Domains of Multivariable Functions

Consider the function \( f(x, y) = \sqrt{9 - x^2 - y^2} \).

What are the values of \( f(-2, 1) \) and \( f(6, -2) \)?

\[
\begin{align*}
 f(-2, 1) &= \sqrt{9 - (-2)^2 - 1^2} \\
 &= 2
\end{align*}
\]

\[
\begin{align*}
 f(6, -2) &= \sqrt{9 - 6^2 - (-2)^2} \\
 &= \text{not a real number}
\end{align*}
\]

The ordered pair (6, -2) is not in the domain of \( f \).

Find the domain of \( f(x, y) \) and sketch the domain onto Figure 1.

\[
\begin{align*}
\text{Issue:} & \quad 9 - x^2 - y^2 \geq 0 \\
& \quad 9 \geq x^2 + y^2
\end{align*}
\]

**Domain Borders**

\[
\begin{align*}
 x^2 + y^2 &= 9
\end{align*}
\]

Find the domain of the function \( g(x, y) = \sin^{-1}(x - y + 3) \) and sketch the domain onto Figure 2.

\[
\begin{align*}
\text{Issue:} & \quad -1 \leq x - y + 3 \leq 1
\end{align*}
\]

**Boundaries**

\[
\begin{align*}
-1 &= x - y + 3 \\
\sum (x - y + 3) &= 1 \\
-x &= y + 3 \quad \sum \frac{y}{y} = -1
\end{align*}
\]
Find the domain of \( h(x,y) = \ln(x^2 - 9) + \cos^{-1}\left(\frac{x + y}{2}\right) - \tan^{-1}(x^2 + y^2 - 4) \) and sketch the domain onto Figure 3.

\[ h: \quad x^2 - 9 > 0 \implies x^2 > 9 \]

\[ \cos^{-1}: \quad -1 \leq \frac{x + y}{2} \leq 1 \]

\[ \tan^{-1}: \quad \text{issue free: Domain: } \mathbb{R} \]

**Boundaries**

\[ x^2 = 9 \quad \iff \quad x = \pm 3 \]

\[ \frac{x + y}{2} = \pm 1 \quad \iff \quad y = -x \pm 2 \]

What does the domain of the function \( f(x,y,z) = \frac{1}{\sqrt{4 - x^2 - y^2 - z^2}} \) look like?

**Domain:**

\[ 4 - x^2 - y^2 - z^2 > 0 \]

\[ y > x^2 + y^2 + z^2 \]

\[ \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{4} < 1 \]

The domain is the goosy part of a Cadbury egg, i.e., the inside of an ellipsoid (specifically \( \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \)).

Practice on finding domains of multivariable functions

Find and sketch the domain of each of the following functions on the provided worksheet.

a. \( f(x,y) = \sin^{-1}\left(\frac{3}{x}\right) + \ln(x + 2 - y) \)

b. \( f(x,y) = \sqrt{x^2 - 1} - \cos^{-1}(x^2 + y^2 - 1) \)

c. \( f(x,y) = \sqrt{-xy} + \sqrt{x^2 - y} \)
Level curves are parallel to the $xy$-plane. ($z = k$)

Trace curves are perpendicular to the $xy$-plane (i.e. lie in a plane perpendicular to the $xy$-plane).

**Level curves of Multivariable Functions**

Three dimensional functions ($z = f(x, y)$) always have an attendant set of level curves of form $z = k$ where $k$ is a real number. The level curve $z = k$ is the set of all points in the $xy$-plane where the output from the function is $f(x, y) = k$. This idea is illustrated in Figure 1.

![Figure 1: Level curves for $z = x^2 + y^2 + 10$](image)

For each of the functions in Table 1, four level curves are shown in one of the attached figures 1 - 6 and a plot of the actual surface is shown in one of the attached figures 7-12. For each function, identify the figure number that corresponds to its level curve plot, the figure number that corresponds to its surface plot, and the plane equation for each of the four level curves shown in its level curve plot.

<table>
<thead>
<tr>
<th>Function</th>
<th>Surface Plot</th>
<th>Level Curves</th>
<th>Red Curve</th>
<th>Blue Curve</th>
<th>Brown Curve</th>
<th>Green Curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z = \sin^{-1}\left(\frac{1}{x}\right)$</td>
<td>5</td>
<td>8</td>
<td>$z = \frac{-3}{4}$</td>
<td>$z = \frac{3}{4}$</td>
<td>$z = \frac{1}{4}$</td>
<td>$z = \frac{1}{4}$</td>
</tr>
<tr>
<td>$z = \sqrt{x^2 - 1}$</td>
<td>6</td>
<td>12</td>
<td>$z = 1$</td>
<td>$z = 4$</td>
<td>$z = 2$</td>
<td>$z = 3$</td>
</tr>
<tr>
<td>$z = \sqrt{\frac{y}{x}}$</td>
<td>1</td>
<td>11</td>
<td>$z = 3$</td>
<td>$z = 1$</td>
<td>$z = 1$</td>
<td>$z = 2$</td>
</tr>
<tr>
<td>$z = \ln(y - x)$</td>
<td>4</td>
<td>10</td>
<td>$z = 1$</td>
<td>$z = 2$</td>
<td>$z = 0$</td>
<td>$z = 2$</td>
</tr>
<tr>
<td>$z = \cos^{-1}\left(\frac{x}{y}\right)$</td>
<td>3</td>
<td>7</td>
<td>$z = 0$</td>
<td>$z = 2$</td>
<td>$z = 3$</td>
<td>$z = 1$</td>
</tr>
<tr>
<td>$z = \ln\left(\frac{x}{y}\right)$</td>
<td>2</td>
<td>9</td>
<td>$z = 0$</td>
<td>$z = 1$</td>
<td>$z = 3$</td>
<td>$z = 2$</td>
</tr>
</tbody>
</table>

One of figures 13-16 shows level curves for the function $z = \frac{y}{x} + \frac{1}{x} + 3$. Which one?
Which of figures 19-22 are the level curves for the surface in Figure 17? How about Figure 18?

Level surfaces of Multivariable Functions

The temperature (degrees Calvin) at a point inside a vat of grease is given by the formula

$$T(x, y, z) = \frac{500z}{x^2 + 2y^2}.$$  

The axis system for the vat has been set up so that one corner of the vat corresponds to the point (2, 2, 2) and all points in the vat have coordinates that are each greater than two. A level surface, $w = k$, for this function is the set of all points in the vat that satisfy $T(x, y, z) = k$. Contextually these level surfaces are called isothermal surfaces. What is the shape of the isothermal surfaces in this vat?

\[
\frac{500 \cdot z}{x^2 + 2y^2} = k \quad \Rightarrow \quad 500 \cdot z = k \cdot x^2 + 2ky^2
\]

\[
\Rightarrow \quad z = \frac{x^2}{\frac{500}{k}} + \frac{y^2}{\frac{250}{k}}
\]

The isothermal surfaces are paraboloids.