ground and rising at a rate of  $0.88 \frac{\text{ft}}{\text{s}}$ .

**Table 8.2.5:** What we know about the variables at the described moment

Variable	What we know
$h$	The value of this variable is 300.
$\theta$	$\tan(\theta) = \frac{3}{4}$ , which implies that $\cos(\theta) = \frac{4}{5}$ .
$\frac{dh}{dt}$	The value of this variable is 0.88.
$\frac{d\theta}{dt}$	This is the variable for which we need to solve.

From our related rates equation and the values in Table 8.2.5,

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{400} \cos^2(\theta) \frac{dh}{dt} \\ \left. \frac{d\theta}{dt} \right|_{\cos(\theta)=\frac{4}{5}, \frac{dh}{dt}=0.88} &= \frac{1}{400} \left( \frac{4}{5} \right)^2 (0.88) \\ &= 0.001408. \end{aligned}$$

When the elevation of the drone is 300 ft and the drone is rising at a rate of  $0.88 \frac{\text{ft}}{\text{s}}$ , the angle of elevation from the tether point to the drone is increasing at a rate of (about)  $0.0014 \frac{\text{rad}}{\text{s}}$ .

## Exercises

Three related rates problems are stated below. Each of the problems can be modeled using a right triangle. The sole task in this problem is to identify which pieces of the triangle correspond to quantity-variables and which piece of the triangle (if any) remains constant as the other pieces change.

For each problem, you need to identify which two or three pieces of the triangle correspond to quantity-variables and which pieces remain constant or are not relevant. For each problem, draw a right triangle and label the pieces appropriately (finally a D for Diagram). When you are done, any given triangle should either have two pieces labeled as variables and one piece labeled with a constant value or three pieces labeled as variables.

1. A jet is climbing at a constant airspeed of 430 mph and at a constant angle of elevation of  $12^\circ$ . At what rate is the elevation of the jet changing when the jet is 2640 ft above the ground? (Assume that the ground elevation below the jet's path does not change as the jet makes its climb.)
2. A 17 ft ladder is propped against a very tall wall but the top of the ladder is slipping down the wall. At the moment the top of the ladder is 12 ft above the ground it is slipping at the rate of  $0.2 \frac{\text{ft}}{\text{s}}$ . At what rate is the distance between the foot of the ladder and the wall changing at that moment?

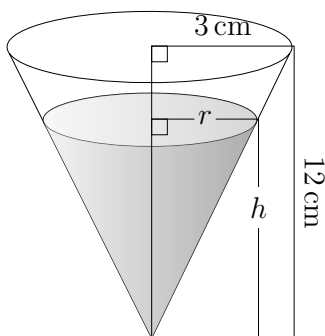
3. In Kansas, Highway A runs due north-south for 125 miles and Highway B runs due east-west for 222 miles. The elevation of both highways is constantly 725 feet their entire lengths. The highways intersect at a point exactly halfway through their runs. One Tuesday a car is travelling east on Highway B at the constant speed of 76 mph and a truck is traveling north on Highway A at a constant speed of 84 mph. The car passes through the point of intersection at 4:12 pm and the truck passes through the point of intersection at 4:17 pm. At what rate is the distance between the two vehicles changing at 4:27 that afternoon? Assume that the speeds of the car and truck do not vary at all between 4:10 pm and 4:30 pm.
4. Go ahead and finish the three problems stated in Exercises 1–3. Remember to complete each step in the problems solving process.
- (a) Draw a properly labeled variable diagram. (D) Note that you completed this step while working Exercises 1–3.
  - (b) Make a variable definition table. Include each variable type (quantity, rate, and time). Include the unit for each variable (R)
  - (c) Write down an equation that relates the two quantity-variables. (E)
  - (d) Differentiate both sides of the equation you just wrote with respect to  $t$ . Simplify the result. This equation is called the **related rates equation**. (D)
  - (e) Write an orientation sentence or phrase and create a summary table for your quantity and rate variables. (preparing for S)
  - (f) Substitute the known values into the related rates equation and solve for the unknown rate. (S)
  - (g) Write a conclusion.
5. When working with “common knowledge” formulas, it is important to identify which variables in the formula are really relevant to the stated problem and to then eliminate any irrelevant variables from the formula. Consider the following problem.

Slushy is flowing out of the bottom of a cup at the constant rate of  $0.9 \frac{\text{cm}^3}{\text{s}}$ . The cup is the shape of a right circular cone. The height of the cup is 12 cm and the radius at the top of the cup is 3 cm. Determine the rate at which the depth of the remaining slushy changes at the instant there are exactly  $8\pi \text{ cm}^3$  of slushy remaining in the cup.

The presence of the cubic-centimeter unit indicates that the problem is rooted in a volume formula. Specifically, we need the volume formula for a right circular cone which is  $V = \frac{\pi}{3}r^2h$  where  $r$  is the radius at the top of the cone and  $h$  is the height of the cone.

In the stated problem, we are told the rate at which the volume of the remaining slushy changes and we are asked to determine the rate at which the depth of the remaining slushy changes. The rates correspond to  $\frac{dV}{dt}$  and  $\frac{dh}{dt}$  where  $V$  and  $h$  are, respectively, the volume ( $\text{cm}^3$ ) and depth (cm) of the slushy that remains in the cup  $t$  seconds after the slushy begins to flow out of the bottom of the cup.

Since the value of  $\frac{dr}{dt}$  is neither given nor desired, we need to eliminate the variable  $r$  from the formula  $V = \frac{\pi}{3}r^2h$ . We do this by exploiting a property of similar triangles. Two triangles are said to be similar if and only if the three angular measurements of the first triangle are identical to the angular measurements of the second triangle. One thing that is always true about similar triangles is that their corresponding sides are proportional. This latter fact is the only reason we are able to define trigonometric functions in terms of the ratios of the sides of a right triangle!



**Figure 8.2.6:** Slushy Cone

A diagram of the cup is shown in Figure 8.2.6. The radius of the cone at the top of the remaining slushy is labeled as  $r$  and the depth of the remaining slushy is labeled as  $h$ . The radius at the top of the cup as well as the height of the cup are also shown.

Use the fact that the two right triangles in Figure 8.2.6 are similar to write a proportion that relates  $r$ ,  $h$ , 3 cm, and 12 cm. Solve that equation for  $r$  and substitute the resultant formula into  $V = \frac{\pi}{3}r^2h$ . Simplify the resultant formula of  $h$ ; that's the actual volume formula you want to use while solving the related rates problem—the formula that relates  $V$  and  $h$ .

Go ahead and solve the related rates problem.

## 8.3 Supplement

### Exercises

1. A weather balloon is rising vertically at the rate of  $10 \frac{\text{ft}}{\text{s}}$ . An observer is standing on the ground 300 ft horizontally from the point where the balloon was released. At what rate is the distance between the observer and the balloon changing when the balloon is 400 ft height?

2. At 2:00 PM one day, Muffin and Rex were sleeping atop one another. A loud noise startled the animals and Muffin began to run due north at a constant rate of  $1.7 \frac{\text{ft}}{\text{s}}$  while Rex ran due east at a constant rate of  $2.1 \frac{\text{ft}}{\text{s}}$ . The critters maintained these paths and rates for several seconds. Calculate the rate at which the distance between the cat and dog was changing three seconds into their run.

3. Schuyler's clock is kaput; the minute hand functions as it should but the hour hand is stuck at 4. The minute hand on the clock is 30 cm long and the hour hand is 20 cm long. Determine the rate of change between the tips of the hands every time the minute hand points directly at 12.

4. It was a dark night and 5.5 ft tall Bahram was walking towards a street lamp whose light was perched 40 ft into the air. As he walked, the light caused a shadow to fall behind Bahram. When Bahram was 80 ft from the base of the lamp he was walking at a pace of  $2 \frac{\text{ft}}{\text{s}}$ . At what rate was the length of Bahram's shadow changing at that instant? Make sure that each stated and calculated rate value has the correct sign. Make sure that your conclusion sentence clearly communicates whether the length of the shadow was increasing or decreasing at the indicated time.

5. The gravitational force,  $F$ , between two objects in space with masses  $m_1$  and  $m_2$  is given by the formula  $F = \frac{Gm_1m_2}{r^2}$  where  $r$  is the distance between the objects' centers of mass and  $G$  is the universal gravitational constant.

Two pieces of space junk, one with mass 500 kg and the other with mass 3000 kg, were drifting directly toward one another. When the objects' centers of mass were 250 km apart the lighter piece was moving at a rate of  $0.5 \frac{\text{km}}{\text{h}}$  and the heavier piece was moving at a rate of  $0.9 \frac{\text{km}}{\text{h}}$ . Leaving  $G$  as a constant, determine the rate at which the gravitational force between the two objects was changing at that instant. Make sure that each stated and calculated rate value has the correct sign. Make sure that your conclusion sentence clearly communicates whether the force was increasing or decreasing at that given time. The unit for  $F$  is Newtons (N).

6. An eighteen inch pendulum sitting atop a table is in the downward part of its motion. The pivot point for the pendulum is 30 in above the table top. When the pendulum is  $45^\circ$  away from vertical, the angle formed at the pivot is decreasing at the rate of  $25 \frac{^\circ}{\text{s}}$ . At what rate is the end of the pendulum approaching the table top at this instant? Make sure that each stated and calculated rate value

has the correct sign. Make sure that you use appropriate units when defining your variables.



# Lab 9

## Critical Numbers and Graphing from Formulas

### 9.1 Motivation

In the first activity of this lab you are going to discuss a few questions with your group mates that will hopefully motivate you for one of the topics covered in the lab.

### Exercises

1. Discuss how you could use the first derivative formula to help you determine the vertex of the parabola  $y = -2x^2 + 18x - 7$  and then determine the vertex. Remember that the vertex is a point in the  $xy$ -plane and as such is identified using an ordered pair.

2. The curves in Figures 9.1.1 and 9.1.2 were generated by two of the four functions given below. Use the given functions along with their first derivatives to determine which functions generated the curves. Please note that the  $y$ -scales have deliberately been omitted from the graphs and that different scales were used to generate the two graphs. Resist any temptation to use your calculator; use of your calculator totally obviates the point of the exercise.

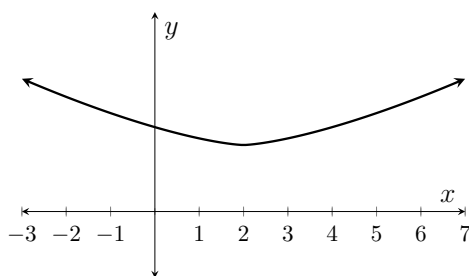


Figure 9.1.1: mystery curve 1

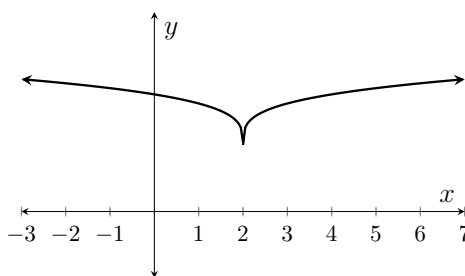


Figure 9.1.2: mystery curve 2

The potential functions are

$$f_1(x) = \frac{1}{(x-2)^{10/7}} + C_1$$

$$f_2(x) = \frac{1}{(x-2)^{2/7}} + C_2$$

$$f_3(x) = (x-2)^{2/7} + C_3$$

$$f_4(x) = (x-2)^{10/7} + C_4$$

where  $C_1, C_2, C_3, C_4$  are unknown constants.

## 9.2 Local Extrema

Referring back to Exercises 9.1, the vertex of the parabola in Exercise 1 is called a **local maximum point** and the points  $(2, C_3)$  and  $(2, C_4)$  in Exercise 2 are called **local minimum points**. Collectively, these points are called **local extreme points**.

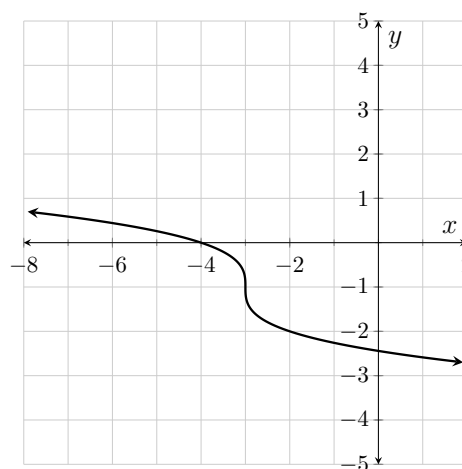
While working through Exercises 9.1 you hopefully came to the conclusion that the local extreme points had certain characteristics in common. In the first place, they must occur at a number in the domain of the function (which eliminated  $f_1$  and  $f_2$  from contention in Exercise 2). Secondly, one of two things must be true about the first derivative when a function has a local extreme point; it either has a value of zero or it does not exist. This leads us to the definition of a **critical number** of a function.

**Definition 9.2.1** (Critical Numbers). If  $f$  is a function, then we define the **critical numbers** of  $f$  as the numbers in the domain of  $f$  where the value of  $f'$  is either zero or does not exist.

### Exercises

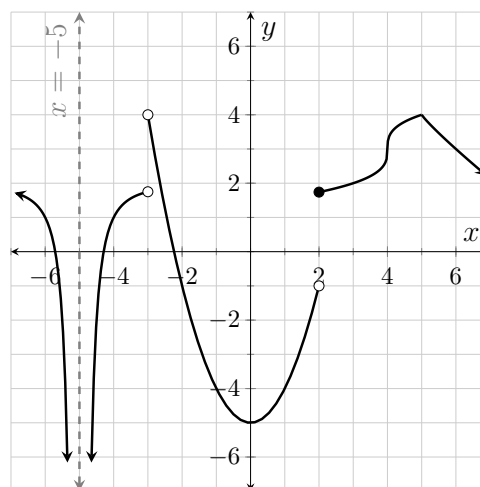
1. The function  $g$  shown in Figure 9.2.2 has a vertical tangent line at  $-3$ .

Veronica says that  $-3$  is a critical number of  $g$  but Tito disagrees. Tito contends that  $-3$  is not a critical number because  $g$  does not have a local extreme point at  $-3$ . Who is right?



**Figure 9.2.2:**  $y = g(x)$

Answer each of the following questions (using complete sentences) in reference to the function  $f$  shown in Figure 9.2.3.



**Figure 9.2.3:**  $y = f(x)$

2. What are the critical numbers of  $f$ ?
3. What are the local extreme points on  $f$ ? Classify the points as local minimums or local maximums and remember that points on the plane are represented by ordered pairs.
4. Find the absolute maximum value of  $f$  over the interval  $(-7, 7)$ . Please note that the function value is the value of the  $y$ -coordinate at the point on the curve and as such is a number.

Decide whether each of the following statements is *True* or *False*.

5. True or False? A function always has a local extreme point at each of its critical numbers.
6. True or False? If the point  $(t_1, h(t_1))$  is a local minimum point on  $h$ , then  $t_1$  must be a critical number of  $h$ .
7. True or False? If  $g'(2.7) = 0$ , then  $g$  must have a local extreme point at 2.7.
8. True or False? If  $g'(2.7) = 0$ , then 2.7 must be a critical number of  $g$ .
9. True or False? If  $g'(9)$  does not exist, then 9 must be a critical number of  $g$ .

### 9.3 Formal Identification of Critical Numbers

When finding critical numbers based upon a function formula, there are three issues that need to be considered; the domain of the function, the zeros of the first derivative, and the numbers in the domain of the function where the first derivative is undefined. When writing a formal analysis of this process each of these questions must be explicitly addressed. The following outline shows the work you need to show when you are asked to write a formal determination of critical numbers based upon a function formula.

**Algorithm 9.3.1** (A process for formal determination of critical numbers). *Please note that you should present your work in narrative form using complete sentences that establish the significance of all stated intervals and values. For example, for the function  $g(t) = \sqrt{t-7}$  the first sentence you should write is “The domain of  $g$  is  $[7, \infty)$ .”*

1. *Using interval notation, state the domain of the function.*
2. *Differentiate the function and completely simplify the resultant formula. Simplification includes:*
  - (a) *Manipulating all negative exponents into positive exponents.*
  - (b) *If any rational expression occurs in the formula, all terms must be written in rational form and a common denominator must be established for the terms. The final expression should be a single rational expression.*
  - (c) *The final expression must be completely factored.*

3. *State the values where the first derivative is equal to zero; show any non-trivial work necessary in making this determination. Make sure that you write a sentence addressing this issue even if the first derivative has no zeros.*
4. *State the values in the domain of the function where the first derivative does not exist; show any non-trivial work necessary in making this determination. Make sure that you write a sentence addressing this issue even if no such numbers exist.*
5. *State the critical numbers of the function. Make sure that you write this conclusion even if the function has no critical numbers.*

*Because the domain of a function is such an important issue when determining critical numbers, there are some things you should keep in mind when determining domains.*

- *Division by zero is never a good thing.*
- *Over the real numbers, you cannot take even roots of negative numbers.*
- *Over the real numbers, you can take odd roots of negative numbers.*
- *$\sqrt[n]{0} = 0$  for any positive integer  $n$ .*
- *Over the real numbers, you can only take logarithms of positive numbers.*
- *The sine and cosine function are defined at all real numbers. You should check with your lecture instructor to see if you are expected to know the domains of the other four trigonometric functions and/or the inverse trigonometric functions.*

## Exercises

Formally establish the critical numbers for each of the following functions following the procedure outlined in Algorithm 9.3.1.

1.  $f(x) = x^2 - 9x + 4$

2.  $g(t) = 7t^3 + 39t^2 - 24t$

3.  $p(t) = (t + 8)^{2/3}$

4.  $z(x) = x \ln(x)$

5.  $y(\theta) = e^{\cos(\theta)}$

6.  $T(t) = \sqrt{t - 4}\sqrt{16 - t}$

The first derivative of the function  $m(x) = \frac{\sqrt{x-5}}{x-7}$  is

$$m'(x) = \frac{3-x}{2\sqrt{x-5}(x-7)^2}.$$

7. Roland says that 5 is a critical number of  $m$  but Yuna disagrees. Who is correct and why?
8. Roland says that 7 is a critical number of  $m$  but Yuna disagrees. Who is correct and why?
9. Roland says that 3 is a critical number of  $m$  but Yuna disagrees. Who is correct and why?

## 9.4 Sign Tables for the First Derivative

Once you have determined the critical numbers of a function, the next thing you might want to determine is the behavior of the function at each of its critical numbers. One way you could do that involves a sign table for the first derivative of the function.

### Exercises

1. The first derivative of  $f(x) = x^3 - 21x^2 + 135x - 24$  is  $f'(x) = 3(x-5)(x-9)$ .

The critical numbers of  $f$  are trivially shown to be 5 and 9. Copy Table 9.4.1 onto your paper and fill in the missing information. Then state the local minimum and maximum points on  $f$ . Specifically address both minimum and maximum points even if one and/or the other does not exist. Remember that points on the plane are represented by ordered pairs. *Make sure that you state points on  $f$  and not  $f'$ !*

Interval	Sign of $f'$	Behavior of $f$
$(-\infty, 5)$		
$(5, 9)$		
$(9, \infty)$		

**Table 9.4.1:**  $f'(x) = 3(x-5)(x-9)$

The first derivative of the function  $g(t) = \frac{\sqrt{t-4}}{(t-1)^2}$  is

$$g'(t) = \frac{-3(t-5)}{2(t-1)^3\sqrt{t-4}}.$$

2. State the critical numbers of  $g$ ; you *do not* need to show a formal determination of the critical numbers. You *do* need to write a complete sentence.
3. Copy Table 9.4.2 onto your paper and fill in the missing information. Then state the local minimum and maximum points on  $g$ .

Specifically address both minimum and maximum points even if one and/or the other does not exist.

Interval	Sign of $g'$	Behavior of $g$
$(4, 5)$		
$(5, \infty)$		

**Table 9.4.2:**  $g'(t) = \frac{-3(t-5)}{2(t-1)^3\sqrt{t-4}}$

4. Why did we not include any part of the interval  $(-\infty, 4)$  in Table 9.4.2?
5. Formally, we say that  $g(t_0)$  is a local minimum value of  $g$  if there exists an open interval centered at  $t_0$  over which  $g(t_0) < g(t)$  for every value of  $t$  on that interval (other than  $t_0$ , of course). Since  $g$  is not defined to the left of 4, it is impossible for this definition to be satisfied at 4; hence  $g$  does not have a local minimum value at 4. State the local minimum and maximum points on  $g$ . Specifically address both minimum and maximum points even if one and/or the other does not exist.
6. Write a formal definition for a local maximum point on  $g$ .

The first derivative of the function  $k(x) = \frac{\sqrt[3]{(x-2)^2}}{x-1}$  is

$$k'(x) = \frac{4-x}{3(x-1)^2\sqrt[3]{x-2}}.$$

7. State the critical numbers of  $k$ ; you *do not* need to show a formal determination of the critical numbers. You *do* need to write a complete sentence.

8. Copy Table 9.4.3 onto your paper and fill in the missing information.

**Table 9.4.3:**  $k'(t) = \frac{4-x}{3(x-1)^2 \sqrt[3]{x-2}}$

Interval	Sign of $k'$	Behavior of $k$
$(-\infty, 1)$		
$(1, 2)$		
$(2, 4)$		
$(4, \infty)$		

9. The number 1 was included as an endpoint in Table 9.4.3 even though 1 is not a critical number of  $k$ . Why did we have to include the intervals  $(-\infty, 1)$  and  $(1, 2)$  in the table as opposed to just using the single interval  $(-\infty, 2)$ ?

Perform each of the following for the functions in Exercises 10–12.

- Formally establish the critical numbers of the function.
- Create a table similar to Tables 9.4.1–9.4.3. Number the tables. Don't forget to include table headings and column headings.
- State the local minimum points and local maximum points on the function. Make sure that you explicitly address both types of points even if there are none of one type and/or the other.

10.  $k(x) = x^3 + 9x^2 - 10$

11.  $g(t) = (t + 2)^3(t - 6)$

12.  $F(x) = \frac{x^2}{\ln(x)}$

Consider a function  $f$  whose first derivative is  $f'(x) = (x - 9)^4$ .

13. Is 9 definitely a critical number of  $f$ ? Explain why or why not.
14. Other than at 9, what is always the sign of  $f'(x)$ ? What does this sign tell you about the function  $f$ ?
15. What type of point does  $f$  have at 9? (Hint, draw a freehand sketch of the curve.)



16. How could the second derivative of  $f$  be used to confirm your conclusion in Exercise 15? Go ahead and do it.

## 9.5 Inflection Points

When searching for inflection points on a function, you can narrow your search by identifying numbers where the function is continuous (from both directions) and the second derivative is either zero or undefined. (By definition an inflection point cannot occur at a number where the function is not continuous from both directions.) You can then build a sign table for the second derivative that implies the concavity of the given function.

When performing this analysis, you need to simplify the second derivative formula in the same way you simplify the first derivative formula when looking for critical numbers and local extreme points.

### Exercises

1. Identify the inflection points for the function shown in Figure 9.2.3.

The first two derivatives of the function  $y(x) = \frac{(x+2)^2}{(x+3)^3}$  are

$$y'(x) = \frac{-x(x+2)}{(x+3)^4} \quad y''(x) = \frac{2(x+\sqrt{3})(x-\sqrt{3})}{(x+3)^5}.$$

- Yolanda was given this information and asked to find the inflection points on  $y$ . The first thing Yolanda wrote was, “The critical numbers of  $y$  are  $\sqrt{3}$  and  $-\sqrt{3}$ .” Explain to Yolanda why this is not true.
- What are the critical numbers of  $y$  and in what way are they important when asked to identify the inflection points on  $y$ ?
- Copy Table 9.5.1 onto your paper and fill in the missing information.

**Table 9.5.1:**  $y''(x) = \frac{2(x+\sqrt{3})(x-\sqrt{3})}{(x+3)^5}$

Interval	Sign of $y''$	Behavior of $y$
$(-\infty, -3)$		
$(-3, -\sqrt{3})$		
$(-\sqrt{3}, \sqrt{3})$		
$(\sqrt{3}, \infty)$		

5. State the inflection points on  $y$ ; you may round the dependent coordinate of each point to the nearest hundredth.
6. The function  $y$  has a vertical asymptote at  $-3$ . Given that fact, it was impossible that  $y$  would have an inflection point at  $-3$ . Why, then, did we never-the-less break the interval  $(-\infty, -\sqrt{3})$  at  $-3$  when creating our concavity table?

Perform each of the following for the functions in Exercises 7–9.

- State the domain of the function.
- Find, and completely simplify, the formula for the second derivative of the function. It is *not* necessary to simplify the formula for the first derivative of the function.
- State the values in the domain of the function where the second derivative is either zero or does not exist.
- Create a table similar to Table 9.5.1. Number the table. Don't forget to include table headings and column headings.
- State the inflection points on the function. Make sure that you explicitly address this question even if there are no inflection points.

7.  $f(x) = x^4 - 12x^3 + 54x^2 - 10x + 6$

8.  $g(x) = (x - 2)^2 e^x$

9.  $G(x) = \sqrt{x^3} + 6\sqrt{x}$

10. The second derivative of the function  $w(t) = t^{1.5} - 9t^{0.5}$  is  $w''(t) = \frac{3(t+3)}{4t^{1.5}}$  yet  $w$  has no inflection points. Why is that?

## 9.6 End Behavior

We are frequently interested in a function's "end behavior." That is, what is the behavior of the function as the input variable increases without bound or decreases without bound.

Many times a function will approach a horizontal asymptote as its end behavior. If the horizontal asymptote  $y = L$  represents the end behavior of the function  $f$  both as  $x$  increases without bound and as  $x$  decreases without bound, we write  $\lim_{x \rightarrow \infty} f(x) = L$  and  $\lim_{x \rightarrow -\infty} f(x) = L$ .

While working Limits and Continuity you investigated strategies for formally establishing limit values as  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ . In this activity you are going to investigate a more informal strategy for determining these type limits.

Consider  $\lim_{x \rightarrow \infty} \frac{4x-2}{3+20x}$ . When the value of  $x$  is really large, we say that the term  $4x$  dominates the numerator of the expression and the term  $20x$  dominates the denominator. We actually call those terms **the dominant terms** of the numerator and denominator. The dominant terms are significant because when the value of  $x$  is really large, the other terms in the expression contribute almost nothing to the value of the expression. That is, for really large values of  $x$ :

$$\begin{aligned}\frac{4x-2}{3+20x} &\approx \frac{4x}{20x} \\ &= \frac{1}{5}\end{aligned}$$

For example, even if  $x$  has the paltry value of 1000,

$$\begin{aligned}\frac{4(1000)-2}{3+20(1000)} &= \frac{3998}{20003} \\ &\approx 0.19987 \dots \approx \frac{1}{5}\end{aligned}$$

This tells us that  $\lim_{x \rightarrow \infty} \frac{4x-2}{3+20x} = \frac{1}{5}$  and that  $y = \frac{1}{5}$  is a horizontal asymptote for the graph of  $y = \frac{4x-2}{3+20x}$ .

## Exercises

The formulas used to graph Figures 9.6.1–9.6.5 are given below. Focusing first on the dominant terms of the expressions, match the formulas with the functions ( $f_1$  through  $f_5$ ).

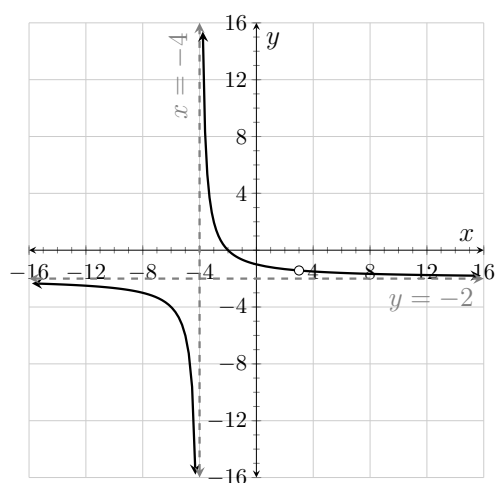
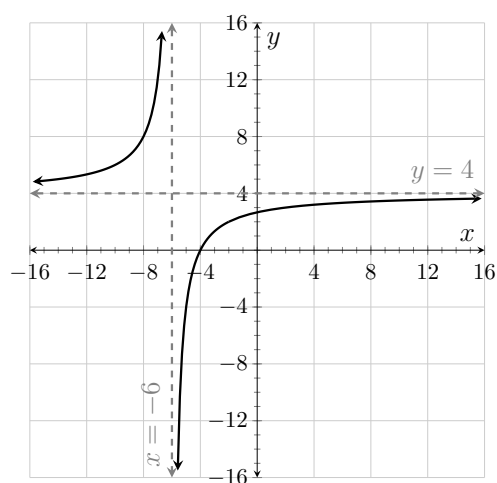
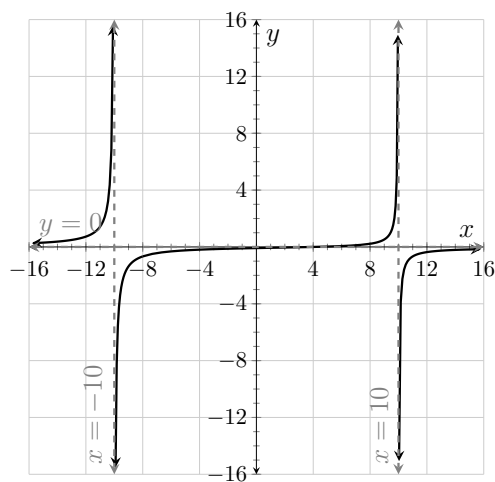
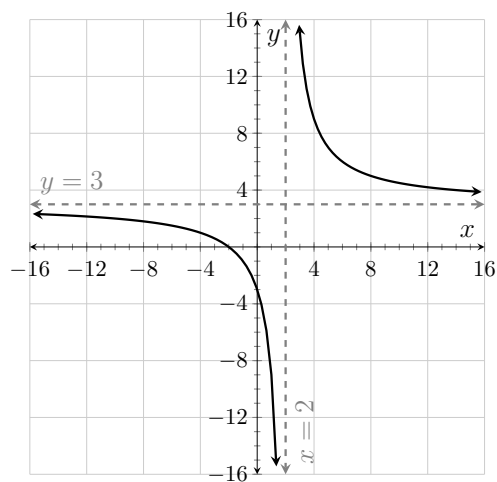
1.  $y = \frac{3x+6}{x-2}$

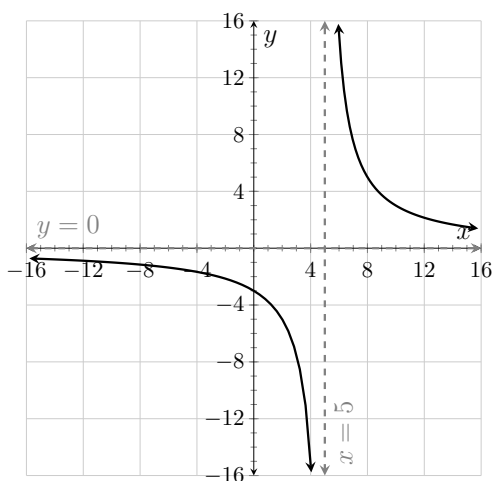
2.  $y = \frac{16+4x}{6+x}$

3.  $y = \frac{6x^2-6x-36}{36-3x-3x^2}$

4.  $y = \frac{-2x+8}{x^2-100}$

5.  $y = \frac{15}{x-5}$

Figure 9.6.1:  $y = f_1(x)$ Figure 9.6.2:  $y = f_2(x)$ Figure 9.6.3:  $y = f_3(x)$ Figure 9.6.4:  $y = f_4(x)$



**Figure 9.6.5:**  $y = f_5(x)$

Use the concept of dominant terms to informally determine the value of each of the following limits.

6.  $\lim_{x \rightarrow -\infty} \frac{4 + x - 7x^3}{14x^3 + x^2 + 2}$

7.  $\lim_{t \rightarrow -\infty} \frac{4t^2 + 1}{4t^3 - 1}$

8.  $\lim_{\gamma \rightarrow \infty} \frac{8}{2\gamma^3}$

9.  $\lim_{x \rightarrow \infty} \frac{(3x + 1)(6x - 2)}{(4 + x)(1 - 2x)}$

10.  $\lim_{t \rightarrow \infty} \frac{4e^t - 8e^{-t}}{e^t + e^{-t}}$

11.  $\lim_{t \rightarrow -\infty} \frac{4e^t - 8e^{-t}}{e^t + e^{-t}}$

## 9.7 Making Graphs

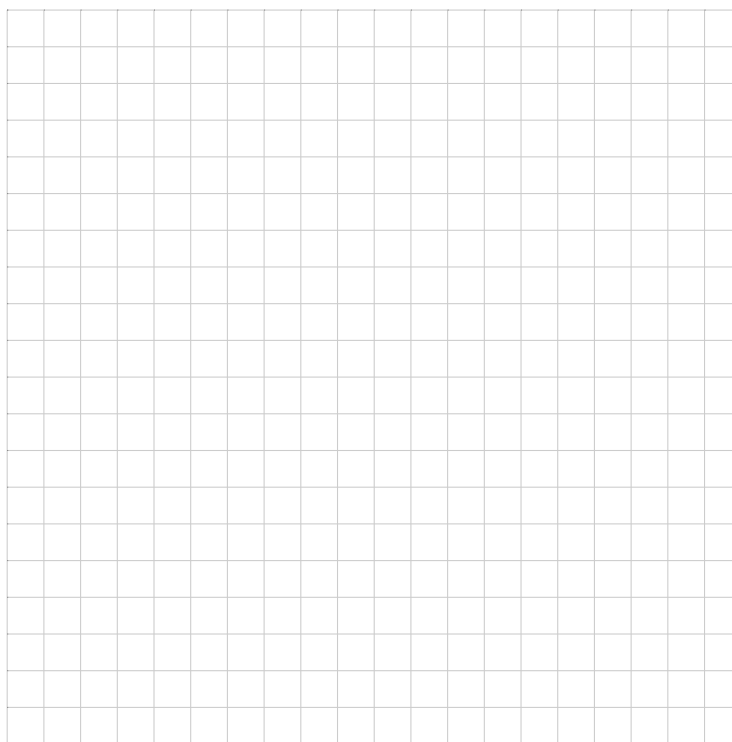
Let's put it all together and produce some graphs.

### Exercises

Consider the function  $f(x) = \frac{8x^2 - 8}{(2x - 4)^2}$ .

- Evaluate each of the following limits:  $\lim_{x \rightarrow \infty} f(x)$ ,  $\lim_{x \rightarrow -\infty} f(x)$ ,  $\lim_{x \rightarrow 2^-} f(x)$ , and  $\lim_{x \rightarrow 2^+} f(x)$ .

2. What are the horizontal and vertical asymptotes for  $f$ 's graph?
3. What are the horizontal and vertical intercepts for  $f$ 's graph?
4. Use the formulas  $f'(x) = \frac{4(1-2x)}{(x-2)^3}$  and  $f''(x) = \frac{4(4x+1)}{(x-2)^4}$  to help you accomplish each of the following.
  - State the critical numbers of  $f$ .
  - Create well-documented increasing/decreasing and concavity tables for  $f$ .
  - State the local minimum, local maximum, and inflection points on  $f$ . Make sure that you explicitly address all three types of points whether they exist or not.
5. Graph  $y = f(x)$  onto Figure 9.7.1. Choose a scale that allows you to clearly illustrate each of the features found in Exercises 1–4. Label all axes and asymptotes well and write the coordinates of each local extreme point and inflection point next to the point.



**Figure 9.7.1:**  $y = f(x)$

6. Check your graph using a graphing calculator.

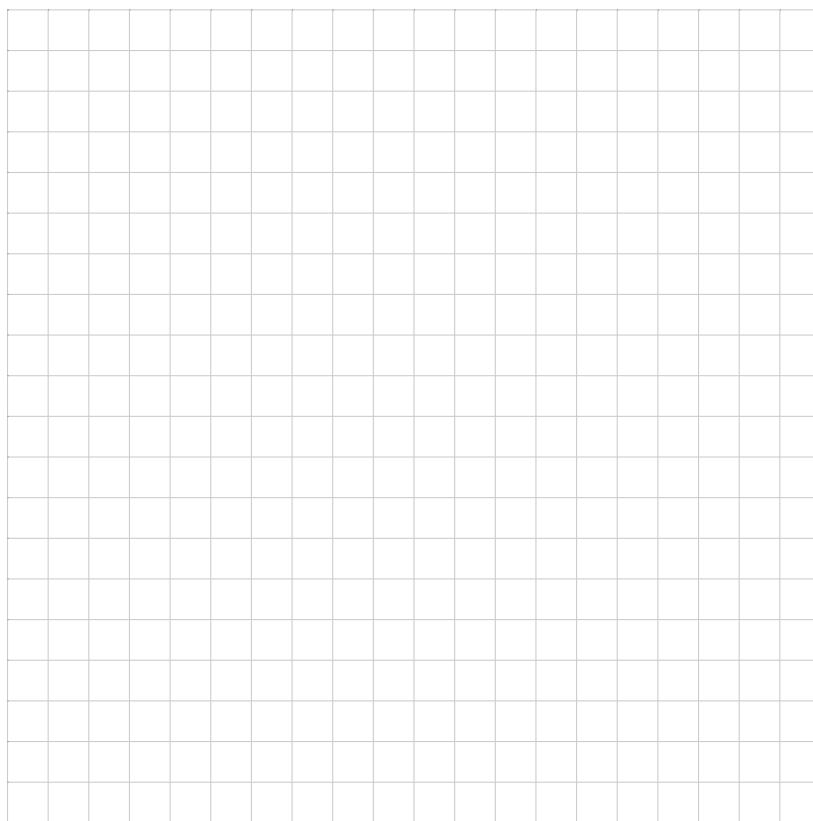
7. Following analysis similar to that implied in Exercises 1–5 graph the function

$$g(t) = \frac{1}{(e^t + 4)^2}.$$

Use the formulas

$$g'(t) = \frac{-2e^t}{(e^t + 4)^3} \quad g''(t) = \frac{4e^t(e^t - 2)}{(e^t + 4)^4}.$$

Check your graph using a graphing calculator.

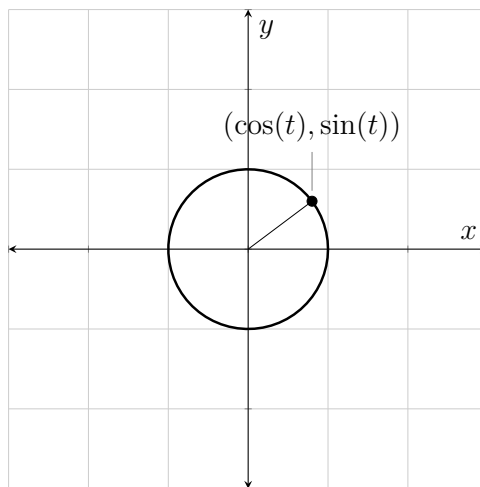


**Figure 9.7.2:**  $y = g(x)$

## 9.8 Supplement

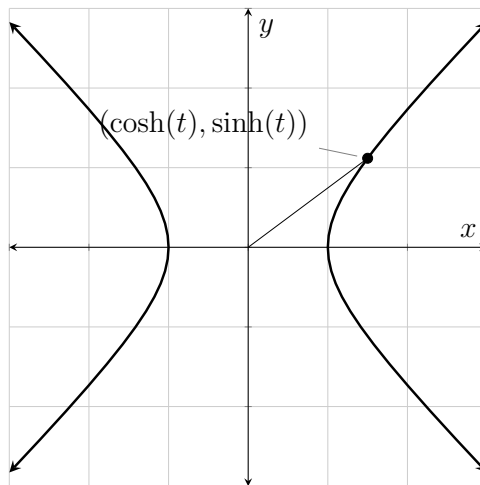
### Exercises

The sine and cosine function are called **circular trigonometric functions** because, for any given value of  $t$  the point  $(\cos(t), \sin(t))$  lies on the circle with equation  $x^2 + y^2 = 1$ . The area of the “pie slice” defined by the origin,  $(1, 0)$ , and the circular arc to  $(\cos(t), \sin(t))$  is exactly  $\frac{1}{2}t$ .



**Figure 9.8.1:**  $(\cos(t), \sin(t))$  on the circle  $x^2 + y^2 = 1$

There are analogous functions called **hyperbolic sine** and **hyperbolic cosine**. As you might suspect, these functions generate points that lie on a hyperbola; specifically, for all values of  $t$ , the point  $(\cosh(t), \sinh(t))$  lies on the hyperbola  $x^2 - y^2 = 1$ . The area of the “shark fin” defined by the origin,  $(1, 0)$ , and the hyperbolic arc to  $(\cosh(t), \sinh(t))$  is exactly  $\frac{1}{2}t$ .



**Figure 9.8.2:**  $(\cosh(t), \sinh(t))$  on the hyperbola  $x^2 - y^2 = 1$

It turns out that there are formulas for these hyperbolic trigonometric functions



in terms of functions we already know. Specifically:

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

1. Use the exponential formulas to verify the identity  $\cosh^2(t) - \sinh^2(t) = 1$ .
2. Use the exponential formulas to determine the first derivatives (with respect to  $t$ ) of  $\cosh(t)$  and  $\sinh(t)$ . Determine, by inference, the second derivative formulas for each of these functions.
3. Determine the critical numbers of  $\cosh$  and  $\sinh$ .
4. Determine the intervals over which  $\cosh$  and  $\sinh$  are increasing/decreasing and over which they are concave up/concave down.
5. Use the exponential formulas to determine each of the following limits:  
 $\lim_{t \rightarrow -\infty} \cosh(t)$ ,  $\lim_{t \rightarrow \infty} \cosh(t)$ ,  $\lim_{t \rightarrow -\infty} \sinh(t)$ , and  $\lim_{t \rightarrow \infty} \sinh(t)$ .
6. Use the information you determined in Exercises 3–5 to help you draw freehand sketches of  $y = \cosh(t)$  and  $y = \sinh(t)$ .
7. There are four more hyperbolic functions that correspond to the four additional circular functions; e.g.  $\tanh(t) = \frac{\sinh(t)}{\cosh(t)}$ . Find the exponential formulas for these four functions.
8. What would you guess to be the first derivative of  $\tanh(t)$ ? Take the derivative of the formula for  $\tanh(t)$  to verify your suspicion.
9. Let  $f(t) = \tanh(t)$ . What is the sign on  $f'(t)$  at all values of  $t$ ? What does this tell you about the function  $f$ ? What are the values of  $f(0)$  and  $f'(0)$ ?
10. Use the exponential formula for  $\tanh$  to determine  $\lim_{t \rightarrow -\infty} \tanh(t)$  and  $\lim_{t \rightarrow \infty} \tanh(t)$ .
11. Use the information you determined in Exercises 9 and 10 to help you draw a freehand sketch of  $y = \tanh(t)$ .

Consider the function  $k$  defined by  $k(t) = \sqrt[3]{t^8} - 256\sqrt[3]{t^2}$ .

12. What are the critical numbers of  $k$ ? Remember to show all relevant work! Remember that your formula for  $k'(t)$  needs to be a single, completely factored, fraction!
13. Create an increasing/decreasing table for  $k$ .
14. State each local minimum and maximum point on  $k$ .

Consider the function  $f$  defined by  $f(x) = \cos^2(x) + \sin(x)$  over the restricted domain  $[0, 2\pi]$ .

- 15. What are the critical numbers of  $f$ ? Remember to show all relevant work! Remember that your formula for  $f'(x)$  needs to be a single, completely factored, fraction!
- 16. Create an increasing/decreasing table for  $f$ .
- 17. State each local minimum and maximum point on  $f$ .

18. Consider  $g(t) = \frac{t+9}{t^3}$ . Find (and completely simplify)  $g''(t)$  and state all numbers where  $g''(t)$  is zero or undefined. Then construct a concavity table for  $g$  and state all of the inflection points on  $g$ .

For each of the following functions build increasing/decreasing tables and concavity tables and then state all local minimum points, local maximum points, and inflection points on the function. Also determine and state all horizontal asymptotes and vertical asymptotes for the function. Finally, draw a detailed sketch of the function.

19.  $f(x) = \frac{x-3}{(x+2)^2}$

20.  $g(x) = \sqrt[3]{x^2}(x+5)$

21.  $k(x) = \frac{(x-4)^2}{x+3}$

# Appendix A

## Limit Laws

**Replacement Limit Laws** The following laws allow you to replace a limit expression with an actual value. For example, the expression  $\lim_{t \rightarrow 5} t$  can be replaced with the number 5, the expression  $\lim_{t \rightarrow 7^-} 6$  can be replaced with the number 6, and the expression  $\lim_{z \rightarrow \infty} \frac{12}{e^z}$  can be replaced with the number 0.

In all cases both  $a$  and  $C$  represent real numbers.

**Theorem** (Limit Law R1).

$$\lim_{x \rightarrow a} x = \lim_{x \rightarrow a^-} x = \lim_{x \rightarrow a^+} x = a$$

**Theorem** (Limit Law R2).

$$\lim_{x \rightarrow a} C = \lim_{x \rightarrow a^-} C = \lim_{x \rightarrow a^+} C = \lim_{x \rightarrow \infty} C = \lim_{x \rightarrow -\infty} C = C$$

**Theorem** (Limit Law R3).

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) = \infty &\implies \lim_{x \rightarrow \infty} \frac{C}{f(x)} = 0 \\ \lim_{x \rightarrow \infty} f(x) = -\infty &\implies \lim_{x \rightarrow \infty} \frac{C}{f(x)} = 0 \\ \lim_{x \rightarrow -\infty} f(x) = \infty &\implies \lim_{x \rightarrow -\infty} \frac{C}{f(x)} = 0 \\ \lim_{x \rightarrow -\infty} f(x) = -\infty &\implies \lim_{x \rightarrow -\infty} \frac{C}{f(x)} = 0\end{aligned}$$

**Infinite Limit Laws (limits as the output increases or decreases without bound)** Many limit values do not exist. Sometimes the non-existence is caused by the function value either increasing without bound or decreasing without bound. In these special cases we use the symbols  $\infty$  and  $-\infty$  to communicate the non-existence of the limits.

Some examples of these type of non-existent limits follow.

- $\lim_{x \rightarrow \infty} x^n = \infty$  (if  $n$  is a positive real number)
- $\lim_{x \rightarrow -\infty} x^n = \infty$  (if  $n$  is an even positive integer)
- $\lim_{x \rightarrow -\infty} x^n = -\infty$  (if  $n$  is an odd positive integer)
- $\lim_{x \rightarrow \infty} e^x = \infty$ ,  $\lim_{x \rightarrow -\infty} e^{-x} = \infty$ ,  $\lim_{x \rightarrow \infty} \ln(x) = \infty$ ,  $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$

**Algebraic Limit Laws** The following laws allow you to replace a limit expression with equivalent limit expressions. For example, the expression

$$\lim_{x \rightarrow 5} (2 + x^2)$$

can be replaced with the expression

$$\lim_{x \rightarrow 5} 2 + \lim_{x \rightarrow 5} x^2.$$

*Limit Laws A1–A6 are valid if and only if every limit in the equation exists.*

In all cases  $a$  can represent a real number, a real number from the left, a real number from the right, the symbol  $\infty$ , or the symbol  $-\infty$ .

**Theorem** (Limit Law A1).  $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$

**Theorem** (Limit Law A2).  $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$

**Theorem** (Limit Law A3).  $\lim_{x \rightarrow a} (Cf(x)) = C \lim_{x \rightarrow a} f(x)$

**Theorem** (Limit Law A4).  $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

**Theorem** (Limit Law A5).  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if and only if  $\lim_{x \rightarrow a} g(x) \neq 0$

**Theorem** (Limit Law A6).  $\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$  when  $f$  is continuous at  $\lim_{x \rightarrow a} g(x)$

**Theorem** (Limit Law A7). If there exists an open interval centered at  $a$  over which  $f(x) = g(x)$  for  $x \neq a$ , then  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$  (provided that both limits exist).

Additionally, if there exists an open interval centered at  $a$  over which  $f(x) = g(x)$  for  $x \neq a$ , then  $\lim_{x \rightarrow a} f(x) = \infty$  if and only if  $\lim_{x \rightarrow a} g(x) = \infty$ , and, similarly,  $\lim_{x \rightarrow a} f(x) = -\infty$  if and only if  $\lim_{x \rightarrow a} g(x) = -\infty$ .

## Rational Limit Forms

- $\frac{\text{real number}}{\text{non-zero real number}}$ . Example:  $\lim_{x \rightarrow 3} \frac{2x+16}{5x-4}$  (the form is  $\frac{22}{11}$ )

The value of the limit is the number to which the limit form simplifies; for example,  $\lim_{x \rightarrow 3} \frac{2x+16}{5x-4} = 2$ . You can immediately begin applying limit laws if you are “proving” the limit value.

- $\frac{\text{non-zero real number}}{\text{zero}}$ . Example:  $\lim_{x \rightarrow 7^-} \frac{x+7}{7-x}$  (the form is  $\frac{14}{0}$ )

The limit value doesn’t exist. The expression whose limit is being found is either increasing without bound or decreasing without bound (or possibly both if you have a two-sided limit). You may be able to communicate the non-existence of the limit using  $\infty$  or  $-\infty$ . For example, if the value of  $x$  is a little less than 7, the value of  $\frac{x+7}{7-x}$  is positive. Hence you could write  $\lim_{x \rightarrow 7^-} \frac{x+7}{7-x} = \infty$ .

- $\frac{\text{zero}}{\text{zero}}$ . Example:  $\lim_{x \rightarrow -2} \frac{x^2-4}{x^2+3x+2}$  (the form is  $\frac{0}{0}$ )

This is an **indeterminate form limit**. You do not know the value of the limit (or even if it exists) nor can you begin to apply limit laws. You need to manipulate the expression whose limit is being found until the resultant limit no longer has indeterminate form. For example:

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{x^2-4}{x^2+3x+2} &= \lim_{x \rightarrow -2} \frac{(x+2)(x-2)}{(x+2)(x+1)} \\ &= \lim_{x \rightarrow -2} \frac{x-2}{x+1}. \end{aligned}$$

The limit form is now  $\frac{-4}{-1}$ ; you may begin applying the limit laws.

- $\frac{\text{real number}}{\text{infinity}}$ . Example:  $\lim_{x \rightarrow 0^+} \frac{1-4e^x}{\ln(x)}$  (the form is  $\frac{-3}{-\infty}$ )

The limit value is zero and this is justified by logic similar to that used to justify Limit Law R3.

- $\frac{\text{infinity}}{\text{real number}}$ . Example:  $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{1-4e^x}$  (the form is  $\frac{-\infty}{-3}$ )

The limit value doesn’t exist. The expression whose limit is being found is either increasing without bound or decreasing without bound (or possibly both if you have a two-sided limit). You may be able to communicate the non-existence of the limit using  $\infty$  or  $-\infty$ . For example, you can write  $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{1-4e^x} = \infty$ .

- $\frac{\text{infinity}}{\text{infinity}}$ . Example:  $\lim_{x \rightarrow \infty} \frac{3+e^x}{1+3e^x}$  (the form is  $\frac{\infty}{\infty}$ )

This is an **indeterminate form limit**. You do not know the value of the limit (or even if it exists) nor can you begin to apply limit laws. You need

to manipulate the expression whose limit is being found until the resultant limit no longer has indeterminate form. For example:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{3 + e^x}{1 + 3e^x} &= \lim_{x \rightarrow \infty} \left( \frac{3 + e^x}{1 + 3e^x} \cdot \frac{1/e^x}{1/e^x} \right) \\ &= \lim_{x \rightarrow \infty} \frac{\frac{3}{e^x} + 1}{\frac{1}{e^x} + 3}.\end{aligned}$$

The limit form is now  $\frac{1}{3}$ ; you may begin applying the limit laws.

There are other indeterminate limit forms, although the two mentioned here are the only two **rational indeterminate forms**. The other indeterminate forms are discussed in future calculus courses.

## Appendix B

### Derivative Formulas

$k$ ,  $a$ , and  $n$  represent constants;  $u$  and  $y$  represent functions of  $x$ .

**Table B.0.1:**

Basic Formulas	Chain Rule Format	Implicit Derivative Format
$\frac{d}{dx}(k) = 0$		
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(u^n) = nu^{n-1} \frac{d}{dx}(u)$	$\frac{d}{dx}(y^n) = ny^{n-1} \frac{dy}{dx}$
$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$	$\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}} \frac{d}{dx}(u)$	$\frac{d}{dx}(\sqrt{y}) = \frac{1}{2\sqrt{y}} \frac{dy}{dx}$
$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\frac{d}{dx}(\sin(u)) = \cos(u) \frac{d}{dx}(u)$	$\frac{d}{dx}(\sin(y)) = \cos(y) \frac{dy}{dx}$
$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\frac{d}{dx}(\cos(u)) = -\sin(u) \frac{d}{dx}(u)$	$\frac{d}{dx}(\cos(y)) = -\sin(y) \frac{dy}{dx}$
$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\frac{d}{dx}(\tan(u)) = \sec^2(u) \frac{d}{dx}(u)$	$\frac{d}{dx}(\tan(y)) = \sec^2(y) \frac{dy}{dx}$
$\frac{d}{dx}(\sec(x)) = \sec(x) \tan(x)$	$\frac{d}{dx}(\sec(u)) = \sec(u) \tan(u) \frac{d}{dx}(u)$	$\frac{d}{dx}(\sec(y)) = \sec(y) \tan(y) \frac{dy}{dx}$
$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$	$\frac{d}{dx}(\cot(u)) = -\csc^2(u) \frac{d}{dx}(u)$	$\frac{d}{dx}(\cot(y)) = -\csc^2(y) \frac{dy}{dx}$
$\frac{d}{dx}(\csc(x)) = -\csc(x) \cot(x)$	$\frac{d}{dx}(\csc(u)) = -\csc(u) \cot(u) \frac{d}{dx}(u)$	$\frac{d}{dx}(\csc(y)) = -\csc(y) \cot(y) \frac{dy}{dx}$
$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\tan^{-1}(u)) = \frac{1}{1+u^2} \frac{d}{dx}(u)$	$\frac{d}{dx}(\tan^{-1}(y)) = \frac{1}{1+y^2} \frac{dy}{dx}$
$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}} \frac{d}{dx}(u)$	$\frac{d}{dx}(\sin^{-1}(y)) = \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx}$
$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{ x \sqrt{x^2-1}}$	$\frac{d}{dx}(\sec^{-1}(u)) = \frac{1}{ u \sqrt{u^2-1}} \frac{d}{dx}(u)$	$\frac{d}{dx}(\sec^{-1}(y)) = \frac{1}{ y \sqrt{y^2-1}} \frac{dy}{dx}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(e^u) = e^u \frac{d}{dx}(u)$	$\frac{d}{dx}(e^y) = e^y \frac{dy}{dx}$
$\frac{d}{dx}(a^x) = \ln(a)a^x$	$\frac{d}{dx}(a^u) = \ln(a)a^u \frac{d}{dx}(u)$	$\frac{d}{dx}(a^y) = \ln(a)a^y \frac{dy}{dx}$
$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$	$\frac{d}{dx}(\ln(u)) = \frac{1}{u} \frac{d}{dx}(u)$	$\frac{d}{dx}(\ln(y)) = \frac{1}{y} \frac{dy}{dx}$
$\frac{d}{dx}( x ) = \frac{ x }{x}$	$\frac{d}{dx}( u ) = \frac{ u }{u} \frac{d}{dx}(u)$	$\frac{d}{dx}( y ) = \frac{ y }{y} \frac{dy}{dx}$

**Theorem** (Constant Factor Rule of Differentiation).

$$\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x))$$

Alternatively, if  $y = f(x)$ , then  $\frac{d}{dx}(ky) = k \frac{dy}{dx}$ .

**Theorem** (Sum/Difference Rule of Differentiation).

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

**Theorem** (Product Rule of Differentiation).

$$\frac{d}{dx}(f(x) \cdot g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x))$$

**Theorem** (Quotient Rule of Differentiation).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - f(x) \cdot \frac{d}{dx}(g(x))}{[g(x)]^2}$$

**Theorem** (Chain Rule of Differentiation).

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Alternatively, if  $u = g(x)$ , then  $\frac{d}{dx}(f(u)) = f'(u) \cdot \frac{d}{dx}(u)$ .

Alternatively, if  $y = f(u)$ , where  $u = g(x)$ , then  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ .

## Appendix C

### Some Useful Rules of Algebra

- For positive integers  $m$  and  $n$ ,  $\sqrt[n]{x^m} = x^{m/n}$ . This is universally true when  $x$  is positive. When  $x$  is negative, things are more complicated. It boils down to a choice. Some computer algebra systems choose to define an expression like  $(-8)^{1/3}$  as a certain negative real number (namely,  $-2$ ). Others choose to define an expression like  $(-8)^{1/3}$  as a complex number in the upper half plane of the complex plane ( $1 + i\sqrt{3}$ ). Because of the ambiguity, some choose to simply declare expressions like  $(-8)^{1/3}$ , with a negative base, to be undefined. The point is, if your computer tells you that say,  $(-8)^{1/3}$  is undefined, you should realize that you may be expected to interpret  $(-8)^{1/3}$  as  $-2$ .
- For real numbers  $k$  and  $n$ , and  $x \neq 0$ ,  $\frac{k}{x^n} = kx^{-n}$ .
- For a real number  $k \neq 0$ ,  $\frac{f(x)}{k} = \frac{1}{k}f(x)$ .
- For positive real numbers  $A$  and  $B$  and all real numbers  $n$ ,
  - $\ln(AB) = \ln(A) + \ln(B)$
  - $\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$
  - $\ln(A^n) = n \ln(A)$



# Appendix D

## Units of Measure

**Table D.0.1:** Distance

Unit	Stands For	Definition	Roughly
m	meter	the distance light travels in $1/299792458$ of a second	half the height of a typical doorway
mm	millimeter	exactly $1/1000$ of a meter	the thickness of a credit card
cm	centimeter	exactly $1/100$ of a meter	the width of your pinky finger
km	kilometer	exactly 1000 meters	in parts of Portland where the city blocks are square, about 12.5 blocks
in	inch	exactly 0.0254 meters	diameter of a bottle cap
ft	foot	exactly 12 inches	the exterior length of an average adult male's shoe
mi	mile	exactly 5280 feet	in parts of Portland where the city blocks are square, about 20 blocks

**Table D.0.2:** Area

Unit	Stands For	Definition	Roughly
cm <sup>2</sup>	square centimeter	the area of a square whose sides are each one centimeter	a little less than the circular area of a new pencil eraser
in <sup>2</sup>	square inch	the area of a square whose sides are each one inch	the area of a photograph on an ID card

**Table D.0.3:** Volume

Unit	Stands For	Definition	Roughly
cm <sup>3</sup>	cubic centimeter	the volume of a cube whose sides are each one centimeter	the volume of a blueberry
L	liter	exactly 1000 cubic centimeters	a bottle of wine is $\frac{3}{4}$ of a liter
mL	milliliter	exactly 1 cubic centimeter; $\frac{1}{1000}$ of a liter	the volume of a blueberry
in <sup>3</sup>	cubic inch	the volume of a cube whose sides are each one inch	the volume of an shelled whole walnut
gal	liter	exactly 231 cubic inches	the volume of the larger type of plastic milk containers sold in supermarkets

**Table D.0.4:** Time

Unit	Stands For	Definition	Roughly
s	second	the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom	the time it takes you to say the phrase “differential calculus”
min	minute	exactly 60 seconds	how long it takes to microwave a full dinner plate from the refrigerator
h	hour	exactly 3600 seconds; exactly 60 minutes	the length of one episode of a premium cable television show

**Table D.0.5:** Speed

Unit	Stands For	Definition	Roughly
mph	mile per hour	exactly one mile per hour	a typical walking speed is about three miles per hour

**Table D.0.6:** Mass

Unit	Stands For	Definition	Roughly
kg	kilogram	the mass of a certain cylindrical object made of a platinum-iridium alloy called the International Prototype of the Kilogram, which is stored in Sèvres, France	the mass of a 500-page paperback book
mg	milligram	exactly $1/1000000$ of a kilogram	the mass of a small snowflake

**Table D.0.7:** Angle

Unit	Stands For	Definition	Roughly
rad	radian	the angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle	a little less than the angles on an equilateral triangle
°	degree	the angle subtended at the center of a circle by an arc whose length is $1/360$ the circumference of the circle	a very small angle; if you could look directly at the bottom of the sun, and then moved your eyes to look at the top of the sun, you would have moved your eyes about half of one degree

**Table D.0.8:** Force

Unit	Stands For	Definition	Roughly
N	newton	the force needed to accelerate one kilogram from rest up to a speed of one meter per second with one second of time	if you hold an apple in the palm of your hand, it applies about two newtons of force

**Table D.0.9:** Power

Unit	Stands For	Definition	Roughly
kW	kilowatt	the power to transfer one thousand joules of energy in one second (one joule is the amount of energy used when one newton of force pushes an object one meter)	the power used by a nicer microwave oven

## Appendix E

# Solutions to Supplemental Exercises

### 1.4 Exercises

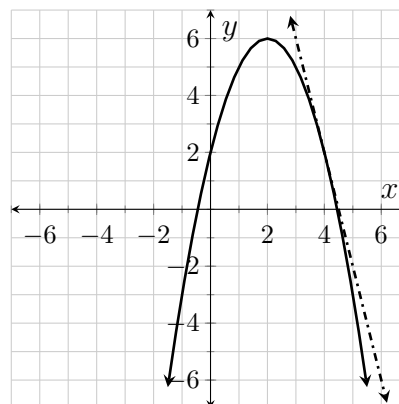
1.  $4 - 2x - h$  for  $h \neq 0$

2. The slope is 3.

3.  $-4 - h$

4.  $-4$

5.



6.  $-7$  for  $h \neq 0$

7.  $\frac{-7}{(x+h+4)(x+4)}$  for  $h \neq 0$

8.  $0$  for  $h \neq 0$

9.  $3t^2 + 3th + h^2 - 1$  for  $h \neq 0$

10.  $\frac{t^2+th-64}{t(t+h)}$  for  $h \neq 0$

11.  $60 - 32t - 16h$  for  $h \neq 0$

12.  $-71.2 \frac{\text{ft}}{\text{s}}$

13. The average rate of change in the coaster's velocity over the first 7.5 s of descent is  $-13\frac{1}{3} \frac{\text{ft/s}}{\text{s}}$ .

14. The average velocity experienced

by the ball between the 1.5th second of play and the third second of play is  $0 \frac{\text{m}}{\text{s}}$ .

15. Between the first swing and the 29th swing the average rate of change in the pendulum's period is  $-0.1 \frac{\text{s}}{\text{swing}}$ .

16. During the second minute of flight, the average rate of change in the rocket's acceleration is  $0.13 \frac{\text{mph}}{\text{ss}}$ .

## 2.14 Exercises

1.  $\lim_{t \rightarrow -\infty} g(t) = \infty$ ;  $\lim_{t \rightarrow -\infty} g(t)$  does not exist.

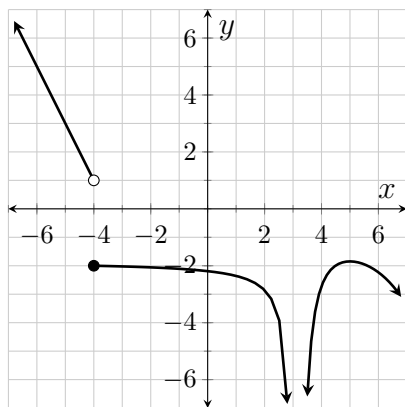
2.  $\lim_{x \rightarrow \infty} f(x) = 0$ ;  $\lim_{x \rightarrow 0} f(x)$  exists.

3.  $\lim_{t \rightarrow \frac{1}{3}^-} z(t) = \frac{2}{3}$ ;  $\lim_{t \rightarrow \frac{1}{3}^-} z(t)$  exists.

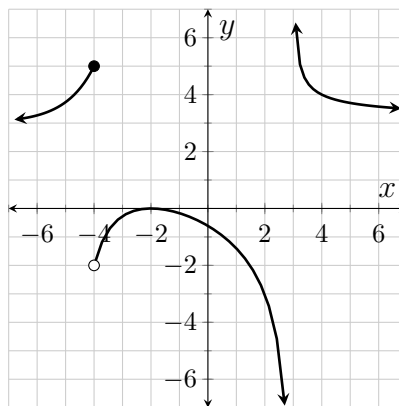
4.  $\lim_{\theta \rightarrow -1^+} g(\theta) = 3,000,000$ ;  $\lim_{\theta \rightarrow -1^+} g(\theta)$  exists.

5.  $\lim_{t \rightarrow \frac{7}{9}^+} T(t) = \infty$ ;  $\lim_{t \rightarrow \frac{7}{9}^+} T(t)$  does not exist.

6.



7.



8. Discontinuities at 0, 3, 4, and  $2\pi$ . Continuous from the left at 3. Removable discontinuities at 4 and  $2\pi$ .

9.  $k = 0$

10.  $\lim_{x \rightarrow 4^-} (5 - \frac{1}{x-4}) = \infty$ . This limit does not exist.

11.  $\lim_{x \rightarrow \infty} \frac{e^{2/x}}{e^{1/x}} = 1$ . This limit exists.

12.  $\lim_{x \rightarrow 2^+} \frac{x^2-4}{x^2+4} = 0$ . This limit exists.

13.  $\lim_{x \rightarrow 2^+} \frac{x^2-4}{x^2-4x+4} = \infty$ . This limit does not exist.

14.  $\lim_{x \rightarrow \infty} \frac{\ln(x) + \ln(x^6)}{7 \ln(x^2)} = \frac{1}{2}$ . This limit exists.

15.  $\lim_{x \rightarrow -\infty} \frac{3 + \frac{2}{x^2}}{\frac{3}{x^2} - 2} = -\frac{3}{2}$ . This limit exists.

16.  $\lim_{x \rightarrow \infty} \sin\left(\frac{\pi e^{3x}}{2e^x + 4e^{3x}}\right) = \frac{1}{\sqrt{2}}$ . This limit exists.      29.  $\lim_{x \rightarrow 0} e^{-x} = 1$ . This limit exists.
17.  $\frac{\ln(1/x)}{\ln(x/x)}$  does not exist.      30.  $\lim_{x \rightarrow \infty} \ln(x) = \infty$ . This limit does not exist.
18.  $\lim_{x \rightarrow 5} \sqrt{\frac{x^2 - 12x + 35}{5 - x}} = \sqrt{2}$ . This limit exists.      31.  $\lim_{x \rightarrow 1} \ln(x) = 0$ . This limit exists.
19.  $\lim_{h \rightarrow 0} \frac{4(3+h)^2 - 5(3+h) - 21}{h} = 19$ . This limit exists.      32.  $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$ . This limit does not exist.
20.  $\lim_{h \rightarrow 0} \frac{5h^2 + 3}{2 - 3h^2} = \frac{3}{2}$ . This limit exists.      33.  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ . This limit exists.
21.  $\lim_{h \rightarrow 0} \frac{\sqrt{9-h} - 3}{h} = -\frac{1}{6}$ . This limit exists.      34.  $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$ . This limit exists.
22.  $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin(\theta + \frac{\pi}{2})}{\sin(2\theta + \pi)} = -\frac{1}{2}$ . This limit exists.      35.  $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$ . This limit does not exist.
23.  $\lim_{x \rightarrow 0^+} \frac{\ln(x^e)}{\ln(e^x)} = -\infty$ . This limit does not exist.      36.  $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$ . This limit does not exist.
24.  $\lim_{x \rightarrow \infty} e^x = \infty$ . This limit does not exist.      37.  $\lim_{x \rightarrow \infty} e^{1/x} = 1$ . This limit exists.
25.  $\lim_{x \rightarrow -\infty} e^x = 0$ . This limit exists.      38.  $\lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$ . This limit exists.
26.  $\lim_{x \rightarrow 0} e^x = 1$ . This limit exists.      39.  $\lim_{x \rightarrow -\infty} \frac{1}{e^x} = \infty$ . This limit does not exist.
27.  $\lim_{x \rightarrow \infty} e^{-x} = 0$ . This limit exists.      40.  $\lim_{x \rightarrow \infty} \frac{1}{e^{-x}} = \infty$ . This limit does not exist.
28.  $\lim_{x \rightarrow -\infty} e^{-x} = \infty$ . This limit does not exist.      41.  $\lim_{x \rightarrow -\infty} \frac{1}{e^{-x}} = 0$ . This limit exists.

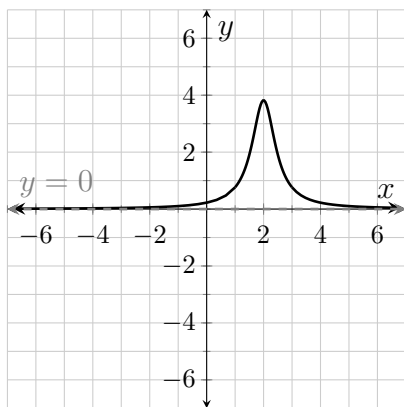
### 3.5 Exercises

1. Both methods yield  $f'(x) = 2x$ .      5.  $-71.2 \frac{\text{ft}}{\text{s}}$ .
2. Both methods yield  $f'(x) = \frac{1}{2\sqrt{x}}$ .      6.  $-32 \frac{\text{ft}}{\text{s}^2}$ .
3. Both methods yield  $f'(x) = 0$ .      7.  $\frac{\text{beats/min}}{\text{ft/min}}$ .
4. If  $f(x) = \sin(x)$ , then  $f'(x) = \cos(x)$ .      8.  $\frac{\text{gal/mi}}{\text{mi/hr}}$ .
5.  $\frac{\text{mi/hr}}{\text{s}}$       9.  $\frac{\text{mi}}{\text{s}}$ .
10.  $\frac{\text{mi}}{\text{s}}$

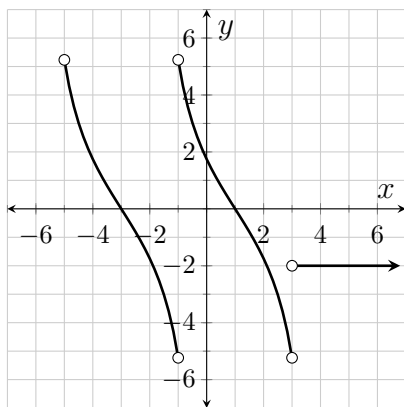
11. When Carl jogs at a pace of  $300 \frac{\text{ft}}{\text{min}}$  his heart rate is  $84 \frac{\text{beats}}{\text{min}}$ .
12. At the pace of  $300 \frac{\text{ft}}{\text{min}}$ , Carl's heart rate changes relative to his pace at a rate of  $0.02 \frac{\text{beats}/\text{min}}{\text{ft}/\text{min}}$ .
13. When Hanh drives her pick-up on level ground at a constant speed of 50 mph, the truck burns fuel at the rate of  $0.03 \frac{\text{gal}}{\text{mi}}$ .
14. (On level ground), at the speed of 50 mph, the rate at which Hanh's truck burns fuel changes relative to the speed at a rate of  $-0.006 \frac{\text{gal}/\text{mi}}{\text{mi}/\text{hr}}$ .
15. Twenty seconds after lift-off the space shuttle is cruising at the rate of  $266 \frac{\text{mi}}{\text{h}}$ .
16. Twenty seconds after lift-off the velocity of the space shuttle is increasing at the rate  $18.9 \frac{\text{mi}/\text{hr}}{\text{s}}$ .
17. Twenty seconds after lift-off the space shuttle is at an elevation of 0.7 mi.
18. Twenty seconds after lift-off the space shuttle's elevation is increasing at a rate of  $0.074 \frac{\text{mi}}{\text{s}}$ .
19.  $y = \frac{7}{32}(x - 1) + \frac{5}{8}$
20.  $\frac{2}{5}$

## 4.7 Exercises

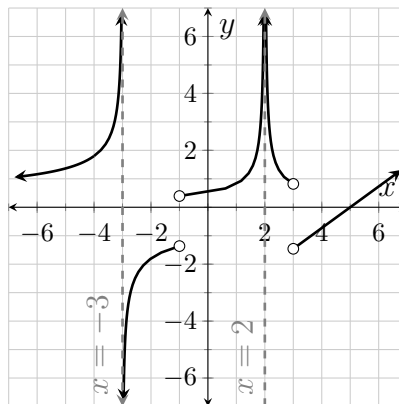
1.



2.



3.



4. 4.7.1(c), 4.7.1(d), 4.7.1(e)

5. 4.7.1(b), 4.7.1(d)

6. 4.7.1(a), 4.7.1(c)

7. 4.7.1(c), 4.7.1(d), 4.7.1(e)

8. 4.7.1(c), 4.7.1(d), 4.7.1(e)

9. 4.7.1(e), 4.7.1(f), 4.7.1(g)

10. 4.7.1(g)

11. 4.7.1(e), 4.7.1(f), 4.7.1(g)

12. 4.7.1(a), 4.7.1(c), 4.7.1(e), 4.7.1(f), 4.7.1(g)

13. 4.7.1(e), 4.7.1(f), 4.7.1(g)

14. The unit for  $V'(45)$  is  $\frac{\text{gal}}{\text{s}}$  and the unit for  $V''(45)$  is  $\frac{\text{gal/s}}{\text{s}}$ . The most realistic value for  $V'(45)$  is  $-1.2 \frac{\text{gal}}{\text{s}}$ .

15. The unit for  $R'(45)$  is  $\frac{\text{gal/s}}{\text{s}}$ , and the unit for  $R''(45)$  is  $\frac{\text{gal/s}}{\text{s}}$ . The most realistic value for  $R'(45)$  is  $-0.001$ .

16. The most realistic value for  $V''(45)$  is 0.001.

17. The statement is true for some functions and false for other functions.

18. The statement is true for some functions and false for other functions.

19. The statement is true for some functions and false for other functions.

20. The statement is true for some functions and false for other functions.

21. The statement is true for some functions and false for other functions.

22. Lisa is right.

23. Janice is right.

24. False

25. True

26. False

27. False

28. False

29. False

30. True

31. True

32. False

33. False

34. True

35. True

36. See the solutions section.

37.  $-1, 1, 4$

38. nowhere

39.  $(-5, 6)$

40.  $(-6, -3)$  and  $(-1, 1)$

41.  $(4, 6)$

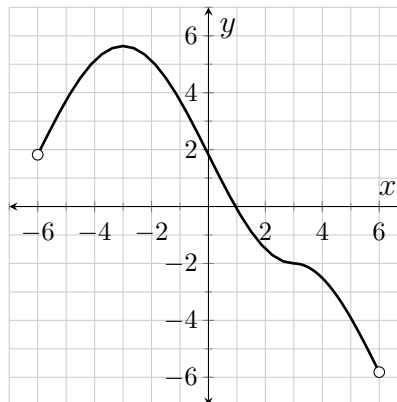
42.  $(-3, -1)$ ,  $(1, 4)$ , and  $(4, 6)$

43.  $(-6, -5)$

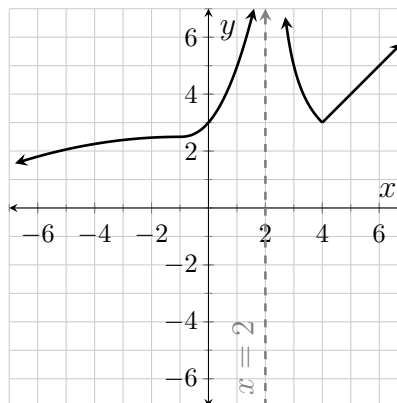
44. There is no way to tell.

45.  $-5$

46.  $y = -4(x + 3) + 14$



47.



48.



## 5.12 Exercises

1.  $\frac{1}{11\sqrt[11]{x^5}}$
2.  $\frac{35}{t^8}$
3.  $\frac{4}{\sqrt[3]{u^2}}$
4.  $e\pi\alpha^{\pi-1}$
5.  $-\frac{28}{\sqrt{t^9}}$
6.  $\frac{4}{3\sqrt[3]{t^2}}$
7.  $5[\cos^2(x) - \sin^2(x)]$  or  $5\cos(2x)$
8.  $\frac{5}{7}te^t(2+t)$
9.  $4(\ln(x) + 1)$
10.  $2x\sin^{-1}(x) + \frac{x^2}{\sqrt{1-x^2}}$
11.  $2t\tan^{-1}(t) + 1$
12.  $\frac{x^6 7^x}{3}(7 + x\ln(7))$
13.  $\frac{4e^\theta}{(e^\theta + 1)^2}$
14.  $\frac{2-8\ln(x)}{x^5}$
15.  $\frac{-9\sqrt{t}+10}{2(3\sqrt{t^3}-5t)^2}$
16.  $\frac{\sec^2(x)\tan^{-1}(x)(1+x^2)-\tan(x)}{(\tan^{-1}(x))^2(1+x^2)}$
17.  $\frac{1-2t\tan^{-1}(t)}{(1+t^2)^2}$
18.  $\frac{-4(\cos(\beta)+2\sin(\beta))}{(\sin(\beta)-2\cos(\beta))^2}$
19. See the solutions section.
20.  $xe^x(2\sin(x)\cos(x)+x\sin(x)\cos(x)+2x\cos^2(x)-x)$
21.  $y = 11(x - 5) + 30$
22.  $y = x + 4$
23.  $y = \frac{1}{2}$
24.  $x = 0$
25.  $\frac{d^2y}{dt^2} = 99\sqrt{t^7}$
26.  $f'$  positive over  $(-\infty, -8)$  and  $(2, \infty)$ ; negative over  $(-8, 2)$
27.  $f'$  increasing over  $(-3, \infty)$ ; decreasing over  $(-\infty, -3)$
28.  $f''$  positive over  $(-3, \infty)$ ; negative over  $(-\infty, -3)$
29.  $f'(x) = x^2 + 6x - 16$ ,  $f''(x) = 2x + 6$
30.  $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$ , and  $\frac{11\pi}{6}$
31.  $g'(t) = 3\cos(t)(1 + 2\sin(t))$ , with zeros at  $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$ , and  $\frac{11\pi}{6}$
32.  $y = 2$
33. 24,749
34. 860
35.  $\frac{2}{3}$
36. 537
37. -20
38. 494
39.  $\frac{289}{4}$
40. 2

**6.5 Exercises**

1. All three functions have derivatives that simplify to  $3e^{3x}$ .
2.  $k'(\theta) = 1$  over  $(-\frac{\pi}{2}, \frac{\pi}{2})$
3. See the solutions section.
4.  $-1$
5.  $k'(\frac{\pi}{2})$  is undefined.
6.  $(0, 2)$  and about  $(3.2, \infty)$
7.  $(2, \infty)$
8.  $(-2, 3)$  and  $(3, \infty)$
9.  $h$  is differentiable at  $-3$  with  $h'(-3) = 0$
10. yes
11. no
12. no
13. no
14. yes
15. yes
16.  $\frac{1}{2(1+x)\sqrt{x}}$
17.  $e^{e^{\sin(x)} + \sin(x)} \cos(x)$
18.  $-\frac{\sin(x)}{|\sin(x)|}$
19.  $\sec^2(x \sec(x)) \sec(x) (1 + x \tan(x))$
20.  $\sec(x) \sec(\sec(x)) (\sec(x) + \tan^2(x) \tan(\sec(x)))$
21.  $\frac{4x \cos(x^2)}{3 \sqrt[3]{\sin(x^2)}}$
22.  $4 \sin(x) (\sin(x) + 2x \cos(x))$
23.  $\frac{1}{x} \left( 1 + \frac{1}{\ln(x)} \right)$
24.  $-1 - \frac{1}{x}$
25.  $\frac{2}{3x} + \frac{4 \sec^2(x)}{3 \tan(x)}$
26.  $1 - \frac{1}{2(x+2)}$

27.  $\frac{ex^{e-1}}{x^e + e}$
28.  $4 \sec^4(e^x) \tan(e^x) e^x$
29.  $\frac{1}{\sqrt{e^{2x} - 1}}$
30.  $\frac{1}{2\sqrt{x^3}} \csc\left(\frac{1}{\sqrt{x}}\right) \cot\left(\frac{1}{\sqrt{x}}\right)$
31.  $\frac{1}{2\sqrt{x}} \cos(\sqrt{x})$
32.  $\frac{1}{1+4x^2}$
33.  $\frac{x^2}{3} \left(9 \sin\left(\frac{x}{3}\right) + x \cos\left(\frac{x}{3}\right)\right)$
34.  $\frac{14}{\sqrt{3}} \sqrt{x^5}$
35.  $e^{xe^x} \frac{xe^x + x^2 e^x - 1}{x^2}$
36.  $e^{xe^2} (1 + xe^2)$
37.  $\left(4 \sin^3(x) + \frac{1}{2\sqrt{\sin^3(x)}}\right) \cos(x)$
38.  $4(\sin(x) \cos(x^2) + x \cos(x) \cos(x^2) - 2x^2 \sin(x) \cdot \sin(x^2))$
39.  $\cos(x \cos^2(x)) \cos(x) (\cos(x) - 2x \sin(x))$
- 

## 7.4 Exercises

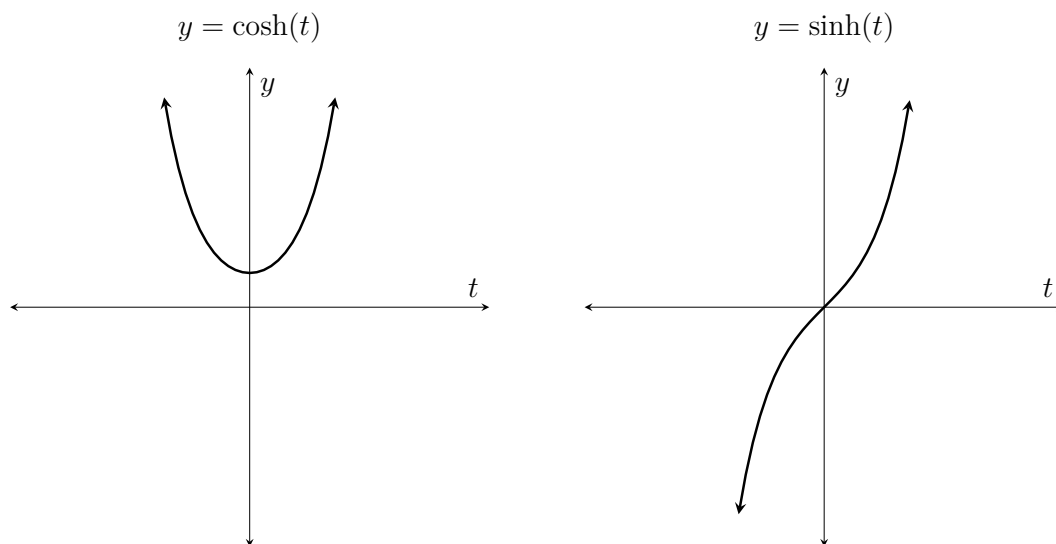
1.  $-1$  and  $1$
2.  $\frac{6+\pi\sqrt{3}}{12-2\pi\sqrt{3}}$
3.  $y = \frac{1}{3}(x-1) - 1$
4.  $\frac{x \sin(x)}{\sqrt{x-1}} \left(\frac{1}{x} + \cot(x) - \frac{1}{2x-2}\right)$
5.  $\frac{e^{2x}}{\sin^4(x) \sqrt[4]{x^5}} \left(2 - 4 \cot(x) - \frac{5}{4x}\right)$
6.  $\frac{\ln(4x^3)}{x^5 \ln(x)} \left(\frac{3}{x \ln(4x^3)} - \frac{5}{x} - \frac{1}{x \ln(x)}\right)$
- 

## 8.3 Exercises

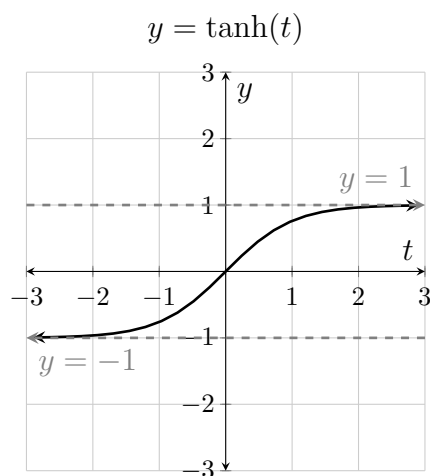
1.  $8 \frac{\text{ft}}{\text{s}}$
2. approximately  $2.701 \frac{\text{ft}}{\text{s}}$
3. approximately  $1.248 \frac{\text{cm}}{\text{min}}$
4. decreasing at approximately  $0.3188 \frac{\text{ft}}{\text{s}}$
5.  $0.2688G$  newtons per our, for  $G$  measured in  $\frac{\text{Nkm}^2}{\text{kg}^2}$
6. decreasing at approximately  $5.553 \frac{\text{in}}{\text{s}}$
-

## 9.8 Exercises

1. See the solutions section.
2.  $\frac{d}{dt}(\cosh(t)) = \sinh(t)$ ,  $\frac{d}{dt}(\sinh(t)) = \cosh(t)$ ,  $\frac{d^2}{dt^2}\cosh(t) = \cosh(t)$ , and  $\frac{d^2}{dt^2}\sinh(t) = \sinh(t)$
3. The only critical number for  $\cosh$  is 0. The function  $\sinh$  has no critical numbers.
4.  $\cosh$  is increasing on  $(0, \infty)$ , decreasing on  $(-\infty, 0)$ , and concave up on  $(-\infty, \infty)$ ;  $\sinh$  is increasing on  $(-\infty, \infty)$ , concave up on  $(0, \infty)$ , and concave down on  $(-\infty, 0)$
5.  $\lim_{t \rightarrow -\infty} \cosh(t) = \lim_{t \rightarrow \infty} \cosh(t) = \lim_{t \rightarrow \infty} \sinh(t) = \infty$ ;  $\lim_{t \rightarrow -\infty} \sinh(t) = -\infty$
- 6.



7.  $\tanh(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}$ ,  $\operatorname{sech}(t) = \frac{2}{e^t + e^{-t}}$ ,  $\operatorname{csch}(t) = \frac{2}{e^t - e^{-t}}$ , and  $\operatorname{coth}(t) = \frac{e^t + e^{-t}}{e^t - e^{-t}}$
8.  $\operatorname{sech}^2(t)$
9. positive;  $f$  is always increasing;  $f(0) = 0$ ;  $f'(0) = 1$
10.  $\lim_{t \rightarrow -\infty} \tanh(t) = -1$  and  $\lim_{t \rightarrow \infty} \tanh(t) = 1$



11.

12. 8,  $-8$ , and 0

13. See the solutions section.

14. local minimum points at  $(-8, -768)$  and  $(8, -768)$ ; local maximum point at  $(0, 0)$ 15.  $\frac{\pi}{6}$ ,  $\frac{\pi}{2}$ ,  $\frac{5\pi}{6}$ , and  $\frac{3\pi}{2}$ 

16. See the solutions section.

17. local minimum points at  $(0, 1)$ ,  $(\frac{\pi}{2}, 1)$ , and  $(\frac{3\pi}{2}, -1)$ ; local maximum points at  $(\frac{\pi}{6}, \frac{5}{4})$ ,  $(\frac{5\pi}{6}, \frac{5}{4})$ , and  $(2\pi, 0)$ 18.  $g''(t) = \frac{6(t+18)}{t^5}$ ;  $g''(t) = 0$  at  $-18$  and  $g''(t)$  is undefined at 0;

Interval	$g''$	$g$
$(-\infty, -18)$	positive	concave up
$(-18, 0)$	negative	concave down
$(0, \infty)$	positive	concave up

inflection point at  $(-18, \frac{1}{648})$ 

19. See the solutions section.

20. See the solutions section.

21. See the solutions section.